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On an adjacency property of almost all graphs

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Abstract

A graph is called n-existentially closed or n-e.c. if it satisfies the following adjacency property: for every n-element subset S of the vertices, and for every subset T of S, there is a vertex not in S which is joined to all of T and to none of $S\backslash T$. The unique countable random graph is known to be n-e.c. for all n. Equivalently, for any fixed n, almost all finite graphs are n-e.c. However, few examples of n-e.c. graphs are known other than large Paley graphs and examples of 2-e.c. graphs given in (Cacetta, et al., *Ars Combin*. 19 (1985) 287–294).

An n-e.c. graph is critical if deleting any vertex leaves a graph which is not n-e.c. We classify the 1-e.c. critical graphs. We construct 2-e.c. critical graphs of each order $\geqslant 9$, and describe a 2-e.c.-preserving operation: replication of an edge. We also examine which of the well-known binary operations on graphs preserve n-e.c. for $n=1,2,3$.

MSC: 05C35; 05C80; 05C75

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1. Introduction

Fagin [11], and later Blass and Harary [4], studied graphs with certain adjacency properties as an instance of their work on the asymptotic probabilities of first-order sentences over the class of finite graphs (and more generally over classes of finite relational structures). Central to their arguments was the use of the graph 'extension axioms', which were shown to hold for almost all finite graphs. A problem they posed (as yet unsolved in general) was to find the minimal order of graphs satisfying a single extension axiom. For related work, see [1–3,5–7,9,10].

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In this article, we continue the investigation of properties of finite graphs satisfying a certain adjacency condition. We study the class of n-e.c. graphs, which are defined as follows.

Definition 1. Fix an integer $n \geqslant 1$. A graph G is *n-existentially closed or n-e.c.* if for for every n-element subset S of the vertices, and for every subset T of S, there is a vertex not in S which is joined to every vertex in T and to no vertex in $S \backslash T$.

N-e.c. graphs were first studied in [7], where they were called graphs with property $P(n)$. Let $f(n)$ be the largest integer for which there is a graph on n vertices with property $P(f(n))$. It was proved in [7] that $\log n - (2 + o(1)) \log \log n < f(n) \log 2 < \log n$. Further, explicit examples of graphs with property $P(2)$ were given for all orders $\geqslant 9$.

Our emphasis is on the cases $n = 1, 2$. First, we examine the 1-e.c. and 2-e.c. graphs, and classify the minimal graphs with these properties. Inspired by the Fagin–Blass–Harary problem, we introduce the notion of an n-e.c. critical graph. We present a complete classification of the 1-e.c. criticals, and produce 2-e.c. criticals of each order $\geqslant 9$. In the last part of the paper, we study the n-e.c. preserving properties of certain well-known graph operations (including Cartesian product, categorical product, and lexicographic product).

Throughout, all graphs are finite and simple. For a graph G, $V(G)$ denotes its vertex-set and $E(G)$ denotes its edge-set. The order of G is $|V(G)|$. We denote an edge joining x and y by xy or sometimes $(x)(y)$ for clarity. If $U \subseteq V(G)$, $G \restriction U$ is the subgraph of G induced by U; for $x \in V(G)$, $G - x = G \restriction (V(G) \backslash \{x\})$. For a fixed vertex $x \in V(G)$, $N(x) = N_1(x)$ is the set of vertices joined to x; $N_0(x)$ is the set of vertices not joined to and not equal to x. The union of q disjoint copies of G is denoted by qG.

2. Minimal and critical n-e.c. graphs

A 1-e.c. graph is one such that for every vertex s there is a vertex joined to s and a vertex not joined to s; that is, a graph with no isolated or universal vertices. A 2-e.c. graph G is one such that for each pair of distinct vertices x, y there is a vertex joined to both x and y, a vertex joined to neither x nor y, a vertex joined to x but not to y, and a vertex joined to y but not to x.

We derive the name n-e.c. from the model theoretic notion of an existentially closed or e.c. graph. An infinite graph is e.c. if and only if it is n-e.c. for each $n \geqslant 1$. There is only one countably infinite e.c. graph, the random graph R. (See [8] for a definition and survey of results on R.) It follows from the results of [4] that for a first-order sentence φ in the language of graphs, R satisfies φ if and only if almost all finite graphs satisfy φ.

The following lemma follows from the definitions.

Lemma 2. *Let G be a n-e.c. graph for some fixed $n>1$. For a fixed $x \in V(G)$, the following graphs are $(n-1)$-e.c.*:

1. $G - x$,
2. $G \restriction N(x)$,
3. $G \restriction N_0(x)$.

It follows from the results of [4] that almost all graphs are n-e.c. for a given $n \geqslant 1$. Hence, there is an n-e.c. graph G with smallest order; one property of such a graph G is that for any $x \in V(G)$, $G - x$ is not n-e.c. These facts motivate the following definition.

Definition 3. Fix $n \geqslant 1$.

1. *A* graph G is *n-e.c. minimal* if it is n-e.c. and it has the smallest order of any n-e.c. graph.
2. A graph G is *n-e.c. critical* if it is n-e.c. and for each $x \in V(G)$, $G - x$ is not n-e.c.

An easy observation is that complementation preserves the properties of being n-e.c., n-e.c. minimal, and n-e.c. critical. In the next two subsections, we attempt to classify the 1-e.c. and 2-e.c. critical graphs.

2.1. The 1-e.c. critical graphs

The classification of the 1-e.c. critical graphs is complete. We define a *$*$-vertex* in a graph G to be one that is either universal (i.e. joined to every other vertex) or isolated in G.

Theorem 4. *Let G be a 1-e.c. critical graph. Then G is one of the following graphs*:

1. qK_2, *where q is some integer* $\geqslant 2$,
2. *the complement of a graph in* (1),
3. P_4 (*the path with three edges*).

Proof. We leave it to the reader to check that each of the listed graphs is 1-e.c. critical. Now, let G be a fixed 1-e.c. critical graph. If $|V(G)| = 4$, then it can be verified that G is one of $2K_2$, $C_4 = \overline{2K_2}$, or P_4. Therefore, we assume $|V(G)| \geqslant 5$.

Case 1. *G has a connected component equalling K_2*. In this case, we show that G is one of the graphs in (1) above. To see this, define $X = \{\text{components of } G \text{ equalling } K_2\}$, $Y = V(G) \backslash X$. Then $X \neq \emptyset$ by hypothesis; to obtain a contradiction, we assume that $Y \neq \emptyset$. Then $|Y| \geqslant 2$ (otherwise, G has an isolated vertex thus is not 1-e.c.).

Fix $x \in Y$. Then $G - x$ has a $*$-vertex z. Then $z \notin X$, as every vertex in X is joined to some vertex "of X" and is not joined to some other vertex (different from itself). As $z \in Y$ and $X \neq \emptyset$, z must be isolated in $G - x$. Hence, as G is 1-e.c., xz is an edge.

Now, in $G - z$ there is a $*$-vertex y. Again, $y \in Y$ and y is isolated in $G - z$, so that yz is an edge. If $y \neq x$, then z is not isolated in $G - x$. Hence, $y = x$. But then xz is a component of G in Y, which is a contradiction.

Case 2. *The graph* $\bar{G}$ *has a component equalling* K_2. In this case, apply Case 1 to $\bar{G}$. Then $\bar{G} = qK_2$ for some $q \geqslant 1$, so that G is a graph in (2).

Case 3. *Neither* G *nor* $\bar{G}$ *have a component equalling* K_2. We show that this case produces a contradiction. Let $V(G) = \{x_1, \ldots, x_n\}$, for some fixed $n \geqslant 5$. In $G - x_1$, there is a $*$-vertex z. Without loss of generality, we may assume $z = x_2$.

Case 3.1. *The vertex* x_2 *is isolated in* $G - x_1$. In this case, $x_1x_2 \in E(G)$. If x_1 is joined only to x_2, then x_1x_2 is a component of G. Hence, there is a $y \in V(G - x_2)$ so that $yx_1 \in E(G)$. Without loss of generality, $y = x_3$.

Now, in $G - x_3$ there is a $*$-vertex z. $z \neq x_1$ since if x_1 is universal in $G - x_3$, then x_1 is universal in G and x_1 cannot be isolated in $G - x_3$ as $x_1x_2 \in E(G)$. Clearly, $z \neq x_2$. Without loss of generality, assume that $z = x_4$. As x_2x_4 is not an edge, x_4 must be isolated in $G - x_3$. Thus, $x_3x_4 \in E(G)$. Note that $G \restriction \{x_1, x_2, x_3, x_4\} \cong P_4$. Thus, since G is 1-e.c. critical, $|V(G)| > 5$.

In $G - x_4$ there is a $*$-vertex w. Then $w \neq x_2, x_3$. Hence, $w = x_1$ or $w = x_i$, for some $i \geqslant 5$.

Case 3.1.1. $w = x_1$. As $x_1x_2 \in E(G)$, x_1 is universal in $G - x_4$; note that x_1 is not joined to x_4. There is a $*$-vertex y in $G - x_5$. Then $y \neq x_i$, $i = 1, \ldots, 4$. (The reader can verify that each such vertex is not $*$ in $G - x_5$.) Without loss of generality, $y = x_6$. But then x_6 is joined to x_1 and not joined to x_2, which is a contradiction.

Hence, Case 3.1.1 fails.

Case 3.1.2. $w = x_i$ *for some* $i \geqslant 5$. Without loss of generality, assume that $i = 5$. As x_2 is not joined to x_5, x_5 must be isolated in $G - x_4$, so that $x_4x_5 \in E(G)$. But x_4 is isolated in $G - x_3$. Hence, Case 3.1.2 fails.

Therefore, Case 3.1 fails. The following case must hold.

Case 3.2. *The vertex* x_2 *is universal in* $G - x_1$. In this case, x_1 is not joined to x_2. Observe that $\bar{G}$ satisfies the hypotheses of Cases 3 and 3.1. Further, $\bar{G}$ is 1-e.c. critical. So for Case 3.2 apply Case 3.1 to $\bar{G}$. Hence, Case 3 fails and the result follows. □

2.2. *The 2-e.c. minimal and critical graphs*

Recall that for two graphs G and H, the *Cartesian product* of G and H, written $G \square H$, has vertices $V(G) \times V(H)$ and edges $(a,b)(c,d) \in E(G \square H)$ iff $ac \in E(G)$ and $b = d$, or $a = c$ and $bd \in E(H)$. The graph $K_3 \square K_3$ is shown in Fig. 6.

Theorem 5. *The graph* $K_3 \square K_3$ *is the unique* 2-*e.c. minimal graph.*

As was first shown in [7], $K_3 \square K_3$ is a 2-e.c. minimal graph, so we must show uniqueness. We note that $K_3 \square K_3$ is isomorphic to the line graph of $K_{3,3}$, to the lattice graph $L_2(3)$, and to the 9-vertex Paley graph (which adorns the cover of Bollobás' book [6]). The proof of Theorem 5 rests on the following simple lemma.

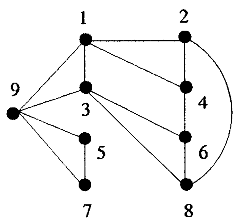

Fig. 1. $6,8 \in N_1(3)$.

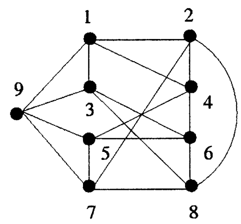

Fig. 2. $54 \in E(G)$, $56 \in E(G)$.

Lemma 6. *If G is a 2-e.c. minimal graph, then G is 4-regular.*

Proof. Fix $x \in G$. Then by Lemma 2, $G \restriction N_0(x)$ and $G \restriction N_1(x)$ are 1-e.c., and so $|N_0(x)|$, $|N_1(x)| \geqslant 4$. As $\{x\} \cup N_0(x) \cup N_1(x)$ is a partition of $V(G)$, and as $|V(G)| = 9$, we must have $|N_0(x)| = |N_1(x)| = 4$. □

Proof of Theorem 5. Let G be a 2-e.c. minimal graph.

Claim. For every $v \in V(G), G \restriction N_1(v) \cong 2K_2$ and $G \restriction N_0(v) \cong C_4$.

To prove the claim, first note that by Lemma 6, G is 4-regular and since $|V(G)| = 9$, $|N_1(v)| = |N_0(v)| = 4$. Thus, in G, the degree-sum of vertices in $N_1(v)$ is the same as the degree-sum of vertices in $N_0(v)$. Edges meeting v contribute a total of 4 to the degree-sum of vertices in $N_1(v)$ (and nothing to the degree-sum of vertices in $N_0(v)$). Edges between $N_1(v)$ and $N_0(v)$ contribute equally to the degree-sum of vertices in $N_1(v)$ and the degree-sum of vertices in $N_0(v)$. Now considering $G_1 = G \restriction N_1(v)$ and $G_0 = G \restriction N_0(v)$, we see that the degree-sum of G_1 must be 4 less than the degree-sum of G_0. Note that by Lemma 2, both G_1 and G_0 are 1-e.c. There are only three 1-e.c. graphs on four vertices, and only the degree-sum of $2K_2$ and C_4 differ by 4. The claim follows.

Let $V(G) = \{1, \ldots, 9\}$. Without loss of generality suppose $N_0(9) = \{2, 4, 6, 8\}$ and $N_1(9) = \{1, 3, 5, 7\}$, and suppose $E(G \restriction N_0(9)) = \{24, 46, 68, 82\}$ and $E(G \restriction N_1(9)) = \{13, 57\}$.

Consider vertex 1. We have that $3, 9 \in N_1(1)$, $39 \in E(G)$, 5, $7 \in N_0(1)$. Thus, since $G \restriction N_1(1) = 2K_2$ by the claim, 1 must be joined to two joined vertices of $\{2, 4, 6, 8\}$. Without loss of generality, assume 1 is joined to 2 and 4. Since $\deg(3) = 4$ and $2, 4, 5, 7 \notin N_1(3)$ we must have that $6, 8 \in N_1(3)$ (see Fig. 1).

Since G is 4-regular, G has four more edges: vertex 5 must be joined to two of $\{2, 4, 6, 8\}$ and 7 must be joined to the other two. Since $G \restriction N_1(5) \cong 2K_2$ by the Claim, the two other vertices of $\{2, 4, 6, 8\}$ that 5 is joined to are joined.

Since $N_0(1) = \{5, 6, 7, 8\}, G \restriction N_0(1) \cong C_4$, and $57, 68 \in E(G)$, it follows that either $56, 78 \in E(G)$ or $58, 76 \in E(G)$. So either 5 is joined to 4 and 6 (see Fig. 2) or 5 is joined to 2 and 8 (see Fig. 3). In either case $G \cong K_3 \square K_3$. □

We define a graph $G = G(k)$ where k is even and $k \geqslant 6$ as follows (arithmetic is mod $2k$): $V(G) = \{1, \ldots, 2k + 1\}$. There is a pairing of the even vertices $2, 4, \ldots, 2k$.

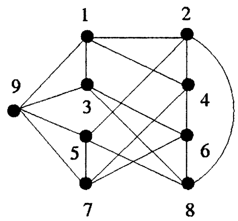

Fig. 3. $52 \in E(G)$, $58 \in E(G)$.

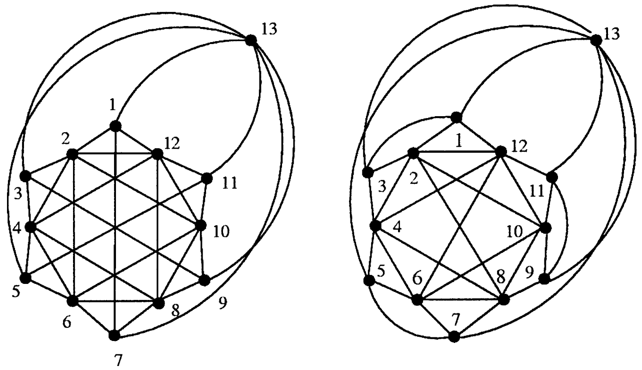

Fig. 4. Two graphs $G(6)$.

For an even vertex i, i is paired with a vertex called $\bar{m}(i)$. There is a pairing of the odd vertices $1,3,\ldots,2k-1$ (excluding $2k+1$). For an odd vertex $i \neq 2k+1$, i is paired with a vertex called $m(i)$. Each $i \in \{2,4,\ldots,2k\}$ is joined to $i-1$ and $i+1$ and all even vertices j except itself and $\bar{m}(i)$. Each $i \in \{1,3,\ldots,2k-1\}$ is joined to $i-1$, $i+1$, $2k+1$ and $m(i)$. The vertex $2k+1$ is joined to $1,3,\ldots,2k-1$.

In other words: start with a k-circuit C with vertices $2,4,\ldots,2k$, with $k \geqslant 6$. Extend each edge of this circuit to a triangle by adding vertices $1,3,\ldots,2k-1$ and joining each of these, call it i, to $i-1$ and $i+1$. Extend C to a complete graph minus a matching. The edges of the matching are $\{i\ \bar{m}(i)\colon i=2,4,\ldots,2k\}$; these are non-edges of G. Add a matching M between $1,3,\ldots,2k-1$. The edges of M are $\{i\ m(i)\colon i=1,3,\ldots,2k-1\}$. Join $2k+1$ to all other odd i; that is, to $1,3,\ldots,2k-1$.

Remark 7. There are several graphs $G(k)$ for a fixed $k \geqslant 6$, depending on the pairings of odd and even vertices that are chosen (see Fig. 4).

Theorem 8. *The graphs $G(k)$, $k \geqslant 6$, are 2-e.c. critical.*

Proof. Fix $k \geqslant 6$. To prove that $G(k)$ is 2-e.c. we provide Table 1. By symmetry we may omit the first two rows of the '2nd only' column.

Table 1

Vertices	Joined to Both	Neither	1st only	2nd only
i, j odd $\neq$ $2k+1$	$2k+1$	odd $\notin$ $\{i, j, m(i), m(j)\}$	$i-1$ if $j \neq i-2$; $i+1$ else	
i, j even	even $\notin$ $\{i, j, \bar{m}(i), \bar{m}(j)\}$	$2k+1$	$i-1$ if $j \neq i-2$; $i+1$ else	
i even, j odd $\neq$ $2k+1$	$j-1$ if $i \neq \bar{m}(j-1)$; $j+1$ else	odd $\notin$ $\{j, m(j),\ i-1,$ $i+1,\ 2k+1\}$	even $\notin$ $\{i, \bar{m}(i),$ $j-1,\ j+1\}$	$2k+1$
i odd $\neq$ $2k+1$, $2k+1$	$m(i)$	even $\notin$ $\{i-1,\ i+1\}$	$i+1$	odd $\notin$ $\{i, 2k+1, m(i)\}$
i even, $2k+1$	$i-1$	$\bar{m}(i)$	even $\neq \bar{m}(i)$	$i+3$

Note that in the table, the specified vertex always exists since $k \geqslant 6$. To see that $G(k)$ is 2-e.c. critical note that:

1. The vertex $i \neq 2k+1$, i odd, is the only vertex joined to both $m(i)$ and $2k+1$.
2. The vertex i, i even, is the only vertex joined to neither $2k+1$ nor $\bar{m}(i)$.
3. The vertex $2k+1$ is the only vertex joined to both 1 and j odd $\notin \{1, 2k+1, 3, 2k-1\}$ (and more generally, $2k+1$ is the only vertex joined to both i odd $\neq 2k+1$ and j odd $\notin \{i, 2k+1, i+2, i-2\}$). □

Remark 9. Graphs $G(4)$ can be defined in a similar fashion as $G(k)$ for $k \geqslant 6$ (see Fig. 5). There is only one way to extend the circuit C of a $G(4)$ to a clique minus a matching: add no edges between the vertices 2, 4, 6, 8 of C. There are two (non-isomorphic) matchings between 1, 3, 5, 7. Adding the matching $\{\{1,5\},\{3,7\}\}$ we obtain $K_3 \square K_3$ (see Fig. 6). Adding matching $\{\{1,3\},\{5,7\}\}$ gives a graph which is not 2-e.c. since, for example, there is no vertex joined to neither 1 nor 5 (see Fig. 7).

Consider the graph we will call $G^*(k)$, which is $G(k)$ with the 'standard matching' $m(i) = i+k$, $i = 1, 3, \ldots, 2k-1$, and the 'standard nonmatching', $\bar{m}(i) = i+k$, $i = 2, 4, \ldots, 2k$. Since $G^*(k)$ is 2-e.c. critical by Theorem 8, so is its complement $\overline{G^*(k)}$. We note that $\overline{G^*(k)}$ is very similar in structure to $G^*(k)$ (see Fig. 8).

The graph $\overline{G^*(k)}$ is isomorphic to a graph consisting of $G^*(k)$ along with $k(k-4)$ additional edges, which we call *special* edges. The isomorphism from $G^*(k)$ to

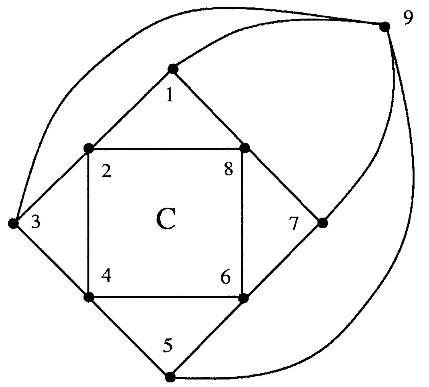

Fig. 5. The circuit C in $G(4)$.

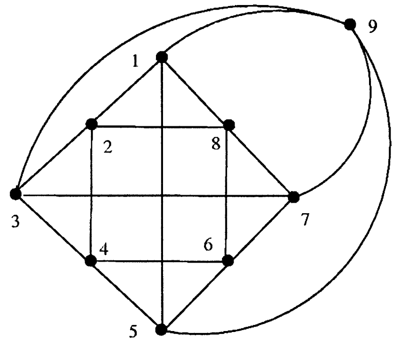

Fig. 6. $K_3 \square K_3$.

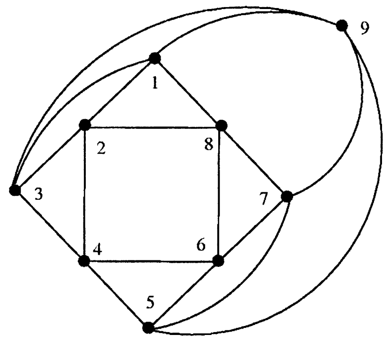

Fig. 7. A non-2-e.c. graph.

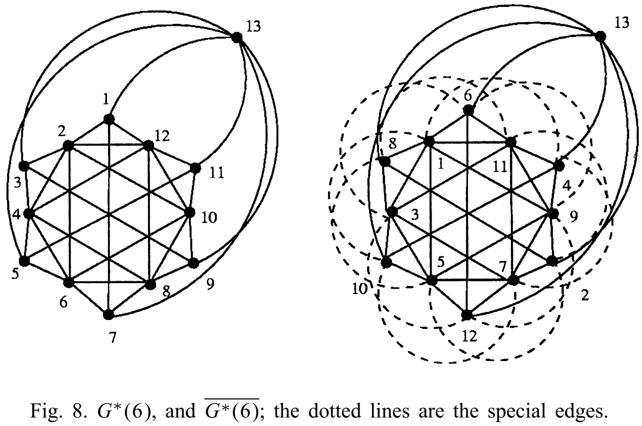

Fig. 8. $G^*(6)$, and $\overline{G^*(6)}$; the dotted lines are the special edges.

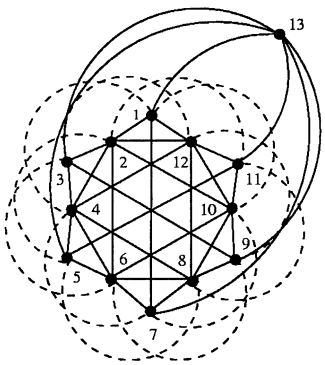

Fig. 9. $G^*(6)^+$.

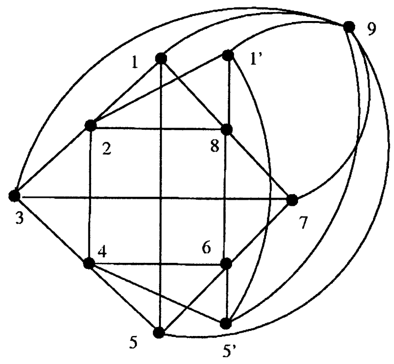

Fig. 10. $R(K_3\Box K_3, 15)$.

$\overline{G^*(k)}$-{special edges} is

$$f(i) = \begin{cases} i+1 & \text{for } i \text{ odd, } i \neq 2k+1, \\ i+k+1 \pmod{2k} & \text{for } i \text{ even,} \\ 2k+1 & \text{for } i = 2k+1. \end{cases}$$

The graph $\overline{G^*(k)}$ is isomorphic to a graph $G^*(k)^+$ consisting of $G^*(k)$ plus the following edges: in $G^*(k)$, an odd vertex $i \neq 2k+1$ is joined to $i-1, i+1, m(i) = i+k$, and $2k+1$. In $G^*(k)^+$, odd vertex $i \neq 2k+1$ is joined to all even vertices i except $i+k-1$ and $i+k+1$, as well as $m(i) = i+k$ and $2k+1$ (see Fig. 9).

Theorem 10. *For a fixed $k \geqslant 6$, a graph F containing $G^*(k)$ and contained in $G^*(k)^+$ is* 2-*e.c.*

Proof. To prove that F is 2-e.c., first note that for vertices $i, j \in V(F)$, since there is a vertex joined to both i, j in $G^*(k)$, there is a vertex joined to both in F; since there is a vertex joined to neither in $\overline{G^*(k)} \cong G^*(k)^+$, there is a vertex joined to neither in F. For the remaining cases, we provide Table 2. Again by symmetry we omit the first two rows of the '2nd only' column. □

2.3. Replication

The 2-e.c. critical graphs presented in Section 2.2 are all of order $\equiv 1 \pmod 4$. In this section, using the replication operation, we give 2-e.c. critical graphs of orders $\equiv 0, 2$, and $3 \pmod 4$.

Definition 11. Let G be a graph and let $e = ab \in E(G)$. The *replicate*, $R = R(G, e)$, is the graph with vertices $V(G) \cup \{a', b'\}$ and edges $E(G) \cup \{a'b'\} \cup \{a'c\colon ac \in E(G)$ and $c \neq b\} \cup \{b'c\colon bc \in E(G)$ and $c \neq a\}$ (in other words, add a new edge $a'b'$ to G, join a' to $N(a) \setminus \{b\}$ and do the analogous for b').

Table 2

Vertices	Joined to	
	1st only	2nd only
i, j odd $\neq 2k+1$	$m(i)$ if $j \neq m(i)$; $i+1$ if $j = m(i) = i+k$	
i, j even	$\bar{m}(j)$ if $i \neq \bar{m}(j)$; $i+1$ if $i = \bar{m}(j) = j+k$	
i even, j odd $\neq 2k+1$	$v \in \{j+k-1,$ $j+k+1\} - \{\bar{m}(i)\}$	$2k+1$
i odd $\neq 2k+1$, $2k+1$	$i+1$	odd $\notin$ $\{i, 2k+1, m(i)\}$
i even, $2k+1$	$i+2$	$i+k-1$

Fig. 10 gives an example of replicate $R(G,e)$ where $G = K_3 \square K_3$ and $e = 15$. To prove the next lemma, we introduce some new notation which will also be useful in Section 3.

Definition 12. Let G be a graph, and let $n \geqslant 1$ be fixed.

1. An *n-e.c. problem* in G is a $2 \times n$ matrix $\begin{pmatrix} x_1 & \dots & x_n \\ i_1 & \dots & i_n \end{pmatrix}$, where $\{x_1, \dots, x_n\}$ is an n-element subset of $V(G)$, and for $1 \leqslant j \leqslant n$, $i_j \in \{0,1\}$.
2. A *solution* to an n-e.c. problem $\begin{pmatrix} x_1 & \dots & x_n \\ i_1 & \dots & i_n \end{pmatrix}$ is a vertex $y \in V(G)$ so that y is joined to x_j if $i_j = 1$ and y is not joined to x_j and $y \neq x_j$ if $i_j = 0$.

Observe that a graph G is n-e.c. if and only if each n-e.c. problem in G has a solution.

Lemma 13. *If G is 2-e.c. then $R = R(G,e)$ is 2-e.c. for every $e \in E(G)$.*

Proof. Fix $e = ab$. The proof proceeds by cases. Fix distinct $x, y \in V(R)$. We show that each problem $\begin{pmatrix} x & y \\ i & j \end{pmatrix}$ has a solution in R.

Case 1. $\{a', b'\} \cap \{x, y\} = \emptyset$. In this case, a solution to the problem in G is a solution to the problem in R.

Case 2. $\{a', b'\} \cap \{x, y\} \neq \emptyset$. We consider two subcases.

Case 2.1. $|\{x, y\} \cap \{a', b'\}| = 1$. Without loss of generality, $x = a'$ and $y \neq b'$. First suppose $y = a$. If $(i, j) = (1, 1)$ a neighbour of a solves the problem; $(0, 0)$ is solved similarly. $(1, 0)$ is solved by b' and $(0, 1)$ is solved by b.

If $y \neq a$, first solve $\binom{a\ y}{i\ j}$ by say, c, in G. If $c \neq b$, then c also solves $\binom{a'\ y}{i\ j}$. If $c = b$, then $i = 1$ (as $ab \in E(G)$) and $y \neq b$. Hence, the problem $\binom{a'\ y}{i\ j}$ is solved by b'.

Case 2.2. $|\{x, y\} \cap \{a', b'\}| = 2$. A solution to $\binom{a\ b}{i\ j}$ in G is a solution to $\binom{a'\ b'}{i\ j}$ in R. □

Note that adding one or more of the edges aa', bb' to R will still preserve 2-e.c.

Remark 14. In Theorem 4 of [7] it has been proven that a 2-e.c. graph exists for all orders $n \geqslant 9$. We wish to point out that this result follows quickly from Lemma 13. Replicating edges of $K_3 \square K_3$ gives 2-e.c. graphs of all odd orders > 9. Replicating edges of a 10-element 2-e.c. graph gives 2-e.c. graphs of all even orders > 10. An example of a 10-element 2-e.c. graph is the following: add a vertex to $K_3 \square K_3$ that is joined precisely to vertices 2,3,6,7.

We now see that in some cases, replication also preserves 2-e.c. criticality.

Definition 15. An edge $e = ab \in E(G)$ is *good* if:

1. There is a $c \in V(G)$ and $i \in \{0, 1\}$ so that $\binom{a\ c}{1\ i}$ has b as its unique solution, and there is a $d \in V(G) \backslash \{a, b\}$ and $j \in \{0, 1\}$ so that $\binom{b\ d}{1\ j}$ has a as its unique solution;
2. For each $x \in V(G) \backslash \{a, b\}$, one of the following holds:
   (a) The vertex x is the unique solution to $\binom{a\ b}{1\ 1}$,
   (b) The vertex x is the unique solution to a problem $\binom{f\ g}{i\ j}$ with $f \neq a, b$ so that $i = 1$ and f is not joined to a or b, or $i = 0$ and f is joined to both a and b,
   (c) The vertex x is the unique solution to a problem $\binom{f\ g}{i\ j}$ with $f \neq a, b$ and $g \neq a, b$ for some $i, j \in \{0, 1\}$.

In the graphs $G(k)$, $k \geqslant 6$, or $K_3 \square K_3$, an edge $e = \mathrm{im}(i)$, where $i \neq 2k + 1$ is odd is good: for (1), $m(i)$ is the unique solution to $\binom{i\ \ 2k+1}{1\ \ 1}$ and i is unique for $\binom{m(i)\ \ 2k+1}{1\ \ 1}$. For (2), $2k+1$ is unique for $\binom{i\ \ m(i)}{1\ \ 1}$; an odd $j \notin \{2k+1, i, m(i)\}$ is unique for $\binom{m(j)\ \ 2k+1}{1\ \ 1}$ and $m(j)$ is not joined to i or $m(i)$; an even j is unique for $\binom{2k+1\ \ \bar{m}(j)}{0\ \ 0}$ and $2k+1$ is joined to both i and $m(i)$.

Lemma 16. *Let G be a 2-e.c. critical graph and $e \in E(G)$. If e is good then $R = R(G,e)$ is 2-e.c. critical.*

Proof. By Lemma 13, R is 2-e.c. Fix $x \in V(R)$, and let $e = ab$.

Case 1. $x \neq a,b,a',b'$. As e is good, (2a), (2b) or (2c) holds. If x is the unique solution to $\begin{pmatrix} a & b \\ 1 & 1 \end{pmatrix}$ in G, then neither a' nor b' can solve this problem in R, so x is the unique solution to this problem in R.

If x is the unique solution to a problem $\begin{pmatrix} f & g \\ i & j \end{pmatrix}$ in G with $f \neq a,b$, then if $i = 1$, f is not joined to a or b. Hence, f is not joined to a' or b' in R, so neither of them can solve the problem in R. The case $i = 0$ is similar.

If x is a unique solution to a problem $\begin{pmatrix} f & g \\ i & j \end{pmatrix}$ in G with $f \neq a,b$ and $g \neq a,b$, then if a' solves this problem, so would a which is a contradiction. The argument for b' is similar.

Case 2. $x = a$ or $x = b$. By (1), there is a $c \in V(G)\backslash\{a,b\}$ and i so that $\begin{pmatrix} a & c \\ 1 & i \end{pmatrix}$ has as its unique solution $x = b$. But then a' and b' cannot solve this problem as they are not joined to a. The argument for $x = a$ is similar.

Case 3. $x = a'$ or $x = b'$. By (1), b is the unique solution to $\begin{pmatrix} a & c \\ 1 & i \end{pmatrix}$ in G. We claim that b' is the unique solution to $\begin{pmatrix} a' & c \\ 1 & i \end{pmatrix}$ in R. Otherwise, say the problem is solved by $z \neq b'$. As $a'z \in E(R)$, $z \neq a,b,a'$. It follows from the definition of R that $az \in E(G)$. But then z is a solution to $\begin{pmatrix} a & c \\ 1 & i \end{pmatrix}$ in G, which contradicts the fact that b is the unique solution to this problem. The argument for a' is similar. □

By Lemma 16, it follows that the graphs $R(G(k), \mathrm{im}(i))$, where $k \geqslant 6$ is even and $i \neq 2k+1$ is odd, are 2-e.c. critical. We have therefore discovered 2-e.c. critical graphs of every odd order $\geqslant 9$.

We now give examples of 2-e.c. criticals of all even orders $\geqslant 10$. Define a graph H by deleting the edge 59 in $K_3 \square K_3$, adding a new vertex 10, and joining 10 to 1, 5, 7, and 9.

Lemma 17. *H is 2-e.c. critical.*

Proof. We leave it to the reader to verify that H is 2-e.c. For criticality, we provide Table 3. □

Theorem 18. *There are 2-e.c. critical graphs of all even orders $\geqslant 10$.*

Table 3

Vertex	1	2	3	4	5
Uniquely solves	$\begin{pmatrix} 2 & 8 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 8 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 9 & 10 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 5 & 6 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 9 & 10 \\ 0 & 1 \end{pmatrix}$
Vertex	6	7	8	9	10
Uniquely solves	$\begin{pmatrix} 7 & 8 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 6 & 8 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 6 & 7 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 3 & 7 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 5 \\ 1 & 1 \end{pmatrix}$

Proof. Define $H_0 = H$. If H_n has been defined, define $H_{n+1} = R(H_n, 12)$, with replication edge $e_{n+1} = 1_{n+1}2_{n+1}$. Then H_n is 2-e.c. of order $10 + 2n$. We show that H_n is 2-e.c. critical. H_0 is 2-e.c. critical by Lemma 17.

We proceed by induction on n. (Note that 12 is not good in H.) Assume H_n is 2-e.c. critical, so that 1_j uniquely solves $\begin{pmatrix} 2_j & 2 \\ 1 & 0 \end{pmatrix}$ and 2_j uniquely solves $\begin{pmatrix} 1_j & 1 \\ 1 & 0 \end{pmatrix}$, for $1 \leqslant j \leqslant n$. For ease of notation, let $1_{n+1} = a$ and $2_{n+1} = b$. Neither a nor b can solve the 2-e.c. problems that vertices 1 to 10 uniquely solve in the proof of Lemma 17: a and b cannot solve the problems for vertices 1, 2, and 10 as they are not joined to 1 and 2. The vertex a cannot solve the problems for 3–9 otherwise 1 would also (a similar argument holds for b).

The vertices a and b cannot solve the problems that 1_j and 2_j uniquely solve in the induction hypothesis, as a and b are not joined to either 1_j or 2_j. Finally, a uniquely solves $\begin{pmatrix} b & 2 \\ 1 & 0 \end{pmatrix}$ and b uniquely solves $\begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix}$. $\square$

We have found a 3-e.c. critical graph of order 28. This was done by searching through the vertex-transitive graphs of order 20 and up listed on Gordon Royle's website. Angie Ho did the programming, and we thank her for her work.

Note that it follows from Lemma 2 and Theorem 5 that a 3-e.c. graph has at least 19 vertices; if it had 19 vertices it would have to be 9-regular which is impossible. Thus, a 3-e.c. graph has at least 20 vertices. In [2], it is proved that the Paley graph on 29 vertices is 3-e.c.

3. N-e.c.-preserving operations

In this section, we investigate the n-e.c.-preserving properties of some familiar graph operations. First, we recall some binary operations on graphs (see [12]).

Definition 19. Let G and H be graphs.

1. The *disjuction* of G and H, $G \vee H$, has vertices $V(G) \times V(H)$ and edges $(a,b)(c,d) \in E(G \vee H)$ iff $ac \in E(G)$ or $bd \in E(H)$ (or both).

Table 4

	Operation/preserving	a	b	c	d
		1-e.c.	2-e.c.	3-e.c.	4-e.c.
(i)	Complementation	Yes	Yes	Yes	Yes
(ii)	Cartesian product	Yes	No	No	No
(iii)	Disjunction	Yes	Yes	No	No
(iv)	Lexicographic product	Yes	Yes	No	No
(v)	Symmetric difference	Yes	Yes	Yes	No
(vi)	Categorical product	Yes	Yes	No	No
(vii)	Disjoint union	Yes	No	No	No
(viii)	Total join	Yes	No	No	No

2. The *lexicographical product* of G and H, $G[H]$, has vertices $V(G)\times V(H)$ and edges $(a,b)(c,d)\in E(G[H])$ iff $ac\in E(G)$ or $a=c$ and $bd\in E(H)$.
3. The *symmetric difference* of G and H, $G\triangle H$, has vertices $V(G)\times V(H)$ and edges $(a,b)(c,d)\in E(G\triangle H)$ iff exactly one of $ac\in E(G)$ or $bd\in E(H)$.
4. The *categorical product* of G and H, $G\times H$, has vertices $V(G)\times V(H)$ and edges $(a,b)(c,d)\in E(G\times H)$ iff $ac\in E(G)$ and $bd\in E(H)$.
5. The *total join* of disjoint graphs G and H, $G+H$, has vertices $V(G)\cup V(H)$ and edges those of G and those of H and edges connecting every vertex of G to every vertex of H.

Table 4 lists whether these graph operations preserve n-e.c., for $1\leqslant n\leqslant 4$.

To verify column (a) and rows (i), (vii), (viii) is straightforward. For (iib), let G be 2-e.c. graph containing distinct vertices a,b,c,d so that a is not joined to c. We claim there is no solution in $G\square G$ to the 2-e.c. problem

$$\begin{pmatrix} (a,b) & (c,d) \\ 1 & 1 \end{pmatrix}. \tag{1}$$

To see this, let (y_1,y_2) be a solution to (1) in $G\square G$. Now, $(a,b)(y_1,y_2)\in E(G\square G)$ implies that $a=y_1$ and $by_2\in E(G)$, or $b=y_2$ and $ay_1\in E(G)$; $(c,d)(y_1,y_2)\in E(G\square G)$ implies that $c=y_1$ and $dy_2\in E(G)$, or $d=y_2$ and $cy_1\in E(G)$. Since both $(a,b)(y_1,y_2)$, $(c,d)(y_1,y_2)\in E(G\square G)$, it follows that $a=c$, $ac\in E(G)$ or $b=d$, a contradiction.

For (iiib)–(vb), note that if A,B are graphs, if C is any of

$$\{A[B], A\vee B, A\triangle B\},$$

then $(a,b)(a,c)\in E(C)$ iff $bc\in E(B)$ and $(b,a)(c,a)\in E(C)$ iff $bc\in E(A)$, and so if $\begin{pmatrix} b & c \\ i & j \end{pmatrix}$ is solved by y_1 in B then $\begin{pmatrix} (a,b) & (a,c) \\ i & j \end{pmatrix}$ is solved by (a,y_1) in C; the solution of $\begin{pmatrix} (b,a) & (c,a) \\ i & j \end{pmatrix}$ is analogous. Hence, we need only consider 2-e.c. problems of the form

$$\begin{pmatrix} (a,b) & (c,d) \\ i_1 & i_2 \end{pmatrix}, \tag{2}$$

where $a \neq c$ and $b \neq d$. If we let y_1 be a solution to the 2-e.c. problem $\begin{pmatrix} a & c \\ i_1 & i_2 \end{pmatrix}$ in A, and y_2 be a solution to the 2-e.c. problem $\begin{pmatrix} b & d \\ i_1 & i_2 \end{pmatrix}$ in B, then (y_1, y_2) solves (2) in case $C \in \{A[B], A \vee B\}$, or in the case $C = A \triangle B$ and $i_1 = i_2 = 0$. The remaining cases are when $C = A \triangle B$ and either $i_1 = i_2 = 1$ or exactly one of i_j equals 1. In these cases, choose y_1 as before, but choose y_2 to be a solution to $\begin{pmatrix} b & d \\ 0 & 0 \end{pmatrix}$ in B.

For (iiic), let A be any 3-e.c. graph containing distinct elements a, b, c. Consider the 3-e.c. problem in $A \vee A$:

$$\begin{pmatrix} (a,b) & (a,c) & (b,c) \\ 0 & 1 & 0 \end{pmatrix}. \tag{3}$$

If (y_1, y_2) solves (3), then from the first column, $ay_1 \notin E(A)$ and $by_2 \notin E(A)$. From the second column, as $ay_1 \notin E(A)$ we must have $cy_2 \in E(A)$. But by the third column $cy_2 \notin E(A)$. Contradiction.

For (ivc), consider a 3-e.c. graph A containing elements a, b so that $ab \in E(A)$. Consider the 3-e.c. problem in $A[A]$:

$$\begin{pmatrix} (a,b) & (a,a) & (b,a) \\ 0 & 1 & 0 \end{pmatrix}. \tag{4}$$

If (y_1, y_2) solves (4), then from the first and third columns, we must have $ay_1 \notin E(A)$ and $by_1 \notin E(A)$. But then by the second column, since $ay_1 \notin E(A)$, $a = y_1$ and $ay_2 \in E(A)$. This contradicts the assumption that $ab \in E(A)$.

For (vc), we proceed by cases. Fix distinct elements $(a,b), (c,d), (f,g) \in A \triangle B$, where A, B are 3-e.c., and fix a 3-e.c. problem in $C = A \triangle B$

$$\begin{pmatrix} (a,b) & (c,d) & (f,g) \\ i_1 & i_2 & i_3 \end{pmatrix}. \tag{5}$$

Define $X_1 = \{a, c, f\}$, $X_2 = \{b, d, g\}$.

Case 1. For some $i \in \{1,2\}$, $|X_i| = 1$. Here we appeal again to the fact that $(a,b)(a,c) \in E(C)$ iff $bc \in E(B)$ and $(b,a)(c,a) \in E(C)$ iff $bc \in E(A)$.

Case 2. $|X_1| = |X_2| = 2$. Without loss of generality, assume $a = c$ and $b = g$.

Case 3. $|X_1| = 2$, $|X_2| = 3$. Without loss of generality, assume that $a = c$.

There are eight subcases to verify in Cases 2 and 3. In Table 5, we provide a solution (y_1, y_2) to (5) in each subcase.

Case 4. $|X_1| = 3$, $|X_2| = 2$. This case follows by symmetry from Case 3.

Case 5. $|X_1| = |X_2| = 3$. Let y_1 be a solution in A to the 3-e.c. problem

$$\begin{pmatrix} a & c & f \\ i_1 & i_2 & i_3 \end{pmatrix},$$

Table 5

Subcases	Case 2 y_1 is a solution to	y_2 is a solution to	Case 3 y_1 is a solution to	y_2 is a solution to
$i_1=i_2=i_3=1$	$\begin{pmatrix} a & f \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} b & d \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} a & f \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} b & d & g \\ 0 & 0 & 0 \end{pmatrix}$
$i_1=i_2=1,\ i_3=0$	$\begin{pmatrix} a & f \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} b & d \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} a & f \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} b & d & g \\ 0 & 0 & 0 \end{pmatrix}$
$i_1=i_3=1,\ i_2=0$	$\begin{pmatrix} a & f \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} b & d \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} a & f \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} b & d & g \\ 0 & 1 & 0 \end{pmatrix}$
$i_1=0,\ i_2=i_3=1$	$\begin{pmatrix} a & f \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} b & d \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} a & f \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} b & d & g \\ 0 & 1 & 1 \end{pmatrix}$
$i_1=1,\ i_2=i_3=0$	$\begin{pmatrix} a & f \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} b & d \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} a & f \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} b & d & g \\ 1 & 0 & 0 \end{pmatrix}$
$i_1=i_3=0,\ i_2=1$	$\begin{pmatrix} a & f \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} b & d \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} a & f \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} b & d & g \\ 0 & 1 & 0 \end{pmatrix}$
$i_1=i_2=0,\ i_3=1$	$\begin{pmatrix} a & f \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} b & d \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} a & f \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} b & d & g \\ 0 & 0 & 1 \end{pmatrix}$
$i_1=i_2=i_3=0$	$\begin{pmatrix} a & f \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} b & d \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} a & f \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} b & d & g \\ 0 & 0 & 0 \end{pmatrix}$

and let y_2 be a solution in B to the 3-e.c. problem

$$\begin{pmatrix} b & d & g \\ 0 & 0 & 0 \end{pmatrix}.$$

The reader can check that (y_1, y_2) solves (5).

For (vd), let A be 4-e.c. graph containing distinct elements a and b. Consider the 4-e.c. problem in $A \triangle A$

$$\begin{pmatrix} (a,a) & (a,b) & (b,b) & (b,a) \\ 1 & 1 & 0 & 1 \end{pmatrix}. \tag{6}$$

If (y_1, y_2) solves (6), then from the third column there are two cases.

Case 1. *The vertex b is not joined to* y_1 or y_2 *in A*. In this case, by the second column, $ay_1 \in E(A)$ and by the fourth column, $ay_2 \in E(A)$. But then $(y_1, y_2)(a,a) \notin E(A \triangle A)$, contradicting the first column.

Case 2. *The vertex b is joined to both* y_1 and y_2 in A. By the second column, $ay_1 \notin E(A)$ and by the fourth column $ay_2 \notin E(A)$. But then $(y_1, y_2)(a,a) \notin E(A \triangle A)$ again contradicting the first column.

For (vib), note that $A \times B = \overline{\bar{A} \vee \bar{B}}$; now use (ib) and (iiib). For (vic) use the fact $A \vee B = \overline{\bar{A} \times \bar{B}}$ and (ic) and (iiic). $\square$

We do not know of a 4-e.c.- but not 5-e.c.-preserving graph operation.

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