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Homomorphisms and amalgamation

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Dedicated to the memory of Paula Bonato

Abstract

We prove that for each finite core graph G, the class of all graphs admitting a homomorphism into G is a pseudo-amalgamation class, in the sense of Fraïssé. This fact gives rise to a countably infinite universal pseudo-homogeneous graph which shares some of the properties of the infinite random graph. Our methods apply simultaneously to G-colourings in several classes of relational structures, such as the classes of directed graphs or hypergraphs.

MSC: 05C15; 05C75

Keywords: Graph; Structure; Colouring; Homomorphism; Amalgamation

1. Introduction

A class $\mathscr{K}$ of finite graphs closed under isomorphisms has the *amalgamation property*, written (AP), if it satisfies the following. For any graphs A, B, and C in $\mathscr{K}$, for any isomorphism f from A onto an induced subgraph of B, and any isomorphism g from A onto an induced subgraph of C, there is a graph D in $\mathscr{K}$, an isomorphism f' from B onto an induced subgraph of D and an isomorphism g' from C onto an induced subgraph of D, so that for all vertices x of A, we have that $(f'f)(x) = (g'g)(x)$. The class of all finite graphs has AP: we may choose D to be the graph $B \cup C$ that has vertices $V(B) \cup V(C)$ and edges $E(B) \cup E(C)$. Many otherwise favourable classes of finite graphs do not have AP, however. For example, the class of 2-colourable graphs does not have AP (this follows directly or by Theorem 3 below).

Throughout, all structures are countable. A *signature* L is a set of finitely many symbols, where each $R \in L$ is assigned a positive natural number called its *arity*. An

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L-structure G consists of a nonempty set, $V(G)$, called the *vertices* of G, and for each $R \in L$ with arity n, $R(G)$ is a set of n-tuples from $V(G)$. For example, a graph is an L-structure, where L consists of a single symbol E of arity 2, and $E(G)$ is irreflexive and symmetric. We will abuse notation and refer to elements of $R(G)$ in an L-structure as *edges*. The order of G is $|V(G)|$. If $U \subseteq V(G)$, then $G \restriction U$ is the *substructure of G induced by U*, with vertices U and edges those of G restricted to U; if H is an induced substructure of G, then we write $H \leqslant G$. For additional background on L-structures, the reader is directed to [9], where L-structures are referred to as *relations* and *multirelations*.

A structure is *homogeneous* if each isomorphism between finite induced substructures extends to an automorphism. Lachlan and Woodrow [11] classified all of the countable homogeneous graphs, while Cherlin [7] classified the countable homogeneous directed graphs. The connection between classes with AP and homogeneous structures is made by Fraïssé's theorem (see [9]). The *age* of a structure G is the set of isomorphism types of finite induced substructures of G. If $\mathscr{K}$ is a class of countable relational structures, then $\mathscr{K}_{\text{fin}}$ is the class of finite structures in $\mathscr{K}$. A structure G is *universal* in $\mathscr{K}$ if each member $\mathscr{K}$ is isomorphic to an induced substructure of G. The class $\mathscr{K}$ has the *joint embedding property* or JEP if for every pair B and C in $\mathscr{K}$, there is a $D \in \mathscr{K}$ so that $B, C \leqslant D$. (If we allow empty structures, then JEP is a special case of AP. Since we only consider nonempty structures, we will not use this convention.)

Theorem 1 (Fraïssé [8]). *Let $\mathscr{K}$ be a class of countable L-structures closed under isomorphisms. Then the following are equivalent.*

(1) *The class $\mathscr{K}$ has AP, JEP, and is closed under taking induced subgraphs.*
(2) *There is a countable universal and homogeneous structure U in $\mathscr{K}$ whose age is $\mathscr{K}$.*

For example, Theorem 1 applies to the case when $\mathscr{K}$ is the class of all countable graphs: the graph U of item (2) is the infinite random graph. There is a generalization of homogeneity, called *pseudo-homogeneity*, where the conditions in item (1) of Theorem 1 are weakened. Let $\mathscr{C}$ be a class of countable structures closed under isomorphisms and closed under taking induced substructures. A structure $M \in \mathscr{C}$ is *universal pseudo-homogeneous* if there is a subclass $\mathscr{C}'$ of finite structures from $\mathscr{C}$ so that (PH1) M embeds each structure in $\mathscr{C}_{\text{fin}}$ as an induced substructure, (PH2) each finite induced substructure of M is contained in a finite induced substructure from $\mathscr{C}'$, and (PH3) each isomorphism between finite induced substructures of M that are in $\mathscr{C}'$ extends to an automorphism of M. It can be shown that M is unique (up to isomorphism) with properties PH1, PH2, and PH3. Pseudo-homogeneous relational structures were first investigated by Calais [5], and Fraïssé discussed them in Chapter 11 of [9]. As proved in Section 6.6 of [9], a universal pseudo-homogeneous structure exists in $\mathscr{C}$ if and only if $\mathscr{C}'$ satisfies AP and JEP, and $\mathscr{C}$ and $\mathscr{C}'$ satisfy the following *cofinality condition*:

(C) For each finite $A \in \mathscr{C}$, there is a finite $B \in \mathscr{C}'$ so that $A \leqslant B$.

The subclass $\mathscr{C}$ is called a *pseudo-amalgamation class*. We note that the infinite random graph R is universal pseudo-homogeneous with $\mathscr{C}$ the class of all countable graphs and $\mathscr{C}'$ the class of all finite graphs.

The class of n-colourable graphs for each integer $n \geqslant 2$ does not have AP. Wheeler [12] studied amalgamation in the class of n-colourable graphs, and proved that the class of *uniquely n-colourable graphs* has AP. (A graph G is *uniquely n-colourable* if it is n-chromatic and each proper n-colouring of G gives rise to the same partition of $V(G)$ into independent sets.) Given two graphs G and H, a function $f : V(H) \to V(G)$ that is also a function from $E(H)$ to $E(G)$ is a *homomorphism*. We write $f : H \to G$ or $H \to G$, and say that H is *G-colourable*. Graph homomorphisms are related to various colouring problems generalizing the fact that a graph is n-colourable if and only if it admits a homomorphism into K_n, the complete graph of order n. Homomorphisms may be studied in any class of relational structures, as we will discuss shortly.

In this paper, we investigate amalgamation properties of G-colourable graphs and structures. In particular, our results prove that if $\mathscr{K}$ is the class of graphs and G is a finite core graph, then even though the class $\mathscr{C}_{\mathscr{K}}(G)$ of all G-colourable graphs does not have AP, it is a pseudo-amalgamation class. See Theorems 3 and 4. The subclass $\mathscr{C}'$ is the class of uniquely G-colourable structures; see Section 2. Hence, for each core graph G, we demonstrate the existence of a highly symmetric countable universal pseudo-homogeneous G-colourable graph that plays the role of the infinite random graph in the theory of G-colourings; see Corollary 1. We prove our results in the generality of relational structures, an approach that has the advantage of applying simultaneously to many different combinatorial classes of structures such as the classes of graphs, directed graphs, and hypergraphs. (We maintain the notation G and H for structures, and call tuples in $R(G)$ for $R \in L$ "edges", for the benefit of those readers more familiar with graphs than relational structures.)

If G and H are L-structures, then a *homomorphism* from H to G is a mapping $f : V(H) \to V(G)$ such that

$$\bar{a} = (a_1, \ldots, a_n) \in R(H) \text{ implies } f(\bar{a}) = (f(a_1), \ldots, f(a_n)) \in R(G). \tag{1.1}$$

An *embedding* is an injective homomorphism where "implies" in (1.1) is replaced by "if and only if". An *isomorphism* is a surjective embedding. If H admits a homomorphism into G, then we write $H \to G$. The set of all homomorphisms from H to G is written $\mathrm{Hom}(H, G)$, and the set of automorphisms of G (that is, isomorphisms from G to G) is written $\mathrm{Aut}(G)$. The composition of two mappings f and g is written fg. If $f : A \to C$ and $g : B \to C$ are mappings, then $f \cup g : A \cup B \to C$ is defined by

$$(f \cup g)(x) = \begin{cases} f(x) & \text{if } x \in A, \\ g(x) & \text{if } x \in B. \end{cases}$$

The identity mapping on a structure G is written id_G. If J is an induced substructure of H, and $f : H \to G$ is a homomorphism, then $f \upharpoonright J$ is the restriction of f to $V(J)$. A finite structure G is a *core* if $\mathrm{Aut}(G) = \mathrm{Hom}(G, G)$. Equivalently, a finite structure G is a core if every mapping in $\mathrm{Hom}(G, G)$ is onto. (This is not an equivalent formulation for a structure G to be a core if G is infinite. See [1,2] for a discussion of infinite core graphs and digraphs.) If $\mathscr{K}$ is a class of L-structures, then the class of all structures in $\mathscr{K}$ that admit a homomorphism into G is written $\mathscr{C}_{\mathscr{K}}(G)$, and is called the *colour class determined by* G. For further information on homomorphisms of graphs the reader is directed to the survey of Hahn and Tardif [10].

2. Uniquely colourable structures

Throughout the remaining sections, we will assume that $\mathscr{K}$ is a class of countable relational structures over a fixed finite signature L, and that $\mathscr{K}$ is closed under taking induced substructures and isomorphisms. All cores we consider are finite.

A *uniquely n-colourable graph* G is an n-chromatic graph in which every proper n-colouring induces the same partition of $V(G)$ into independent sets. This notion has been generalized by Zhu [13,14] to uniquely G-colourable graphs, where G is a core graph. The following definition generalizes all of these notions to relational structures.

Definition 1. Let G be a core structure in $\mathscr{K}$. A structure $H \in \mathscr{K}$ is uniquely G-colourable if

1. $H \to G$;
2. every homomorphism from H to G is onto;
3. for every pair $f, h \in \mathrm{Hom}(H, G)$, there is a $g \in \mathrm{Aut}(G)$ so that $f = gh$.

The subclass of uniquely G-colourable structures in $\mathscr{K}$ is written $\mathscr{C}!_{\mathscr{K}}(G)$. Every core G is uniquely G-colourable; in particular, $\mathscr{C}!_{\mathscr{K}}(G) \neq \emptyset$.

Definition 2. Assume that each $R \in L$ has arity at least 2. Let G be a core structure in $\mathscr{K}$, and fix $H \in \mathscr{C}_{\mathscr{K}}(G)$ and $f \in \mathrm{Hom}(H, G)$.

1. For a fixed integer $n \geqslant 2$, let S be a set of n (not necessarily distinct) elements of $V(G) \cup V(H)$ with the property that $S \cap V(G)$ and $S \cap V(H)$ are nonempty. If S is listed as $\bar{a} = (a_1, \ldots, a_n)$, then we say that the n-tuple $\bar{a}$ is mixed.
2. Define $H' = G(H, f)$ to be the structure with vertices $V(G) \cup V(H)$, and for each $R \in L$, $R(H')$ consists of $R(G) \cup R(H)$ along with the following edges. If $\bar{a} = (a_1, \ldots, a_n)$ is mixed, then $\bar{a} \in R(H')$ if and only if
$$(f \cup id_G)(\bar{a}) = ((f \cup id_G)(a_1), \ldots, (f \cup id_G)(a_n)) \in R(G).$$
The structure $G(H, f)$ is the fixation of H by G and f.

If $\mathscr{K}$ is the class of graphs, then the graph $G(H, f)$ is just the union of G and H, so that for each vertex x of H, x is joined to all the neighbours of $f(x)$ in G. (See Fig. 1.) The mapping $f \cup id_G : G(H, f) \to G$ is clearly a homomorphism.

Fixations were first studied by Wheeler [12] when $\mathscr{K}$ is the class of all graphs and G is a complete graph. The fixation of a structure may or may not be in $\mathscr{K}$. We will say that $\mathscr{K}$ *has fixations* if for every core $G \in \mathscr{K}$, $H \in \mathscr{C}_{\mathscr{K}}(G)$, and $f \in \mathrm{Hom}(H, G)$,

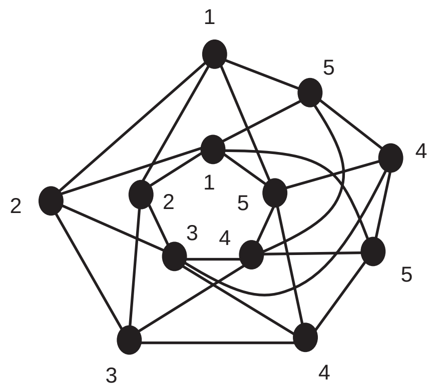

Fig. 1. A fixation of C_7 by C_5. The C_5-colouring of C_7 is shown as the labelling of the vertices of C_7.

we have that $G(H,f) \in \mathscr{K}$ (and is therefore in $\mathscr{C}_{\mathscr{K}}(G)$). It is straightforward to see that if $\mathscr{K}$ is the class of all graphs, directed graphs, or k-uniform hypergraphs where $k \geqslant 3$, then $\mathscr{K}$ has fixations. However, let $\mathscr{K}$ be the class of tournaments, let G consist of the directed edge $(1,2)$, and let H have a single vertex 3. Then $H \in \mathscr{C}_{\mathscr{K}}(G)$ with $f : H \to G$ defined by $f(3) = 1$. But $G(H,f)$ has no directed edge between 1 and 3 and so is not a tournament.

Fixations allow us to extend G-colourable structures to uniquely G-colourable structures, and so will verify property (C) of the Section 1.

Theorem 2. *Assume that L has no symbol of arity* 1 *and that* $\mathscr{K}$ *is a class of L-structures that has fixations. Let G be a core structure, and fix* $H \in \mathscr{C}_{\mathscr{K}}(G)$ *and* $f \in \mathrm{Hom}(H,G)$. *Then* $G(H,f) \in \mathscr{C}!_{\mathscr{K}}(G)$.

Proof. Let $H' = G(H,f)$. Then $H' \in \mathscr{C}_{\mathscr{K}}(G)$, and every homomorphism from H' to G is onto (since $G \leqslant H'$). To prove the theorem, we must verify item (3) of Definition 1.

Let $h : H' \to G$ be a homomorphism. As $G \leqslant H'$ is a core, $h \restriction G$ is an automorphism, so if we let $g = (h \restriction G)^{-1}$, then gh is the identity mapping on G in H'. In order to verify item (3), it is sufficient to show that if $f' = f \cup id_G$, then $f' = gh$. To obtain a contradiction, assume that $(gh) \restriction H \neq f$.

Without loss of generality, we assume that there are distinct $i,k \in V(G)$ and $x \in V(H)$ so that $f(x) = k$ and $(gh)(x) = i$. If $\bar{a}$ and $\bar{b}$ are tuples of vertices in G, and z is a vertex of G, then the notation $(\bar{a}z\bar{b})$ denotes the tuple formed by first listing $\bar{a}$, then z, and then listing $\bar{b}$. If there exist tuples $\bar{a}$ and $\bar{b}$ from G, at least one of which is nonempty, so that $(\bar{a}k\bar{b}) \in R(G)$ for some $R \in L$, then $(\bar{a}x\bar{b}) \in R(H')$ since $(\bar{a}x\bar{b})$ is mixed and by the definition of H'. As gh is a homomorphism and is the identity on G in H', we have that $(\bar{a}i\bar{b}) \in R(G)$.

Hence,

$$(\bar{a}k\bar{b}) \in R(G) \text{ implies } (\bar{a}i\bar{b}) \in R(G). \tag{2.1}$$

Define $f'' : G \to G$ by

$$f''(z) = \begin{cases} i \text{ if } z = k, \\ z \text{ else.} \end{cases}$$

Then f'' is a homomorphism. To see this, let $\bar{a} = (\overline{a_1}k\overline{a_2}\dots\overline{a_{n-1}}k\overline{a_n}) \in R(G)$, where $R \in L$, and each $\overline{a_i}$ is a (possibly empty) tuple from $V(G)\backslash\{k\}$. Since f'' is the identity off of k, it is enough to show that f'' preserves all edges of this form. Then $\bar{a} \in R(G)$ implies that $\bar{b} = (\overline{a_1}x\overline{a_2}k\dots\overline{a_{n-1}}k\overline{a_n}) \in R(H')$, as $\bar{b}$ is mixed (recall that R has arity at least 2) and by the definition of H'. (Note that $\bar{b}$ is chosen so as to replace only one "k" in $\bar{a}$.) By (2.1), we have that $(\overline{a_1}i\overline{a_2}k\dots\overline{a_{n-1}}k\overline{a_n}) \in R(H')$. Proceeding inductively, we obtain that $f''(\bar{a}) = (\overline{a_1}i\overline{a_2}i\dots\overline{a_{n-1}}i\overline{a_n}) \in R(H')$, as desired. But f'' is not onto, contradicting that G is a core. □

The restriction that L contain no symbols of arity 1 in Theorem 2 is essential as the following example demonstrates.

Example 1. An example of a G and $\mathscr{K}$ where there is an $H \in \mathscr{C}_{\mathscr{K}}(G)$ that has no extension to a member of $\mathscr{C}!_{\mathscr{K}}(G)$.

Let $L = \{R, B, E\}$, with R and B of arity 1, and E of arity 2. Let $\mathscr{K}$ be the class of L-structures where E is irreflexive and symmetric. Hence, $\mathscr{K}$ consists of graphs where the vertices may be coloured with one or more of two colours, say red and blue. Let G be the L-structure with vertices 1, 2, and 3, and with edges 12, 23, so that 1 is red and 3 is blue. Then G is a core and $|\mathrm{Aut}(G)| = 1$. Hence, a G-colourable H is in $\mathscr{C}!_{\mathscr{K}}(G)$ if and only if

$$|V(H)| \geqslant |V(G)| \quad \text{and} \quad |\mathrm{Hom}(H, G)| = 1. \tag{2.2}$$

Let $H \in \mathscr{C}_{\mathscr{K}}(G)$ be formed by extending G by a vertex 4 that is joined only to 2, so that 4 is neither red nor blue. Then $H \in \mathscr{C}_{\mathscr{K}}(G)$. However, H has no extension to any $H' \in \mathscr{C}!_{\mathscr{K}}(G)$. Assume otherwise, with $H \leqslant H'$ and $f : H' \to G$ a homomorphism. Then f fixes 1, 2, and 3. Suppose that f maps 4 to 1. (The case when $f(4) = 3$ is similar.) Define a new mapping $h : H' \to G$ that is equal to f except that $h(4) = 3$. It is not hard to see that h is a homomorphism, which contradicts (2.2).

3. Uniquely G-colourable structures have free amalgamation

In this section, we prove that while $\mathscr{C}_{\mathscr{K}}(G)$ may not have AP in general, the subclass $\mathscr{C}!_{\mathscr{K}}(G)$ always does. If B and C are structures with a common induced substructure A so that $V(B) \cap V(C) = V(A)$, then the *free amalgam* or *union of B and C over A*, written $B \cup C$, is the structure with vertices $V(B) \cup V(C)$, and edges $R(B) \cup R(C)$, for each $R \in L$.

We assume throughout that $\mathscr{K}$ is *closed under unions*; that is, whenever $A \leqslant B, C$ and $V(B) \cap V(C) = V(A)$ or $V(B) \cap V(C) = \emptyset$, then $B \cup C \in \mathscr{K}$. For example, the classes of graphs, directed graphs, or k-uniform hypergraphs, where $k \geqslant 3$, are closed under unions.

Definition 3. A structure $A \in \mathscr{K}$ is a free amalgamation base for $\mathscr{K}$ if for all $B, C \in \mathscr{K}$ so that $A \leqslant B, C$ and $V(B) \cap V(C) = V(A)$, we have that $B \cup C \in \mathscr{K}$.

In other words, A is a free amalgamation base for $\mathscr{K}$ if we can always form the free amalgam over A of every pair of structures in $\mathscr{K}$. If $\mathscr{K}$ is the class of all graphs, or the class of all K_n-free graphs, where $n \geqslant 3$, then each graph in $\mathscr{K}$ is a free amalgamation base for $\mathscr{K}$. However, for every $n \geqslant 2$, the class $\mathscr{K}$ of n-colourable graphs contains graphs which are not free amalgamation bases for $\mathscr{K}$. (The class of 1-colourable graphs consists of empty graphs, each of which is a free amalgamation base.) For example, if A is $\overline{K_2}$ with $V(A) = \{x, y\}$, then define the homomorphism $f : \overline{K_2} \rightarrow K_n$ with $f(x) = f(y) = 1$, and define the homomorphism $h : \overline{K_2} \rightarrow K_n$ by $f(x) = 3$ and $f(y) = 2$. The graph $K_n(\overline{K_2}, f)$ consists of the union of $\overline{K_2}$ and K_n, so that x and y are joined to every vertex of K_n except 1, while the graph $K_n(\overline{K_2}, h)$ has y joined to all vertices of K_n except 2 and x joined to every vertex of K_n except 3. The union of $K_n(\overline{K_2}, f)$ and $K_n(\overline{K_2}, h)$ over A is not n-colourable (see Fig. 2).

Wheeler [12] proved that the uniquely n-colourable graphs form the set of free amalgamation bases in the class of n-colourable graphs, for a given positive integer n. The following theorem provides a considerable generalization of that result to many different classes of relational structures.

Theorem 3. *Let G be a core in $\mathscr{K}$ and let $A \in \mathscr{C}_{\mathscr{K}}(G)$. The following are equivalent.*

1. *The structure A is a free amalgamation base for $\mathscr{C}_{\mathscr{K}}(G)$.*
2. *$A \in \mathscr{C}!_{\mathscr{K}}(G)$.*

Proof. For $(1 \Rightarrow 2)$, if $A \notin \mathscr{C}!_{\mathscr{K}}(G)$, then there are $f, h \in \mathrm{Hom}(A, G)$ so that there does not exist $g \in \mathrm{Aut}(A)$ satisfying $f = gh$. (In particular, $f \neq h$, otherwise, we may

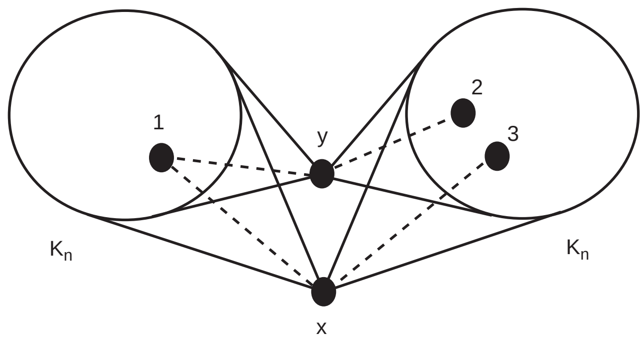

Fig. 2. A union of two n-colourable graphs which is not n-colourable.

choose $g = id_G$.) Let $B = G(A, f)$ and $C = G(A, h)$, and assume, by taking isomorphic copies if necessary, that $V(B) \cap V(C) = V(A)$.

Claim. $B \cup C \notin \mathscr{C}_{\mathscr{K}}(G)$.

To prove the claim, assume, to obtain a contradiction, that $j \in \mathrm{Hom}(B \cup C, G)$. Then $j \restriction B$, and $j \restriction C$ are homomorphisms which are equal on $V(A)$. As $B, C \in \mathscr{C}!_{\mathscr{K}}(G)$, and since $f \cup id_G \in \mathrm{Hom}(B, G)$, and $h \cup id_G \in \mathrm{Hom}(C, G)$, there are $\alpha, \beta \in \mathrm{Aut}(G)$ so that $j \restriction B = \alpha(f \cup id_G)$ and $j \restriction C = \beta(h \cup id_G)$. Hence, $\alpha(f \cup id_G) \restriction A = \beta(h \cup id_G) \restriction A$. Now, for $x \in V(A)$,

$$(\alpha f)(x) = \alpha(f \cup id_G)(x) = \beta(h \cup id_G)(x) = (\beta h)(x),$$

so that $\alpha f = \beta h$. But then $\beta^{-1}\alpha f = h$, and as $\beta^{-1}\alpha \in \mathrm{Aut}(G)$, we have contradicted the choice of f and h.

For $(2 \Rightarrow 1)$, let $A \in \mathscr{C}!_{\mathscr{K}}(G)$ with $B, C \in \mathscr{C}_{\mathscr{K}}(G)$ so that $A \leqslant B, C$, and $V(A) = V(B) \cap V(C)$. Let $f : B \to G$ and $h : C \to G$ be homomorphisms. As $f \restriction A$ and $h \restriction A$ are homomorphisms, and as $A \in \mathscr{C}!_{\mathscr{K}}(G)$, there is a $g \in \mathrm{Aut}(A)$ so that $f \restriction A = g(h \restriction A)$. If we define $h' = gh$, then $h' \in \mathrm{Hom}(C, G)$. Then f and h' are equal on $V(A)$ and hence, $f \cup h' \in \mathrm{Hom}(B \cup C, G)$. Therefore, $B \cup C \in \mathscr{C}!_{\mathscr{K}}(G)$. □

Theorem 2 and the following theorem together prove that $\mathscr{C}_{\mathscr{K}}(G)$ is a *pseudo-amalgamation class*, as defined in the Section 1. In particular, the properties of AP, JEP, and C of Section 1 hold with $\mathscr{C}$ the class of countable structures in $\mathscr{C}_{\mathscr{K}}(G)$ and $\mathscr{C}' = \mathscr{C}!_{\mathscr{K}}(G)_{\mathrm{fin}}$, where $\mathscr{C}!_{\mathscr{K}}(G)_{\mathrm{fin}}$ is the set of finite structures in $\mathscr{C}!_{\mathscr{K}}(G)$.

Theorem 4. *If G is a core structure in $\mathscr{K}$ and $\mathscr{K}$ has fixations, then the subclass $\mathscr{C}!_{\mathscr{K}}(G)_{\mathrm{fin}}$ is closed under unions and has JEP.*

Proof. We prove first that $\mathscr{C}!_{\mathscr{K}}(G)_{\mathrm{fin}}$ has JEP. For this, fix $B, C \in \mathscr{C}!_{\mathscr{K}}(G)_{\mathrm{fin}}$. By taking isomorphic copies, we may assume B and C are disjoint. Then $B \cup C \in \mathscr{C}_{\mathscr{K}}(G)_{\mathrm{fin}}$ (recall that $\mathscr{K}$ is closed under unions). Let $f : B \cup C \to G$ be a homomorphism. Then $D = G(B \cup C, f) \in \mathscr{C}!_{\mathscr{K}}(G)_{\mathrm{fin}}$ by Theorem 2, and $B, C \leqslant D$.

We next verify that $\mathscr{C}!_{\mathscr{K}}(G)_{\mathrm{fin}}$ is closed under unions. Let $A, B, C \in \mathscr{C}!_{\mathscr{K}}(G)_{\mathrm{fin}}$ with $A \leqslant B, C$, and $V(A) = V(B) \cap V(C)$. By the proof of Theorem 3, we have that $B \cup C \in \mathscr{C}_{\mathscr{K}}(G)_{\mathrm{fin}}$. If $f : B \cup C \to G$ is a homomorphism, then $f \restriction A$ is onto G, and hence, f is onto. To show that $B \cup C \in \mathscr{C}!_{\mathscr{K}}(G)_{\mathrm{fin}}$, it remains to verify item (3) of Definition 1.

Now, let $f, h \in \mathrm{Hom}(B \cup C, G)$. Define $f_1 = f \restriction B$, $f_2 = f \restriction C$, $h_1 = h \restriction B$, and $h_2 = h \restriction C$. Then the following properties hold.

1. f_i and h_i are homomorphisms for each $i \in \{1, 2\}$;
2. $f_1 \restriction A = f_2 \restriction A$ and $h_1 \restriction A = h_2 \restriction A$;
3. $f = f_1 \cup f_2$ and $h = h_1 \cup h_2$.

Since $B, C \in \mathscr{C}!_{\mathscr{K}}(G)_{\text{fin}}$ and by item (1), there are $g_1,\ g_2 \in \text{Aut}(G)$ so that

$$f_1 = g_1 h_1 \quad \text{and} \quad f_2 = g_2 h_2. \tag{3.1}$$

We claim that

$$g_1 = g_2. \tag{3.2}$$

To verify (3.2), let $x \in V(G)$ be fixed. There is a $y \in V(A)$ so that $h_1(y) = x$, since every G-colouring of A is onto. Then

$$g_1(x) = g_1(h_1(y)) = f_1(y) = f_2(y) = g_2(h_2(y)) = g_2(x),$$

the second equality holding by (3.1), the third by item (2) above, the fourth by (3.1), and the fifth by item (2).

Let $g = g_1 = g_2$. Hence, by item (3), (3.1), and (3.2), we have that $f = f_1 \cup f_2 = g_1 h_1 \cup g_2 h_2 = g h_1 \cup g h_2 = g(h_1 \cup h_2) = gh$. □

We have the following corollary.

Corollary 1. *Assume that L has no symbol of arity* 1, *and that* $\mathscr{K}$ *is a class countable L-structures that is closed under taking induced substructures, closed under unions, and has fixations. For each core structure G, there is a unique (up to isomorphism) countable universal pseudo-homogeneous G-colourable structure* $M(G)$.

Proof. Let $\mathscr{C}$ be the countable members of $\mathscr{C}_{\mathscr{K}}(G)$ and let $\mathscr{C}' = \mathscr{C}!_{\mathscr{K}}(G)_{\text{fin}}$. Clearly, $\mathscr{C}$ is closed under taking induced substructures. The class $\mathscr{C}'$ satisfies AP and JEP by Theorem 4, and $\mathscr{C}$ and $\mathscr{C}'$ satisfy property C of the Section 1 by Theorem 2. □

Corollary 1 applies to G-colourable structures in the classes of graphs, directed graphs, the classes of k-uniform hypergraphs, where $k \geqslant 3$, and the class of all relational structures over L, where L has no symbols of arity 1. Homomorphisms have been thoroughly investigated in the class of graphs, where we obtain for each core graph G a universal pseudo-homogeneous graph $M(G)$. The graphs $M(G)$ are interesting in their own right, since they hold the promise of playing the role of the infinite random graph R in the class of G-colourable graphs. However, little is known about these graphs. It was proved in [3] that each graph $M(G)$ is *oligomorphic*: the number of orbits in $\text{Aut}(M(G))$ on ordered n-tuples of vertices is finite for all positive integers n. Hence, the graphs $M(G)$ display a large amount of symmetry. The author in [4] began a study of other properties of these graphs, where it is shown that, like R, they contain independent dominating sets, and have Hamiltonian paths if G is connected. In fact, Cameron [6] proved that R has 1-factorizations and has decompositions into one- and two-way Hamiltonian paths. An apparently difficult problem is whether for every nontrivial core graph G, the graph $M(G)$ has a 1-factorization or decomposes into Hamiltonian paths. This problem is open even for $G = K_2$.

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