

Infinite Random Geometric Graphs*

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Abstract. We introduce a new class of countably infinite random geometric graphs, whose vertices V are points in a metric space, and vertices are adjacent independently with probability $p \in (0, 1)$ if the metric distance between the vertices is below a given threshold. For certain choices of V as a countable dense set in \mathbb{R}^n equipped with the metric derived from the L_∞ -norm, it is shown that with probability 1 such infinite random geometric graphs have a unique isomorphism type. The isomorphism type, which we call GR_n , is characterized by a geometric analogue of the existentially closed adjacency property, and we give a deterministic construction of GR_n . In contrast, we show that infinite random geometric graphs in \mathbb{R}^2 with the Euclidean metric are not necessarily isomorphic.

Keywords: graphs, geometric graphs, adjacency property, random graphs, metric spaces, isometry

1. Introduction

The last decade has seen the emergence of the study of large-scale complex networks such as the web graph consisting of web pages and the links between them, the friendship network on Facebook, and networks of interacting proteins in a cell. Several new random graph models were proposed for such networks, and existing models were modified in order to fit the data gathered from these real-life networks. See the books [3, 9] for surveys of such models. A recent trend in stochastic graph modelling is the study of *geometric graph models*. In geometric graph models, vertices are embedded in a metric space, and the formation of edges is influenced by the relative position of the vertices in this space. Geometric graph models have found applications in modelling wireless networks (see [13, 14]), and in modelling the web graph and other complex networks ([1, 5, 12]). In real-world networks, the underlying metric space is a representation of the hidden reality that leads to the formation

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of edges. Thus, for the World Wide Web, web pages are embedded in a high dimensional *topic space*, where pages that are positioned close together in the space contain similar content.

The graph model we study is a variation on the *random geometric graph*, where vertices are chosen at random according to a given probability distribution from a given metric space, and two vertices are adjacent if the distance between the two vertices is no larger than some fixed real number. Random geometric graphs have been studied extensively in their own right (see, for example, [2, 10] and the book [16]).

Analysis of stochastic graph models usually focusses on asymptotic results, which hold for cases where the number of vertices is sufficiently large. An alternative approach is to study the *infinite limit*; that is, the infinite graph that results when the number of vertices reaches infinity. Studying the infinite limit is a well-known tool for studying scientific models, and it can help to recognize large-scale structure and long-term behaviour (see [4, 15]). In this paper, we study the infinite limit of a geometric graph model that is a geometric extension of the classic random graph model $G(\mathbb{N}, p)$.

One of the most studied examples of an infinite limit graph arising from a stochastic model is the infinite random graph. The probability space $G(\mathbb{N}, p)$ consists of graphs with vertices \mathbb{N} , so that each distinct pair of integers is adjacent independently with a fixed probability $p \in (0, 1)$. Erdős and Rényi [11] discovered that with probability 1, all $G \in G(\mathbb{N}, p)$ are isomorphic. A graph G is *existentially closed* (or *e.c.*) if for all finite disjoint sets of vertices A and B (one of which may be empty), there is a vertex $z \notin A \cup B$ adjacent to all vertices of A and to no vertex of B . We say that z is *correctly joined* to A and B . The unique isomorphism type of countably infinite e.c. graph is named the *infinite random graph*, or the *Rado* graph, and is written R . See the surveys [7, 8] for additional background on R .

We now introduce the geometric graph model on which this paper is based. Consider a metric space S with distance function

$$d: S \times S \rightarrow \mathbb{R},$$

$\delta \in \mathbb{R}^+$, a countable subset V of S , and $p \in (0, 1)$. The *Local Area Random Graph* $\text{LARG}(V, \delta, p)$ has vertices V , and for each pair of vertices u and v with $d(u, v) < \delta$, an edge is added independently with probability p . Note that V may be either finite or infinite. The LARG model generalizes well-known classes of random graphs. For example, special cases of the LARG model include the random geometric graphs (where $p = 1$), and the binomial random graph $G(n, p)$ (where S has finite diameter D , and $\delta \geq D$).

In the case V is infinite, our goals are to investigate what adjacency properties are satisfied by graphs generated by the LARG model, and determine when the model generates a unique isomorphism type of countable graph. We prove that with probability 1, graphs in $\text{LARG}(V, \delta, p)$ satisfy a certain adjacency property which is a metric analogue of the e.c. property; see Theorem 2.1. The so-called geometric e.c. property requires that vertices z correctly joined to A and B may be found as close as we like to points in V . Explicit examples of graphs with the geometric e.c. property

are given in Theorem 2.2. For metric spaces such as \mathbb{R}^n with the L_∞ -metric, the geometric e.c. property gives rise to a unique isomorphism type of graph; see Theorems 3.3, 3.6, and 3.7. The main tool here is a generalization of isometry called a step-isometry. Our results are sensitive to the metric used. Non-isomorphism results for other metrics are given in Section 4. In particular, we show in Theorem 4.1 that in \mathbb{R}^2 with the Euclidean metric, there exist non-isomorphic geometric e.c. graphs.

All graphs considered are simple, undirected, and countable unless otherwise stated. If S is a set of vertices in G , then we use the notation $G[S]$ for the subgraph of G induced by S . We use the notation $G \leq H$ if G is an induced subgraph of H . We refer to an isomorphism type as *isotype*, and denote isomorphic graphs by $G \cong H$. Given a metric space S with distance function d , define the (open) *ball of radius δ around x* by

$$B_\delta(x) = \{u \in S : d(u, x) < \delta\}.$$

We will sometimes just refer to $B_\delta(x)$ as a *ball*. A subset V is *dense* in S if for every point $x \in S$, every ball around x contains at least one point from V . We refer to $u \in S$ as points or vertices, depending on the context. Throughout, let \mathbb{N} denote the non-negative natural numbers, and \mathbb{N}^+ denote the positive natural numbers. For a reference on graph theory the reader is directed to [17], while [6] is a reference on metric spaces.

2. Geometrically e.c. Graphs

As noted in the introduction, the unique isotype of the infinite random graph R is characterized by the e.c. property. In this section we define a geometric analogue of this property. As will be demonstrated in Section 3, this property characterizes the unique isotype of graphs obtained from countable dense sets in \mathbb{R}^n , provided we consider a particular metric.

Let $G = (V, E)$ be a graph whose vertices are points in the metric space S with metric d . The graph G is *geometrically e.c. at level δ* (or δ -g.e.c.) if for all $\delta' > 0$ so that $0 < \delta' < \delta$, for all $x \in V$, and for all disjoint finite sets A and B so that $A \cup B \subseteq B_\delta(x)$, there exists a vertex $z \notin A \cup B \cup \{x\}$ so that

- (i) z is correctly joined to A and B ,
- (ii) for all $u \in A \cup B$, $d(u, z) < \delta$, and
- (iii) $d(x, z) < \delta'$.

This definition implies that V is dense in itself, since we may choose A and B to be empty. Further, if G is δ -g.e.c., then G is δ' -g.e.c. for any $\delta' < \delta$.

The geometrically e.c. property bears clear similarities with the e.c. property. The important differences are that a correctly joined vertex must exist only for sets A and B which are contained in an open ball with radius δ and centre x , and it must be possible to choose the vertex z correctly joined to A and B arbitrarily close to x ; see Figure 1.

Theorem 2.1. *Let (S, d) be a metric space and V a countable subset of S which is dense in itself. If $\delta > 0$ and $p \in (0, 1)$, then with probability 1, $\text{LARG}(V, \delta, p)$ is δ -g.e.c.*

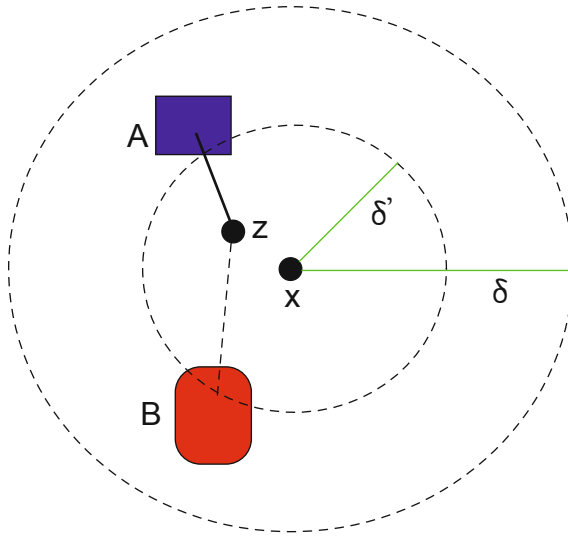


Figure 1: The δ -g.e.c. property.

Proof. Fix $x \in V$, disjoint finite subsets A and B in $B_\delta(x) \cap (V \setminus \{x\})$, and $0 < \delta' < \delta$. Let

$$\beta = \max\{d(x, v) : v \in A \cup B\}.$$

Then $\beta < \delta$. Let $\varepsilon = \min\{\delta - \beta, \delta'\}$. Consider the set $Z = B_\varepsilon(x) \cap V$. Note that ε is chosen so that for any $z \in Z$, $d(z, x) < \delta'$, and for all $u \in A \cup B$,

$$d(u, z) < d(u, x) + d(x, z) < \beta + \varepsilon \leq \delta.$$

For any graph G in $\text{LARG}(V, \delta, p)$, the probability that any vertex $z \in Z$ is correctly joined to A and B equals $p^{|A|}(1-p)^{|B|}$. The probability that no vertex in Z is correctly joined to A and B equals

$$P = \prod_{z \in Z} 1 - \left(p^{|A|}(1-p)^{|B|}\right).$$

Since V is dense in itself, Z contains infinitely many points; hence, $P = 0$. As there are only countably many choices for x , A , and B , and a countable union of measure 0 sets is measure 0, the proof follows. ■

A graph $G = (V, E)$ whose vertices are points in the metric space (S, d) has *threshold* δ if for all edges $uv \in E$, $d(u, v) < \delta$. A graph that is geometrically g.e.c. at level δ and has threshold δ is called a *geometric δ -graph*. By definition, a graph G generated by $\text{LARG}(V, \delta, p)$ has threshold δ , and, if V is countable and dense in itself, G is a geometric δ -graph. Thus, this random graph model generates geometric δ -graphs.

We construct geometric δ -graphs deterministically as follows. Given $\delta > 0$, a countable set V which is dense in itself, and a linear ordering $\sigma: \mathbb{N} \rightarrow V$ of V , define $GR(V, \delta, \sigma)$ as the limit of a chain of finite graphs R_t , where $R_t \leq R_{t+1}$ for any $t > 1$, and $\{\sigma(i) : 1 \leq i \leq t\} \subseteq V(R_t)$. Let R_1 be the trivial graph with vertex set $\sigma(1)$. Assume that R_t is defined and $\{\sigma(i) : 1 \leq i \leq t\} \subseteq V(R_t)$.

We now define R_{t+1} . Enumerate all pairs (A, x) so that $A \subseteq V(R_t)$ and $x \in V(R_t) \setminus A$ so that $A \subseteq B_\delta(x)$, via a lexicographic ordering based on σ . For each pair (A, x) , in order, choose $z = z_{A,x}$ to be the least index point in V (according to σ) such that z has not been chosen for any previous pairs (A, x) ,

$$B_\delta(z) \cap V(R_t) = B_\delta(x) \cap V(R_t), \tag{2.1}$$

and

$$d(z, x) < \min\{1/t, \delta\}.$$

Note that such a vertex exists as V is dense and R_t is finite. Join z to all vertices in A and to no other vertices of R_t . The graph R_{t+1} is formed by adding all the vertices $z_{A,x}$ for all ordered pairs (A, x) and their incident edges. Finally, if $\sigma(t+1)$ is not present already in R_{t+1} , add it as an isolated vertex.

Observe that by (2.1), $GR(V, \delta, \sigma)$ is a δ -threshold graph.

Theorem 2.2. *The graph $GR(V, \delta, \sigma)$ is δ -g.e.c.*

Proof. To show that $GR(V, \delta, \sigma) = (V, E)$ is δ -g.e.c., choose $0 < \delta' < \delta$, a vertex $x \in V$, and disjoint sets $A, B \subseteq V \setminus \{x\}$ so that $A \cup B \subseteq B_\delta(x)$. Let $t > 0$ be chosen so that $A \cup B \cup \{x\} \subseteq V(R_t)$ and $1/t < \delta'$. Let $z = z_{A,x}$ be the vertex in R_{t+1} added to R_t to extend (A, x) . Then z is correctly joined to A and B and $d(x, z) < \delta'$. ■

It should be emphasized that we do not claim that the graphs $GR(V, \delta, \sigma)$ are the unique isotypes of δ -g.e.c. graphs with vertex set V . The theme of when two δ -g.e.c. graphs are isomorphic will be explored in the next two sections.

Balls with radius δ in δ -g.e.c. graphs contain copies of R , and hence, contain isomorphic copies of all countable graphs.

Theorem 2.3. *Let $U \subseteq S$ be so that $U \subseteq B_\delta(x)$ for some $x \in U$. Then a δ -g.e.c. graph with vertex set U is e.c., and so is isomorphic to R .*

Proof. Let G be a graph with vertex set U , where U is as stated. Assume that G is δ -g.e.c. Let A and B be any pair of disjoint, finite subsets of $U \subseteq B_\delta(x)$, and let $0 < \delta' < \delta$. Then by the δ -g.e.c. condition there exists a vertex $z \in B_{\delta'}(x) \cap U$ so that z is correctly joined to A and B . ■

The converse of Theorem 2.3 is false, in general. For example, consider the metric space (\mathbb{R}, d) , where d is the Euclidean metric, and let $\delta = 1$. Fix U an infinite clique in R , and let $U' = V(R) \setminus U$. Embed the vertices of U in \mathbb{R} so that they form a set that is dense in $B_{1/2}(0)$. Embed the vertices of U' so that they form a set that is dense in $B_1(0) \setminus B_{1/2}(0)$. Now choose $y \in U$ so that $d(0, y) < 1/4$, and let $A = \emptyset$, and $B = \{b\}$, where $b \in U \setminus \{y\}$. Let $\delta' = 1/4$. Note that $A \cap B \subseteq B_{\delta'}(y)$. The embedding of the vertices of R is such that all vertices in $B_{\delta'}(y)$ are in U , so they are all adjacent to b .

Thus, $B_{\delta'}(y)$ does not contain any vertex correctly joined to A and B , and thus, this embedding of R is not δ -g.e.c.

We finish with the following theorem which shows that there exists a close relationship between graph distance and metric distance in any graph that is δ -g.e.c. We denote the closure of set V in S by \bar{V} . The set W is *convex* if for every pair of points x and y in W , there exists a point z such that

$$d(x, z) + d(z, y) = d(x, z).$$

Theorem 2.4. *Let $G = (V, E)$ be geometric δ -graph, and let \bar{V} be convex. Let $u, v \in V$ so that $d(u, v) > \delta$. Then the graph distance between u and v in G equals $\lfloor d(u, v)/\delta \rfloor + 1$.*

Theorem 2.4 directly leads to the following corollary, which supplies motivation for proofs of the isomorphism results in the next section, and will be used to prove the non-isomorphism results of the final section.

Corollary 2.5. *If \bar{V} and \bar{W} are convex, and there is a δ -g.e.c. graph with vertices V and a γ -g.e.c. graph with vertex set W which are isomorphic via f , then for every pair of vertices $u, v \in V$,*

$$\lfloor d(u, v)/\delta \rfloor = \lfloor d(f(u), f(v))/\gamma \rfloor.$$

We supply a generalization of isometry, motivated by Theorem 2.4 and Corollary 2.5. Given metric spaces (S, d_S) and (T, d_T) , sets $V \subseteq S$ and $W \subseteq T$, and positive real numbers δ and γ , a *step-isometry at level (δ, γ)* from V to W is a surjective map $f: V \rightarrow W$ with the property that for every pair of vertices $u, v \in V$,

$$\lfloor d_S(u, v)/\delta \rfloor = \lfloor d_T(f(u), f(v))/\gamma \rfloor.$$

Every isometry is a step-isometry, but the converse is false, in general. For example, consider \mathbb{R} with the Euclidean metric, let $\delta = \gamma = 1$, and let $V = [0, 1)$ and $W = [0, 0.5)$. Then $f: V \rightarrow W$ given by $f(x) = x/2$ is a step-isometry, but is not an isometry.

Proof of Theorem 2.4. Let $u, v \in V$. Let $k = \lfloor d(u, v)/\delta \rfloor + 1$. By assumption, $k \geq 2$. Note that the choice of k supplies that

$$(k - 1)\delta \leq d(u, v) < k\delta.$$

Let ℓ be the graph distance of u and v , and note that $\ell > 1$ since G is a δ -threshold graph.

To show that $\ell \geq k$, let $v_0v_1 \cdots v_\ell$, where $v_0 = u, v_\ell = v$, be a shortest path in G from u to v . Since G has threshold δ , $d(v_{i-1}, v_i) < \delta$ for $i = 1, \dots, \ell$. Therefore,

$$\begin{aligned} (k - 1)\delta &\leq d(u, v) \\ &\leq \sum_{i=1}^{\ell} d(v_{i-1}, v_i) \\ &< \ell\delta, \end{aligned}$$

and so $\ell \geq k$.

Next, we show how to construct a path of length k from u to v in G , which will prove that $\ell \leq k$. Let $\varepsilon = (k\delta - d(u, v))/k$, so $d(u, v) = k(\delta - \varepsilon)$.

The set \overline{V} is convex, so for every pair of vertices x, y , there exists a point $z \in \overline{V}$ so that $d(x, z) + d(z, y) = d(x, y)$. Using this property, we can obtain a sequence of points in \overline{V} between u and v whose successive distances add up to $d(u, v)$, and which are at most $\varepsilon/4$ apart. We can then choose vertices x_1, \dots, x_{k-1} from this sequence so that $d(x_i, x_{i+1}) < \delta - 3\varepsilon/4$ for $i = 0, \dots, k-1$, where $x_0 = u$ and $x_k = v$. For $1 \leq i < k$, we may then find $w_i \in V$ so that $d(w_i, x_i) < \varepsilon/8$. Letting $w_0 = u$ and $w_k = v$, we have that for $i = 0, \dots, k-1$,

$$\begin{aligned} d(w_i, w_{i+1}) &\leq d(w_i, x_i) + d(x_i, x_{i+1}) + d(x_{i+1}, w_{i+1}) \\ &< \delta - 3\varepsilon/4 + 2\varepsilon/8 \\ &< \delta - \varepsilon/2. \end{aligned} \tag{2.2}$$

Let $v_0 = w_0 = u$. Now we successively apply the δ -g.e.c. property to choose $v_i \in V$ so that

- (i) $d(v_i, w_i) < \varepsilon/2$, and
- (ii) v_i is adjacent to v_{i-1} in G .

It then follows that $v_0v_1 \dots v_k$ is the desired path of length k . More precisely, fix i , $1 \leq i < k-1$, and assume v_{i-1} exists so that item (i) holds. Note that (i) and (2.2) implies that

$$d(w_i, v_{i-1}) \leq d(w_{i-1}, w_i) + d(w_{i-1}, v_{i-1}) < \delta.$$

Therefore, $v_{i-1} \in B_\delta(w_i)$, and so we can find a vertex v_i in $B_{\varepsilon/2}(w_i)$ which is adjacent to v_{i-1} . To choose the vertex v_{k-1} , let $v_k = w_k = v$. By the same argument as before, $d(w_{k-1}, v_{k-2}) < \delta$. Since $w_k = v_k$, $d(w_{k-1}, v_k) < \delta$. So $\{v_{k-2}, v_k\} \subseteq B_\delta(w_{k-1})$. Therefore, there exist a vertex v_{k-1} which is adjacent to v_{k-2} and $v_k = v$. ■

3. Isomorphism Results

In this section we consider metric spaces where the geometric e.c. property gives a unique isotype of graph. We work in the space \mathbb{R} with the usual metric defined by $d(x, y) = |x - y|$. (We will not mention this explicitly unless there is room for confusion.) The main result of the section—which serves as the template for more general results—is given in Theorem 3.3 below, and it asserts that geometric δ -graphs whose vertex set satisfies some general conditions in \mathbb{R} all are of the same isomorphism type. The conditions on the vertex set is that V must be dense in \mathbb{R} , and it must satisfy the property given by the following definition. A set $V \subseteq \mathbb{R}$ is *integer distance free at level δ* (or δ -idf) if for any two distinct elements u, v in V , $d(u, v)/\delta \notin \mathbb{N}$. An example is \mathbb{Q} , which is δ -idf for any fixed irrational δ . The following two lemmas show that the restriction to δ -idf sets does not lead to a great loss of generality.

Lemma 3.1. *Fix a countable set V that is dense in \mathbb{R} , and fix $\delta > 0$. Then for any $\varepsilon > 0$, there exists a positive $\delta' \in (\delta - \varepsilon, \delta + \varepsilon)$ so that V is δ' -idf.*

Proof. Let

$$D_V = \left\{ \frac{1}{k}d(u, v) : u, v \in V, k \in \mathbb{N} \right\}.$$

Since V and \mathbb{N} are countable, D_V is also countable. Thus, for any $\varepsilon > 0$, the interval $(\delta - \varepsilon, \delta + \varepsilon)$ must contain an element $\delta' \in \mathbb{R} - D_V$. ■

For the proofs in this section, we use a representation of points in terms of integer multiples of δ and the remainders. Precisely, fix $\delta > 0$ and $v_0 \in \mathbb{R}$. Each $v \in \mathbb{R}$ may be uniquely represented as

$$v = v_0 + q(v)\delta + r(v),$$

where $q(v) = \lfloor (v - v_0)/\delta \rfloor$ and $0 \leq r(v) < \delta$. In this representation, we will refer to δ as the *offset*, and to v_0 as the *anchor*. We will omit to state the anchor and offset explicitly wherever it is clear from the context. The term $r(v)$ is called the *representative* of v and $q(v)$ the *quotient*.

Lemma 3.2. *Fix a countable set V dense in \mathbb{R} , and $\delta > 0$. Then V has a countable subset $W \subseteq V$ which is dense in \mathbb{R} and is δ -idf.*

Proof. Let $\{q_i : i \geq 1\}$ be an enumeration of the rationals. We inductively construct a sequence of sets $W_n \subseteq V$ so that W_n is δ -idf, and for every $1 \leq i \leq n$, W_n contains a vertex w so that $d(w, q_i) < 1/n$, and $W_n \subseteq W_{n+1}$ for all $n \geq 1$. Choose $w \in (q_1 - 1, q_1 + 1)$, and let $W_1 = \{w\}$.

For the inductive step, assume W_n is as required. Construct W_{n+1} by iteratively adding points $w_i, i = 1, \dots, n + 1$ so that $d(w_i, q_i) < 1/(n + 1)$ while maintaining the induction hypothesis. Let $\mathcal{R} = \{r(w) : w \in W_n\} \cup \{0, \delta\}$, where the representation has anchor w , the unique element in W_1 , and offset δ . For each i , let $a = \max\{r : r \in \mathcal{R}, r \leq r(q_i)\}$ and $b = \min\{r : r \in \mathcal{R}, r > r(q_i)\}$. Note that $a < b$. Let $a' = w + q(q_i)\delta + a$, where w is the anchor of the representation, and $b' = w + q(q_i)\delta + b$. Now choose

$$w_i \in (a', b') \cap (q_i - 1/(n + 1), q_i + 1/(n + 1)) \cap V.$$

Since V is dense in \mathbb{R} , such an element of V must exist.

Finally, let $W = \bigcup_{i \geq 1} W_n$. By construction, W is δ -idf. To show that W is dense in \mathbb{R} , choose $r \in \mathbb{R}$ and $\varepsilon > 0$. Let n be so that $1/n < \varepsilon/2$, and choose $q \in \mathbb{Q}$ so that $d(q, r) < \varepsilon/2$. Let j be the index of q in the enumeration of \mathbb{Q} , so $q = q_j$. Let $N = \max\{n, j\}$. Then by the induction hypothesis, W_N contains a vertex w so that $d(w, q) < 1/N < \varepsilon/2$. Thus, $d(w, r) < \varepsilon$. ■

Theorem 3.3. *Let V and W be two countable dense subsets of \mathbb{R} , and let $\delta, \gamma > 0$. Let V be δ -idf and W be γ -idf. If G is a geometric δ -graph with vertex set V and H is a geometric γ -graph with vertex set W , then $G \cong H$.*

The proof of the theorem (and others analogous to it in this section) build up an isomorphism as a step-isometry. In the proofs we use an alternative characterization of step-isometries, given by the following lemma.

Lemma 3.4. *Let V and W be subsets of \mathbb{R} , and let δ and γ be two non-negative real numbers. Assume V is δ -idf and W is γ -idf. Then a surjective function $f: V \rightarrow W$ is a step-isometry at level (δ, γ) if and only if the following two conditions hold.*

- (1) *For every $u, v \in V$, if $r(u) < r(v)$ then $r(f(u)) < r(f(v))$.*
- (2) *For every $u \in V$, $q(u) = q(f(u))$,*

where the representation of elements of V has offset δ and that of W has offset γ , and the anchor of the representation of W is the image of the anchor of the representation of V under f .

Proof. Assume first that items (1) and (2) hold, and fix $u, v \in V$. Let $u' = f(u)$ and $v' = f(v)$. Assume without loss of generality that $v > u$, by hypothesis, this implies that $v' > u'$. Then $d(u, v) = v - u = (v - v_0) - (u - v_0)$, and so

$$d(u, v) = (v - v_0) - (u - u_0) = (q(v) - q(u))\delta + (r(v) - r(u)).$$

Hence,

$$\lfloor d(u, v)/\delta \rfloor = q(v) - q(u) - s,$$

where $s = 0$ if $r(v) \geq r(u)$, and $s = 1$ otherwise. Similarly,

$$\lfloor d(u', v')/\gamma \rfloor = q(v') - q(u') - s',$$

where $s' = 0$ if $r(v') \geq r(u')$, and $s' = 1$ otherwise. By hypothesis, we have that $s = s'$ and $q(v) - q(u) = q(v') - q(u')$, and by the idf conditions, $r(u) \neq r(v)$ and $r(u') \neq r(v')$. It follows that f is a step-isometry at level (δ, γ) .

Now assume that f is a step-isometry at level (δ, γ) . Let $v_0 \in V$ and $w_0 \in W$ be so that $f(v_0) = w_0$ and consider the representations of elements of V and W with offsets δ and γ , and anchors v_0 and w_0 , respectively.

Condition (2) follows immediately from the definition of step-isometry. For the proof of (1), fix any $u, v \in V$, and let $u' = f(u)$ and $v' = f(v)$. Then

$$\lfloor d(u, v)/\delta \rfloor = \lfloor q(v) - q(u) + (r(v) - r(u))/\delta \rfloor = q(v) - q(u) - s,$$

where $s = -\lfloor (r(v) - r(u))/\delta \rfloor$; similarly, $\lfloor d(u', v')/\gamma \rfloor = q(v') - q(u') - s'$, where $s' = -\lfloor (r(v') - r(u'))/\delta \rfloor$. Since f is a step-isometry, $s = s'$, and by the idf properties, $s \neq 0$ and $s' \neq 0$. If $s = s' = 0$, then both $r(v) > r(u)$ and $r(v') > r(u')$; if $s = s' = 1$, then $r(v) < r(u)$ and $r(v') < r(u')$. Thus, item (1) holds. ■

Proof of Theorem 3.3. The proof follows using a variant of the back-and-forth method (used to show that R is the unique isotype of e.c. graph). Let $V = \{v_i: i \geq 0\}$ and $W = \{w_i: i \geq 0\}$. For $i \geq 0$, we inductively construct a sequence of pairs of sets (V_i, W_i) and isomorphisms $f_i: G[V_i] \rightarrow H[W_i]$, so that for all $i \geq 1$, $v_i \in V_i$, $w_i \in W_i$, $V_i \subseteq V_{i+1}$ and $W_i \subseteq W_{i+1}$, and f_{i+1} extends f_i . It follows that

$$\bigcup_{i \in \mathbb{N}} f_i: G \rightarrow H$$

is an isomorphism. As an additional induction hypothesis we require that f_i is a step-isometry from V_i to W_i at level (δ, γ) . Specifically, we maintain conditions in items

(1) and (2) from Lemma 3.4, where the representation of elements of V has offset δ and anchor v_0 , and the representation of elements of W has offset γ and anchor w_0 .

Let $V_0 = \{v_0\}$, $W_0 = \{w_0\}$, and define f_0 by $f_0(v_0) = w_0$. Then $q(v_0) = q(w_0) = 0$ and $r(v_0) = r(v'_0) = 0$, so the base case of the induction follows. For the induction step, fix $i \geq 0$. To construct f_{i+1} from f_i we first *go forth* by finding an image of v_{i+1} . In the following, f refers to f_i and $v = v_{i+1}$.

Define

$$a = \max\{r(f(u)) : u \in V_i \text{ and } r(u) < r(v)\} \cup \{0\},$$

$$b = \min\{r(f(u)) : u \in V_i \text{ and } r(u) > r(v)\} \cup \{\delta\}.$$

Since the sets defining a and b are disjoint, and the idf property guarantees that all remainders in W are distinct, we have that $a < b$.

In order to maintain the induction hypothesis, $r(f(v))$ must lie in $[a, b)$, and $q(f(v))$ must equal $q(v)$. Let $k = q(v)$, and consider the interval

$$I = (k\gamma + a, k\gamma + b).$$

Any vertex in I will qualify as a candidate for $f(v)$, so that f_{i+1} is a step-isometry at level (δ, γ) . We must then find a vertex in I that will also guarantee that f is an isomorphism, by making sure it has the correct neighbours. For this, we apply the γ -g.e.c. condition of H .

In order to apply the γ -g.e.c. condition, we need to ensure that the images of all neighbours of v in V_i lie in a γ -ball. Since G has threshold δ , we consider all vertices of V_i that lie in a δ -ball around v . Let $Y = B_\delta(v) \cap V_i$, and fix $x \in I \cap W$. Such a vertex x exists since W is dense in \mathbb{R} . By definition of I , $q(x) = k$. We claim that

$$f(Y) \subseteq B_\gamma(x). \tag{3.1}$$

To prove this, let $u \in Y$. Since $q(v) = k$ and $d(u, v) < \delta$, it follows that $|q(u) - k| \leq 1$. Hence, $q(u)$ is one of $k, k - 1$, or $k + 1$.

If $q(u) = k$, then $q(f(u)) = k$ by induction hypothesis, so $d(f(u), x) < \gamma$. If $q(u) = k - 1$, then $r(u) > r(v)$, so $r(f(u)) > b$ by definition of b . Hence,

$$\begin{aligned} d(f(u), x) &= x - f(u) \\ &< k\gamma + b - (k - 1)\gamma - r(f(u)) \\ &< \gamma. \end{aligned}$$

The final case is when $q(u) = k + 1$. Then $r(f(u)) \leq a$, so we have that

$$\begin{aligned} d(f(u), x) &= f(u) - x \\ &< (k + 1)\gamma + r(f(u)) - k\gamma - a \\ &\leq \gamma. \end{aligned}$$

In all cases, $f(u) \in B_\gamma(x)$, and (3.1) follows.

Since G has threshold δ , $N(v) \cap V_i \subseteq Y$. Now let $A = f(N(v) \cup V_i)$ and $B = (W_i \cap B_\delta(x)) \setminus A$. Then $A \cup B \subseteq B_\gamma(x) \cap W_i$. Let $\varepsilon > 0$ be chosen such that $B_\varepsilon(x) \subseteq I$. We now use the γ -g.e.c. property of H to find a point $z \in B_\varepsilon(x)$ which is adjacent to all vertices in A and no other vertices of (the finite set) W_i . Thus, we can add z to W_i to form W_{i+1} and add v to V_i to form V_{i+1} , and set $f_{i+1}(v) = z$. Observe that f_{i+1} is an isomorphism.

To finish the induction step, if $w_{i+1} \notin W_{i+1}$ then we may *go back*, by finding an image $z = f_{i+1}^{-1}(w_{i+1})$ in an analogous fashion. We then add z to V_{i+1} , and maintain that f_{i+1} is an isomorphism. ■

Combining the above result with Theorem 2.1, the following corollary is now immediate.

Corollary 3.5. *There exists a unique isotype of graph, written GR_1 , such that for all countable dense subsets V of \mathbb{R} , $\delta > 0$ so that V is δ -idf, and $p \in (0, 1)$, the graph $LARG(V, \delta, p)$ is isomorphic to GR_1 .*

The isomorphism type of GR_1 does not depend on the choices of V , δ or p ; moreover, the same result holds for any 1-dimensional normed vector space with the metric derived from the norm. For this reason, we name GR_1 the *infinite random geometric graph of dimension 1*. Note that GR_1 has infinite diameter (unlike R , which has diameter 2). Note that, for any countable set $V \subseteq \mathbb{R}$, any ordering σ of v , and any real $\delta > 0$ so that V is δ -idf, the deterministic construction process $R(V, \sigma, \delta)$ described in the previous section gives explicit representations of GR_1 .

We may extend Theorem 3.3 to sets that are not necessarily dense in all \mathbb{R} . The only additional condition required is that there exists a step-isometry between the two sets. For example, consider the rational intervals $V = [a, b)$ and $W = [a', b')$, where $\lfloor (b - a)/\delta \rfloor = \lfloor (b' - a')/\gamma \rfloor$. Consider the bijective map $f: V \rightarrow W$ defined by

$$f(x) = \begin{cases} a' + q(x)\gamma + r(b')r(x)/r(b), & \text{if } r(x) \leq r(b), \\ a' + (q(x) + 1)\gamma + (\gamma - r(b'))(\delta - r(x))/(\delta - r(b)), & \text{if } r(x) > r(b), \end{cases}$$

where $q(x)$, $r(x)$ and $r(b)$ refer to the representation of elements V with offset δ and anchor a , and $r(b')$ refers to the representation of elements W with offset γ and anchor a' . In other words, f is a convex mapping of the intervals $[a + k\delta, a + k\delta + r(b))$, $k = 0, 1, \dots, q(b)$, to the intervals $[a' + k\gamma, a' + k\gamma + r(b'))$, respectively, and of the intervals $[a + r(b) + k\delta, a + (k + 1)\delta)$ to the intervals $[a' + r(b') + k\gamma, a' + (k + 1)\gamma)$. It is straightforward to verify that f is a step-isometry at level (δ, γ) , so any geometric δ -graph and geometric γ graph with vertex sets V and W , respectively, are isomorphic. Another setting we consider is where V and W are disjoint unions of rational intervals for which there exists a step-isometry between the endpoints of the intervals of V to the endpoints of the intervals of W .

Theorem 3.6. *Let V and W be two countable subsets of \mathbb{R} , and let $\delta, \gamma > 0$. Let F be a bijective step-isometry from V to W at level (δ, γ) , and let V be δ -idf and W be γ -idf. If G is a geometric δ -graph with vertex set V and H is a geometric γ -graph with vertex set W , then $G \cong H$.*

Proof. Let $V = \{v_i : i \geq 0\}$ and $W = \{w_i : i \geq 0\}$, where $w_i = F(v_i)$. As in the proof of Theorem 3.3, we inductively construct a sequence of pairs of sets (V_i, W_i) ($i \geq 0$) and isomorphisms $f_i : G[V_i] \rightarrow H[W_i]$, so that for all $i \geq 1$, $v_i \in V_i$, $w_i \in W_i$, $V_i \subseteq V_{i+1}$ and $W_i \subseteq W_{i+1}$, and f_{i+1} extends f_i . As an additional part of the induction hypothesis, we require that f_i satisfies the following three conditions.

- (1) For every $u, v \in V$, $r(u) \leq r(v)$ if and only if $r(f(u)) \leq r(f(v))$.
- (2) For every $u, v \in V$, $r(u) \leq r(v)$ if and only if $r(f(u)) \leq r(F(v))$.
- (3) For every $u \in V$, $q(u) = q(f(u))$.

The first two conditions are those stated in Lemma 3.4, so this implies that f_i is a step-isometry at level (δ, γ) . We can also conclude from this lemma that for all $u, v \in V$, $r(u) \leq r(v)$ if and only if $r(F(u)) \leq r(F(v))$.

Let $V_0 = \{v_0\}$ and $W_0 = \{w_0\}$, and set $f_0(v_0) = w_0$. Conditions (1) and (3) follow as in the proof of Theorem 3.3. Condition (2) follows from the fact that $w_0 = F(v_0) = f(v_0)$. For the induction step, fix $i \geq 0$. we construct f_{i+1} from f_i by first finding an image of v_{i+1} . In the following, f refers to f_i , and $v = v_{i+1}$.

Let

$$M_a = \{u : u \in V_i \text{ and } r(F(u)) \leq r(F(v))\},$$

$$M_b = \{u : u \in V_i \text{ and } r(F(u)) > r(F(v))\},$$

and

$$a = \max\{x : x = r(f(u)) \text{ or } x = r(F(u)) \text{ where } u \in M_a\},$$

$$b = \min\{x : x = r(f(u)) \text{ or } x = r(F(u)) \text{ where } u \in M_b\}.$$

We have that $a < b$, since the order of the representatives of vertices in V_i is preserved under f and under F . (See the similar argument in the proof of Theorem 3.3.)

In order to maintain conditions (1) and (2) of the induction hypothesis, $r(f(v))$ should lie in $[a, b)$, and because of condition (3), $q(f(v))$ must equal $q(v)$. Let $k = q(v)$, and consider the interval $I = (k\gamma + a, k\gamma + b)$. From the definition of a and b it follows that $F(v) \in I$.

The remainder of the proof is now analogous to the proof of Theorem 3.3 and so is only sketched here. Let $x = F(v)$. We can show that $f(B_\delta(v) \cap V_i) \subseteq B_\gamma(x)$. We can then invoke the γ -g.e.c. condition of H to find a vertex z in I which is correctly joined to the vertices in W_i so that an isomorphism is maintained if we set $f(v) = w$. Finally, we finish the induction step by going back and finding a suitable image $f^{-1}(w_{i+1})$. ■

Finally, we consider the behaviour in higher dimensions. Theorem 3.3 extends to \mathbb{R}^n with $n > 1$, provided we use the *product metric*; that is, the metric derived from the L_∞ norm, defined by:

$$d(u, v) = \max\{|v_i - u_i| : 1 \leq i \leq n\},$$

where u_i denotes the i -th component of u . Hence, we obtain unique isotypes of infinite random geometric graphs in all finite dimensions. For the remainder of the section, n is a positive integer, and d is assumed to be the metric defined above.

Theorem 3.7. Consider the metric space (\mathbb{R}^n, d) , where d is the product metric defined above. Let V and W be two countable sets dense in \mathbb{R}^n , and let $\delta, \gamma > 0$ so that V is δ -idf and W is γ -idf. If G is a geometric δ -graph with vertex set V and H is a geometric γ -graph with vertex set W , then $G \cong H$. In particular, for all choices of V and δ , there is unique isomorphism type of geometric δ -graphs in (\mathbb{R}^n, d) , written GR_n .

Theorem 3.7 is sensitive to the choice of metric. In Section 4, we will show that the conclusion of Theorem 3.7 for d the Euclidean metric fails even for $n = 2$. The following provides the key tool for our proof of Theorem 3.7. As the proof is straightforward generalization of Lemma 3.4, it is omitted.

Lemma 3.8. Let V and W be subsets of \mathbb{R}^n with the product metric, let $v_0 \in V$ and $w_0 \in W$, and let δ and γ be two non-negative real numbers.

Then a surjective function $f: V \rightarrow W$ is a step-isometry at level (δ, γ) if the following two conditions hold for all $u, v \in V$ and for all $i, 1 \leq i \leq n$:

- (1) $r(u_i) \leq r(v_i)$ if and only if $r(f(u)_i) \leq r(f(v)_i)$.
- (2) $q(u_i) = q(f(u)_i)$,

where the representation of the i -th coordinate of elements of V has offset δ and anchor $(v_0)_i$; the representation of the i -th coordinate of elements of W has offset γ and anchor $(w_0)_i$,

Proof of Theorem 3.7. Let $V = \{v_i: i \geq 0\}$ and $W = \{w_i: i \geq 0\}$. We inductively construct a sequence of pairs of sets (V_i, W_i) ($i \geq 0$) and isomorphisms $f_i: G[V_i] \rightarrow H[W_i]$, so that for all $i \geq 1, v_i \in V_i, w_i \in W_i, V_i \subseteq V_{i+1}$ and $W_i \subseteq W_{i+1}$, and f_{i+1} extends f_i . As an additional induction hypothesis we require that f_i satisfies conditions (1) and (2) from Lemma 3.8.

As in the proof of Theorem 3.3, for the base case we take $V_0 = \{v_0\}, W_0 = \{w_0\}$, and $f_0(v_0) = w_0$. For the induction step, fix $i \geq 0$. we construct f_{i+1} from f_i by first finding an image of v_{i+1} . In the following, f refers to f_i , and $v = v_{i+1}$.

For all $j, 1 \leq j \leq n$, define

$$a_j = \max\{r(f(u)_j) : u \in V_i \text{ and } r(u_j) \leq r(v_j)\},$$

$$b_j = \min\{r(f(u)_j) : u \in V_i \text{ and } r(u_j) > r(v_j)\}.$$

In order to maintain the induction hypothesis, for all $j, r(f(v))_j$ should lie in interval $[a_j, b_j)$, and $q(f(v)_j)$ should be equal to $q(v_j)$. Let $k_j = q(v_j)$, and consider the product set

$$I = \prod_{1 \leq j \leq n} (k_j\gamma + a_j, k_j\gamma + b_j).$$

Any vertex in I will qualify as a candidate for $f(v)$ so that f satisfies conditions (1) and (2) from Lemma 3.8. The remainder of the proof is analogous to that of Theorem 3.3, and so is omitted. ■

A *step-isometric isomorphism* is an isomorphism of graphs that is a step-isometry. In the base step in the proof of Theorem 3.7, if we are given induced subgraphs V_0 and W_0 such that $f_0: V_0 \rightarrow W_0$ is a step-isometric isomorphism, then the rest of the proof follows as before. Hence, we have the following corollary, which shows that the graphs GR_n act transitively on step-isometric isomorphic induced subgraphs.

Corollary 3.9. *Let G and H be finite induced subgraphs of GR_n for some positive integer n . A step-isometric isomorphism $f: G \rightarrow H$ extends to an automorphism of GR_n .*

Deleting a point from a dense set V with the idf property in \mathbb{R}^n gives another dense set with the idf property. Hence, we have the following *inexhaustibility* property.

Corollary 3.10. *For all $n > 0$ and vertices x in GR_n , $GR_n - x \cong GR_n$.*

We can combine Theorems 3.6 and 3.7 to obtain a result about isomorphisms between graphs with vertex sets in \mathbb{R}^n if there exists a special type of map between the sets. Given a set $V \subseteq \mathbb{R}^n$, denote the i -th component set of V as

$$V_i = \{x_i : x \in V\}.$$

Theorem 3.11. *Consider the metric space (\mathbb{R}^n, d) , where d is the product metric defined above. Let V and W be two countable sets in \mathbb{R}^n , and let $\delta, \gamma > 0$ so that V_i is δ -idf and W_i is γ -idf for all $1 \leq i \leq n$. Assume that for all $1 \leq i \leq n$, there exists a step-isometry at level (δ, γ) from V_i to W_i . If G is a geometric δ -graph with vertex set V and H is a geometric γ graph with vertex set W , then $G \cong H$.*

The proof is a straightforward generalization of the proofs of Theorems 3.6 and 3.7, and is therefore omitted.

4. Non-Isomorphism Results for Euclidean Space

The choice of metric plays an important role in our isomorphism results in Section 3. We demonstrate that there are non-isomorphic geometrically e.c. graphs in the plane with the usual Euclidean metric (denoted by d).

Theorem 4.1. *Let V be a countable set dense in (\mathbb{R}^2, d) , and let G and H be two graphs generated by the model $LARG(V, p)$, where $0 < p < 1$. Then with probability 1, $G \not\cong H$.*

We have the following corollary, which is the antithesis of the results in the previous section.

Corollary 4.2. *Let V be a countable set dense in \mathbb{R}^2 equipped with the Euclidean metric, and $\delta > 0$ fixed. Then there exist infinitely many pair-wise non-isomorphic δ -g.e.c. graphs with vertex set V .*

For the proof of Theorem 4.1, we rely on the following geometric lemma.

Lemma 4.3. *Let V and W be dense subsets of \mathbb{R}^2 equipped with the Euclidean metric. Then every step-isometry from V to W is an isometry.*

Proof. Assume for a contradiction that there is a step-isometry $f: V \rightarrow W$ at level (δ, γ) that is not an isometry. Without loss of generality, we assume that $\delta = \gamma = 1$. For each $u \in V$, let $u' = f(u) \in W$. Since f is not an isometry, there must exist points x_1 and x_2 so that $d(x_1, x_2) \neq d(x'_1, x'_2)$. Since f is a bijection, we may assume, without loss of generality, that $d(x_1, x_2) < d(x'_1, x'_2)$.

The proof follows by the following two claims. Given $x, y \in V$, define the *discrepancy* of x, y as

$$D(x, y) = |d(x, y) - d(x', y')|.$$

The discrepancy is a measure of the error in the distance between pairs of points and their images under f . Since f is a step-isometry, we have that $D(x, y) < 1$ for all $x, y \in V$.

Claim 4.4. For every $\varepsilon > 0$, if there exist points $x_1, x_2 \in V$ so that

$$D(x_1, x_2) = \varepsilon > 0$$

and $d(x_1, x_2) > 40$, then there exist points $x_3, x_4 \in V$ so that

$$D(x_3, x_4) > 2\varepsilon.$$

Claim 4.5. If there exist points $x_1, x_2 \in V$ so that

$$D(x_1, x_2) = \varepsilon > 0,$$

then there exist points $x_3, x_4 \in V$ so that

$$D(x_3, x_4) > 3/4\varepsilon$$

and $d(x_3, x_4) > 40$.

To see how the lemma follows from the claims, note that by hypothesis, there are two points of V with discrepancy $\varepsilon > 0$. By Claim 4.5, there are two points of V with discrepancy at least $3/4\varepsilon$, and with distance at least 40 apart. By Claim 4.4 there are points with discrepancy at least $3/2\varepsilon$ apart. By induction, we obtain a sequence of pairs of points $\{y_i, z_i\}$ of V whose discrepancy equals $(3/2)^i\varepsilon$ which tends to infinity in i . In particular, there are points y, z of V so that $D(y, z) > 1$, which contradicts the fact that f is a step-isometry.

We now prove Claim 4.4. We first define some constants that will be useful in the proof. Let

$$m = d(x_1, x_2)/2, \text{ and } k = \lfloor m \rfloor + 1.$$

Choose

$$0 < \xi < \varepsilon^2/(2k).$$

Since V is dense in \mathbb{R}^2 , we can find points x_3 and x_4 in V so that

$$k - \xi < d(x_i, x_j) < k \text{ for } i = 1, 2 \text{ and } j = 3, 4.$$

So x_1, x_3, x_2, x_4 are the vertices of a quadrilateral whose sides have length between $k - \xi$ and k . See Figure 2.

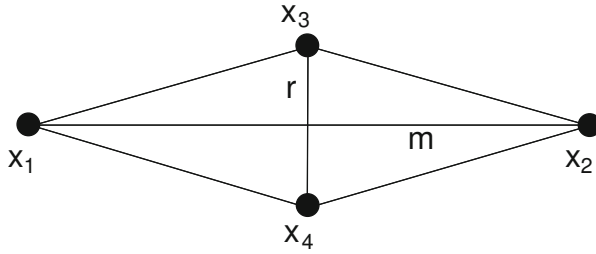


Figure 2: The quadrilateral formed by x_1, x_3, x_2, x_4 .

Let $r = d(x_3, x_4)/2$. The distance between x_3 and x_4 is smallest when all sides of the quadrilateral equal $k - \xi$, so using the Pythagorean theorem we have that

$$r^2 \geq (k - \xi)^2 - m^2 \geq k^2 - 2k\xi - m^2 \geq k^2 - \varepsilon^2 - m^2, \tag{4.1}$$

where the last step follows from the choice of ξ . (Note that the calculation above is only valid if we use the Euclidean metric.)

On the other hand, x_3 and x_4 are furthest when all sides of the quadrilateral equal k . Since $k \leq m + 2$ and $m > 20$, we obtain that

$$r^2 \leq (m + 2)^2 - m^2 = 2m + 1 \leq m^2/4. \tag{4.2}$$

Now consider the quadrilateral formed by the images x'_1, x'_2, x'_3 , and x'_4 . Let

$$m' = d(x'_1, x'_2) / 2 = m + \varepsilon/2, \text{ and } r' = d(x'_3, x'_4) / 2.$$

See Figure 3.

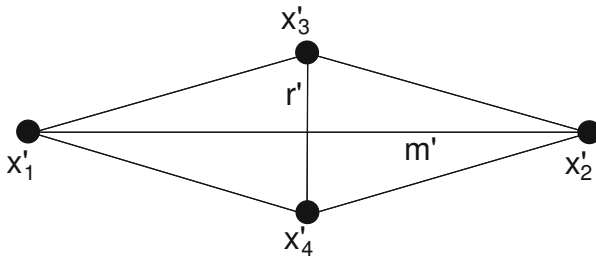


Figure 3: The quadrilateral formed by x'_1, x'_3, x'_2, x'_4 .

Since f is a step-isometry,

$$d(x'_i, x'_j) < k \text{ for } i = 1, 2 \text{ and } j = 3, 4.$$

Now $d(x'_3, x'_4)$ is largest when the quadrilateral has all sides equal to k . It follows that

$$\begin{aligned} (r')^2 &\leq k^2 - (m')^2 \\ &= k^2 - (m + \varepsilon/2)^2 \\ &\leq k^2 - m^2 - m\varepsilon \\ &\leq r^2 - m\varepsilon + \varepsilon^2 \\ &\leq (r - \varepsilon)^2, \end{aligned}$$

where the third inequality follows from (4.1), and the last inequality follows from (4.2). In particular, $r' \leq r - \varepsilon$. As

$$d(x'_3, x'_4) = 2r' \leq 2r - 2\varepsilon = d(x_3, x_4) + 2\varepsilon,$$

the proof of Claim 4.4 follows.

We now prove Claim 4.5. Let $m = d(x_1, x_2)$, and assume $m < 40$. Let $k = 40$. Choose $c > 0$ so that

$$10c/3 + c^2 \leq k\varepsilon/4 \text{ and } c < (2 - \sqrt{3})\varepsilon/8. \tag{4.3}$$

Further, choose points $x_3, x_4, x_5, x_6 \in V$ so that

$$k < d(x_i, x_j) < k + c, \text{ for } i, j \text{ equals } 1, 3 \text{ or } 1, 5, \text{ or } 2, 4 \text{ or } 4, 6, \tag{4.4}$$

$$k - c < d(x_i, x_j) < k, \text{ for } i, j \text{ equals } 3, 5 \text{ or } 4, 6, \tag{4.5}$$

and

$$d(x_3, x_4) < d(x_5, x_6) < d(x_3, x_4) + c. \tag{4.6}$$

The choice of such points is possible since V is dense. See Figure 4.

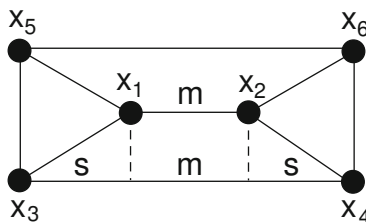


Figure 4: The figure formed by $x_i, 1 \leq i \leq 6$.

Subject to the given constraints, the points x_3 and x_4 are furthest apart when the distances achieve the upper bound of (4.4) and the lower bound from (4.5), and when

$d(x_3, x_4) = d(x_5, x_6)$. Assume this to be the case. Then x_3, x_4, x_5, x_6 form a rectangle, and the line segments x_1x_2, x_3x_4 and x_5x_6 are parallel.

For $i = 1, 2$, let y_i be the orthogonal projection of x_i on the line x_3x_4 . Then $d(y_1, y_2) = d(x_1, x_2) = m$, and $d(x_3, y_1) = d(x_4, y_2)$; we will denote this distance by s . See Figure 4.

Hence, $d(x_3, x_4) = 2s + m$, and

$$s^2 = (k + c)^2 - 1/4(k - c)^2 = 3/4(k^2 + 10kc/3 + c^2),$$

and so

$$d(x_3, x_4) = \sqrt{3(k^2 + (10/3)kc + c^2)} + m.$$

The expression above is based on the case where x_3 and x_4 are furthest apart, and thus, it gives an upper bound for the general case. Combining this upper bound with the condition on c given by (4.3), and with the assumption (4.6) we have for i, j equals 3, 4 and 5, 6 that

$$\begin{aligned} d(x_i, x_j) &\leq m + c + \sqrt{3(k^2 + (10/3)kc + c^2)} \\ &\leq m + c + \sqrt{3(k^2 + k\varepsilon/4)} \\ &< m + c + \sqrt{3}(k + \varepsilon/8) \\ &< m + \sqrt{3}k + \varepsilon/4, \end{aligned}$$

where the first and last steps follow from (4.3).

We next consider the images of these points, and assume without loss of generality that $d(x'_3, x'_4) > d(x'_5, x'_6)$. Since f is a step-isometry, $d(x'_i, x'_j) \geq k$ for i, j equals 1, 3 or 1, 5 or 2, 4 or 2, 6, and $d(x'_i, x'_j) \leq k$ for i, j equals 3, 5 or 4, 6. Moreover, by assumption $d(x'_1, x'_2) = m + \varepsilon$. Now x'_3 and x'_4 are closest together when $d(x'_3, x'_4) = d(x'_5, x'_6)$ and $d(x'_i, x'_j) = k$ for all i, j for which $|i - j|$ is even. As in the previous case, the line segments $x'_5x'_6, x'_1x'_2$, and $x'_3x'_4$ are parallel, and $x'_5x'_6x'_4x'_3$ is a rectangle. Under these assumptions we can compute $d(x'_3, x'_4)$ similarly to the computation for $d(x_3, x_4)$, and obtain that

$$d(x'_3, x'_4) = 2(k\sqrt{3}/2) + (m + \varepsilon).$$

Since our assumptions hold for the case where x'_3 and x'_4 are closest together, we have in general, for i, j equal to 3, 4 and 5, 6, that

$$d(x'_i, x'_j) \geq k\sqrt{3} + m + \varepsilon \geq d(x_i, x_j) + 3\varepsilon/4. \quad \blacksquare$$

A direct consequence of this lemma is the existence of many non-isomorphic g.e.c. graphs with vertex sets dense in \mathbb{R}^2 , equipped with the Euclidean metric. A set V in \mathbb{R}^2 is δ -free if no pair of points in V are distance δ apart. For instance, one

may consider $\delta = 1$, V to be the set of all rational points in \mathbb{R}^2 which are δ -free, and $W = V \cup \{(2^{1/4}, 0)\}$ (which is also δ -free). It is straightforward to see there is no isometry from V onto W . Hence, a 1-g.e.c. graph on V cannot be isomorphic to a 1-g.e.c. graph on W .

In the following proof, we use the notation $\mathbb{P}(A)$ for the probability of an event A .

Proof of Theorem 4.1. An enumeration $\{v_i : i \in \mathbb{N}^+\}$ of V is *good* if $d(v_i, v_{i+1}) < \delta$ for all $i \in \mathbb{N}^+$ and $\{v_1, v_2, v_3\}$ are not collinear. We claim that a countable set V dense in \mathbb{R}^2 has a good enumeration. For a positive integer n , we call $\{v_i : 1 \leq i \leq n\}$ a *partial good enumeration* of V . We prove the claim by constructing a chain of partial good enumerations by induction. Using the density of V , choose three points $\{v_1, v_2, v_3\}$ that are not collinear, so that each are within δ of each other. Let $V_1 = \{v_1, v_2, v_3\}$. Enumerate $V \setminus \{v_1, v_2, v_3\}$ as $\{u_i : i \geq 2\}$. Starting from V_1 , we inductively construct a chain of partial good enumerations $V_n, n \geq 1$, so that for $n \geq 2, V_n$ contains $\{u_i : 2 \leq i \leq n\}$.

We now want to form V_{n+1} by adding $u = u_{n+1}$. If $u \in V_n$, then let $V_{n+1} = V_n$. Assume without loss of generality that $u \notin V_n$. Let $N = |V_n|$. If $d(v_N, u) < \delta$, then let $v_{N+1} = u$ and add it to V_n to form V_{n+1} . Otherwise, by the density of V , choose a shortest finite path $P = p_0, \dots, p_\ell$ of points of $V \setminus V_n$ starting at $v_N = p_0$ and ending at $u = p_\ell$ so that two consecutive points in the path are distance at most δ . Then add the vertices of P to V_n to form V_{n+1} and enumerate them so that $v_{N+i} = p_i$ for $i = 0, 1, \dots, \ell$. Taking the limit of this chain, $\bigcup_{n \geq 1} V_n$ is a good enumeration of V , which proves the claim.

Let $V = \{v_i : i \geq 1\}$ be a good enumeration of V , and for any n , let $V_n = \{v_i : 1 \leq i \leq n\}$. Let G and H be as stated. We say that two pairs $\{v, w\}$ and $\{v', w'\}$ of vertices are *compatible* if $\{v, w\}$ are adjacent in G and $\{v', w'\}$ are adjacent in H or $\{v, w\}$ are non-adjacent in G and $\{v', w'\}$ are non-adjacent in H . For two pairs $\{v, w\}$ and $\{v', w'\}$ such that $d(v, w) = d(v', w')$, the probability that they are compatible equals

$$p^* = \begin{cases} p^2 + (1 - p)^2, & \text{if } d(v, w) < \delta', \text{ and} \\ 1, & \text{otherwise.} \end{cases}$$

By Corollary 2.5 and Lemma 4.3, any isomorphism between subgraphs of G and H must be an isometry. The images of three points in \mathbb{R}^2 that are not collinear determine the isometry. Let A_n be the event that there exists a partial isomorphism f from $G[V_n]$ into H so that $f(\{v_1, v_2, v_3\}) \subseteq V_n$, and let

$$A_n^* = \bigcap_{v \geq n} A_v.$$

Note that $A_n^* \subseteq A_{n+1}^*$ for all n .

Next, we estimate the probability of A_n^* . Note first that $\mathbb{P}(A_n^*) \leq \mathbb{P}(A_v)$ for all $v \geq n$. For any tuple (u_1, u_2, u_3) of three distinct vertices in V_n , let $C_n(u_1, u_2, u_3)$ be the event that there exists a partial isomorphism f from $G[V_n]$ to H so that $f(v_i) = u_i$ for $i = 1, 2, 3$. Since the images of three points that are not collinear determine the isometry f , if C_n happens then all pairs (v_i, v_{i+1}) and $(f(v_i), f(v_{i+1}))$ must be compatible, for $1 \leq i < n$. Thus,

$$\mathbb{P}(C_n(u_1, u_2, u_3)) \leq (p^*)^{n-1}.$$

Now

$$A_n = \bigcup_{u_1, u_2, u_3 \in V_n} C_n(u_1, u_2, u_3),$$

so for $n \geq 3$ we have that $\mathbb{P}(A_n) \leq n^3(p^*)^{n-1}$, and

$$\mathbb{P}(A_n^*) \leq \inf \{v^3(p^*)^{v-1} : v \geq n\} = 0.$$

If B is the event that G and H are isomorphic, then

$$B \subseteq \bigcup_{n \in \mathbb{N}^+} A_n^*.$$

Since the union of countably many sets of measure zero has measure zero, we conclude that $\mathbb{P}(B) = 0$, and thus, with probability 1, $G \not\cong H$. ■

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