Colourings, Generics, and Free Amalgams

by

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Abstract

We study free amalgamation classes over a finite relational language and their applications to the model companions of $\forall_1$ classes over a finite relational language.

If an $\forall_1$ class $\mathcal{K}$ is a free amalgamation class over a finite relational language with edges, the model companion is nqfa(1): non-finitely axiomatizable even modulo axioms asserting "I embed all finite structures in $\mathcal{K}$". Further, there is a structure in $\mathcal{K}$ isometrically embedding each countable structure in $\mathcal{K}$ (relative to the least path metric on the graphs of structures in $\mathcal{K}$).

We study colour classes in $\forall_1$ free amalgamation classes over a finite relational language and their model companions. We find sufficient conditions for the model companion of a colour class to exist; when the model companion exists, it has a theory equal to the theory of a generic structure and is nqfa(1).

Acknowledgements

I would first like to express my appreciation to my supervisor Ross Willard for his support and careful reading of this work. I would like to thank my family, my parents Paul and Anna, and my sister Lisa, for their love and patience given so freely. I also wish to honor the memory of my sister Paula, who passed away in 1992, without whom I would not have chosen to study mathematics...

$$\forall x \exists y (x \text{ is the question, } y \text{ is the answer}).$$

I would like to especially thank my friend and gifted colleague Dejan Delić who taught me many things, including the true meaning of friendship.

To Dejan.

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Chapter 0

Introduction

Existentially closed (or e.c.) structures were first introduced by Abraham Robinson and his school (see [13], for example), and have played an important role in model theory since then. Loosely speaking, e.c. structures are the ultimate generalization of the notion of an algebraically closed field to a first-order context. Parallel to and independent of the development of e.c. structures was the work of Fraïssé [15] on what are now called generic structures (see [26]) or Fraïssé limits of classes of finite structures. Generic structures display rich symmetry (witnessed by their large automorphism groups), and have been actively studied, especially in the guise of so-called random structures (the random graph being the most well-known example; see [8]). Central to the definition of generic structures are amalgamation properties of embeddings of structures. Amalgamation is a somewhat rare property of classes of structures; however, if a class has amalgamation, generics and countable e.c. structures will often coincide, each aspect clarifying the other.

In this work, we take as our starting point classes of structures which already possess amalgamation in the form of free amalgamation (see Definition 2.1 below). Under this hypothesis, the fine structure of the generic becomes transparent and open to thorough investigation. As an example, in Chapter 3, we present some unexpected connections between the theory of colourings of relational structures and free amalgams.

Our investigations are divided into three main lines which describe the logical breakdown of the thesis:

1. $\forall_1$ free amalgamation classes over a finite relational language and their model companions (a class is $\forall_1$ if it can be defined by universally quantified first-order statements; the *model companion* of an $\forall_1$ class $\mathcal{K}$, when it exists, is simply the class of e.c. structures in $\mathcal{K}$).

2. Colour classes in $\forall_1$ free amalgamation classes over a finite relational language and their model companions.

3. Isometric universal structures in $\forall_1$ free amalgamation classes over a finite relational language.

Chapter 1 contains all the statements of the classical results from model theory that we will use in this work.

In Chapter 2, we will investigate the properties of $\forall_1$ free amalgamation classes over a finite relational language. As we will see in Sections 2.2.2 and 2.2.3, if an $\forall_1$ class is a free amalgamation class with edges (see Definition 2.25) then it has a "highly" non-finitely axiomatizable model companion. More precisely, the model companion is nqfa(1): non-finitely axiomatizable even modulo axioms asserting

"I embed all finite structures in the class"; see Theorem 2.27 below. This result extends the result of [30], which proves that the theory of the random graph is non-finitely axiomatizable modulo "I am infinite". Furthermore, we supply new structural and syntactic characterizations of $\forall_1$ free amalgamation classes.

In Chapter 3, we present the central results of the thesis in the study of classes of structures admitting a homomorphism into a fixed finite structure; we call such classes "colour classes". General colour classes are inspired by the well-known classes $C(K_n)$ of n-colourable graphs for $n \geq 2$. Recall that $C(K_n)$ is the class of all graphs whose vertices have partitions into at most n independent sets (that is, vertices with no edges in common), or equivalently, admit a homomorphism into K_n. $C(K_n)$ has an $\aleph_0$-categorical model companion, as first proven by Wheeler in [42]. In Chapter 3, we work with colour classes of structures that are not necessarily graphs. One key point here is that while colour classes fail in general to have amalgamation, the classes of uniquely colourable structures have free amalgamation (see Proposition 3.41). Under certain conditions on the underlying class (see Theorem 3.51), the model companion of the colour class exists, has a theory equal to that of the generic, and is nqfa(1). As of yet, no work on colour classes over $\forall_1$ classes has appeared in the literature.

As proven by Moss in [33], for graphs with the usual "least path" metric, there is a countable graph (distinct from the Fraïssé limit of the finite graphs, the random graph) that isometrically embeds every countable graph. In Chapter 4 we extend this result to $\mathcal{K}$ an $\forall_1$ free amalgamation class. Although some of the discussion of Chapter 4 relies on the analysis of [33], the use of generic structures allows several new advances to be made. In particular, if $\mathcal{K}$ has edges the model companion of the class of distanced structures is again nqfa(1).

Chapter 1

Preliminaries

1.1 $\forall_1$ classes over a relational language

1.1.1 First-order structures

We start with the definition of our main objects of study: first-order structures over a relational language.

As all of the results in this section are known, we are content to give a brief survey of critical results, in most cases merely sketching proofs, and in some cases deferring the reader to appropriate sources.

ω is the set of natural numbers (including 0); $\omega^* = \omega - \{0\}$.

Definition 1.1 *1. A (first-order relational)* ***language*** *or* ***signature*** *is a set of symbols $L = \{R_i : i \in I\}$ along with a function $ar : L \to \omega^*$.*

*2. An L-**structure** A is a pair consisting of a nonempty set dom(A), the **domain** (or **universe**) of A, and an operation $R \mapsto R^A$ defined on all $R \in L$*

so that if $ar(R) = n$ then $R^A \subseteq dom(A)^n$. R^A is the ***interpretation*** *of R in A.*

3. *The* ***cardinality*** *(or* ***order****) of an L-structure A is the cardinality of its domain.*

Remark 1.2 *1.* We abuse notation and let A stand for both a structure and its domain. Furthermore, we suppress mention of the operation $R \mapsto R^A$.

2. We consider only non-empty structures.

3. Given a structure A, $\bar{a} \in A^n$ is a finite (ordered) tuple of length $n \geq 1$. The length of $\bar{a}$ is written $|\bar{a}|$. Given a finite subset S of A, let $\bar{a}$ be an enumeration of S (without repetitions). We will sometimes abuse notation and identify S with $\bar{a}$.

4. In Section 1.2.2 below it will be convenient to allow L to contain a set $\{c_i : i \in I\}$ of constant symbols as well. In this case, an L-structure A carries, for each $i \in I$, an assignment $c_i \mapsto c_i^A \in A$.

Example 1.3 Let $L = \{E\}$, with $ar(E) = 2$.

1. A graph G is an L-structure with E^G irreflexive and symmetric. The domain of G is often called the set of vertices of G; the 2-tuples in E^G are the edges. In practice, we usually write E (without the superscript G) when G is clear from context.

2. An order P is an L-structure with E^P reflexive, anti-symmetric, and transitive. If $(a, b) \in E^P$ we usually write $a \leq b$.

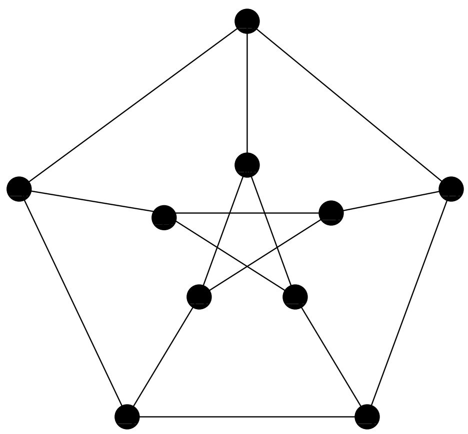

Figure 1.1: An example of a graph: the Petersen graph.

3. Let $L = \{R\}$, $ar(R) = k$, for some $k \geq 2$.

A k-uniform hypergraph H is an L-structure with R^H interpreted as all permutations of a set of k-element subsets of H.

Definition 1.4 *Let A, B be L-structures for a fixed language L.*

*1. A **homomorphism** f from A to B is a map satisfying*

$$\bar{a} \in R^A \text{ implies } f(\bar{a}) \in R^B. \tag{1.1}$$

*2. f is an **embedding** if f is injective and the "implies" in (1.1) is replaced by an "if and only if". f is an **isomorphism** if it is a surjective embedding. We write that A and B are isomorphic as $A \cong B$.*

*An **automorphism** of A is an isomorphism from A to itself; the automorphisms of A form a group, written $aut(A)$.*

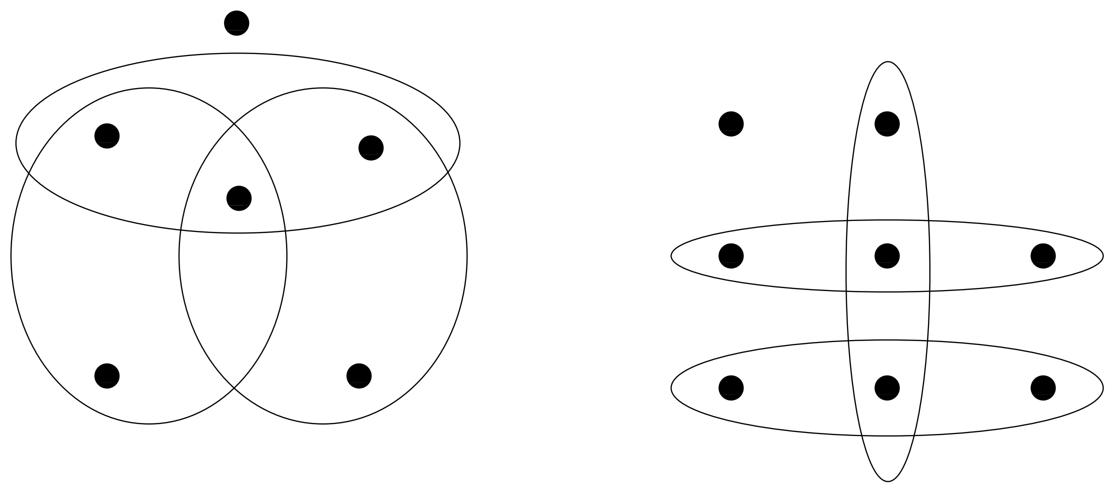

Figure 1.2: 3-uniform hypergraphs; the ovals are hyperedeges.

3. *S is a **substructure** of a structure A if $S \subseteq A$ and the inclusion map $S \hookrightarrow A$ is an embedding. The substructure relation is written $S \leq A$.*

4. *Let $S \subseteq A$. Then the **induced substructure** of A on S, in symbols $A \restriction S$, is the structure with domain S and relations $R^{A \restriction S} = R^A \cap (S^n)$, for $R \in L$ with $ar(R) = n$.*

1.1.2 First-order logic

We give a brief overview of the syntax and semantics of first-order logic. For a more detailed treatment, we refer the reader to Chapter 1 and 2 of [23].

To a first-order language L we add logical connectives $\vee$ (*or*), $\wedge$ (*and*), $\neg$ (*not*), $\rightarrow$ (*implies*), $\leftrightarrow$ (*bi-implication*), $=$ (*equality*), as well as quantifiers $\exists$ (*there exists*), and $\forall$ (*for all*).

First-order formulae are defined recursively in a standard way. A *term* is a variable (or a constant symbol, if L has any). An *atomic formula* is either a relation symbol from L with terms as arguments or the equality of two terms. A general formula over L is obtained by combining formulae by means of connectives, or preceding them with quantifiers. Note that first-order formulae may contain only finitely many conjunctions or disjunctions, and only finitely many quantifiers. A formula is a *sentence* if it has no free (unquantified) variables. A formula is *quantifier-free* if it contains no quantifiers. A formula (sentence) over L is sometimes called an *L-formula* (*L-sentence*).

We will make use of the quantifier hierarchy as stated in the following definition.

Definition 1.5 *Let L be a first-order language.*

1. *L-formulas are $\forall_0$ and $\exists_0$ if they are quantifier-free.*
2. *An L-formula is an $\forall_{n+1}$ formula if it is in the smallest class of L-formulas which contains the $\exists_n$ L-formulas and is closed under $\wedge, \vee$ and adding universal quantifiers at the front.*
3. *An L-formula is an $\exists_{n+1}$ formula if it is in the smallest class of L-formulas which contains the $\forall_n$ L-formulas and is closed under $\wedge, \vee$ and adding existential quantifiers at the front.*

Remark 1.6 *An $\forall_1$ formula is sometimes called* universal*; $\exists_1$ formulas are called* existential*. $\forall_2$ sentences are also called $\forall\exists$ sentences.*

The connection between L-structures and L-sentences is given by Tarksi's definition of truth. If S is an L-structure and φ is an L-sentence we write $S \models \varphi$ is φ

is true in S; S is a *model* of φ. A similar definition holds for sets of L-sentences. If $A, B \in \mathcal{K}(L)$ satisfy the same L-sentences, we write $A \equiv B$.

Definition 1.7 *Let L be finite, $A \in \mathcal{K}(L)$, and let $\bar{a}$ list n distinct elements of A, for some $n \geq 1$. The* ***quantifier-free diagram*** *of $\bar{a}$ in A is the conjunction of all atomic or negated atomic formulas that $\bar{a}$ satisfies in A and is written*

$$qft_{A,\bar{a}}(\bar{x})$$

where $\bar{x}$ is a fixed tuple of variables with $|\bar{x}| = |\bar{a}|$. If A is finite and $\bar{a}$ enumerates A then $qft_{A,\bar{a}}(\bar{x})$ is the ***diagram*** *of A relative to $\bar{a}$.*

1.1.3 Classes of Structures

If L is a relational language, and $\mathcal{K}$ is a class of L-structures, we always assume that $\mathcal{K}$ is closed under isomorphism. The reason for this restriction is that we are interested only in the isomorphism types of structures.

Definition 1.8 *Let L be a first-order language.*

1. *Let $\mathcal{K}(L)$ be the class of all L-structures.*

2. *Given $\mathcal{K} \subseteq \mathcal{K}(L)$, $A \in \mathcal{K}(L)$ is a $\mathcal{K}$-structure if $A \in \mathcal{K}$.*

3. *Given $\mathcal{K} \subseteq \mathcal{K}(L)$, let $\mathcal{K}_{fin}$ be the class of finite structures in $\mathcal{K}$.*

4. *Given a cardinal $\lambda > 0$, let $\mathcal{K}_\lambda$ be the class of $\mathcal{K}$-structures of order λ. $\mathcal{K}_\lambda$ is called the class of $\mathcal{K}$-structures of cardinality λ.*

A detailed discussion of first-order theories can be found in Chapter 2 of [23].

Definition 1.9 *Let L be a first-order language.*

1. *A* ***theory*** *(over L) is a set of consequence closed L-sentences.*
2. *The* ***theory of a class*** *$\mathcal{K}$ is the set of all L-sentences satisfied by each member of $\mathcal{K}$ and is denoted $Th(\mathcal{K})$.*
3. *The theory of $A \in \mathcal{K}(L)$ is the set of all L-sentences satisfied by A, and is denoted $Th(A)$. A theory is* ***complete*** *if it is the theory of some L-structure.*
4. *$\mathcal{K} \subseteq \mathcal{K}(L)$ is* ***elementary*** *if $\mathcal{K}$ is the class of models of some set of L-sentences.*
5. *Fix $n \in \{1, 2\}$. $\mathcal{K} \subseteq \mathcal{K}(L)$ is $\forall_n$ if $\mathcal{K}$ is the class of models in $\mathcal{K}(L)$ of some set of $\forall_n$ L-sentences T. T is an* ***axiomatization*** *of $\mathcal{K}$.*

We will be mainly interested in $\forall_1$ classes of structures. The main point of such a restriction is that in an $\forall_1$ class $\mathcal{K}$, the induced substructures of $A \in \mathcal{K}$ are again in $\mathcal{K}$ (see Theorem 6.5.4 of [23]). Further, we will assume (unless otherwise stated) that L is finite. The key effect of the latter assumption is that for each $n \in \omega^*$, $\mathcal{K}_n$ is finite (in fact, $|\mathcal{K}_n| \leq 2^{mn^\alpha}$, where $m = |L|$, and α is the maximum arity of a symbol of L as the reader can check).

Basic Results of Model Theory

For the convenience of the reader, we collect a few of the basic results of model theory.

Theorem 1.10 (Compactness) *(Theorem 6.1.1 of [23]) A set of L-sentences Σ has a model if and only if every finite subset of Σ has a model.*

Definition 1.11 *Let $A, B \in \mathcal{K}(L)$. A is an* ***elementary substructure*** *of B, written $A \preceq B$, if $A \leq B$ and for all finite tuples $\bar{a}$ from A and all formulas $\phi(\bar{x})$ with $|\bar{x}| = |\bar{a}|$, $A \models \phi(\bar{a})$ if and only if $B \models \phi(\bar{a})$.*

Theorem 1.12 (Löwenheim-Skolem) *(Theorems 6.1.4 and 3.1.5 of [23]) Let $\mathcal{K} \subseteq \mathcal{K}(L)$ be an elementary class over a countable language L.*

1. *For every infinite $A \in \mathcal{K}$ and every cardinal $\lambda > |A|$, there is a model $B \in \mathcal{K}$ with $|B| = \lambda$ so that $A \prec B$.*
2. *If $B \subseteq A \in \mathcal{K}$ and λ is a cardinal so that $|A| \geq \lambda \geq \max(|B|, \aleph_0)$, then there is a $\mathcal{K}$-structure $C \preceq A$ with domain containing B and having cardinality λ.*

Definition 1.13 *Let κ be a cardinal.*

1. *A first-order theory T is κ-**categorical** if there is only one model (up to isomorphism) of T of order κ.*
2. *A structure A is κ-**categorical** if $Th(A)$ is κ-**categorical**.*

Remark 1.14 A theorem of Engeler, Svenonius, and Ryll-Nardzewski characterizes an $\aleph_0$-categorical countable structure A as a countable structure with $aut(A)$ oligomorphic (that is, $aut(A)$ has only finitely many orbits on A^n, for every $n \in \omega^*$). As we do not use this characterization, we direct the interested reader to Ch. 7 of [23] for further information.

1.1.4 Infinitary logic and back-and-forth equivalence

For a thorough discussion of infinitary logics and games, the reader is referred to Ch. 3 of [23] or Ch. 2 of [12].

The definition of the infinitary logic $L_{\infty\omega}$ runs parallel to that of first-order logic, the one difference being that formulas may now contain conjunctions and disjunctions of *sets* of formulas (although only finitely many quantifiers). If two structures $A, B \in \mathcal{K}(L)$ satisfy the same $L_{\infty\omega}$-sentences, we write $A \equiv_{\infty\omega} B$.

$L_{\omega_1\omega}$ is defined similarly and allows only conjunctions or disjunctions of *countable* sets of formulas.

There is a close connection between $L_{\infty\omega}$ and certain games defined on pairs of L-structures.

Let A, B be L-structures. A *back-and-forth game* on A, B is a two player game played in ω steps, with players $\exists$ (*the duplicator*) and $\forall$ (*the spoiler*). At the ith step, for $i \geq 0, \forall$ chooses an element from one of the structures A and B; $\exists$ then chooses an element of the other structure. Apart from $\exists$ having to choose an element from the other structure that $\forall$ chose from, both players have complete freedom to choose elements. Both players are allowed to see and remember all previous moves in the play of the game. At the end of the game, sequences $(a_i : i \in \omega)$ from A and $(b_i : i \in \omega)$ from B have been chosen. $(\bar{a}, \bar{b})$ is a *play* of the game.

$(\bar{a}, \bar{b})$ is a *win for* $\exists$ if there is an isomorphism

$$f : A \restriction \{a_i : i \in \omega\} \to B \restriction \{b_i : i \in \omega\}$$

so that $f(a_i) = b_i$, for $i \in \omega$. A play that is not a win for $\exists$ is a *win for* $\forall$. A and B are said to be *back-and-forth equivalent* if $\exists$ can play so that each play $(\bar{a}, \bar{b})$ is a win for $\exists$.

Theorem 1.15 *(Theorem 3.2.3 of [23]) If $A, B \in \mathcal{K}(L)$ are countable and are back-and-forth equivalent, then $A \cong B$.*

Theorem 1.16 (Karp) *(Theorem 3.5.3 of [23]) $A, B \in \mathcal{K}(L)$ are back-and-forth equivalent if and only if $A \equiv_{\infty\omega} B$.*

Characterizations of $\forall_1$ classes

At this point, we dwell on several characterizations of $\forall_1$ classes (some only true for relational languages) that, while a part of folklore, merit some further attention.

The first of these is the characterization of $\forall_1$ classes as *constrained* classes; namely, classes determined by omitting a set of quantifier-free types.

Definition 1.17 *Let $A, B \in \mathcal{K}(L)$.*

1. *$A \hookrightarrow B$ if there is some embedding from A to B.*

2. *$A \sim B$ if $A \hookrightarrow B$ and $B \hookrightarrow A$.*

Remark 1.18

1. The relation $\hookrightarrow \subseteq \mathcal{K}(L) \times \mathcal{K}(L)$ is a pre-order; namely, it is a reflexive and transitive binary relation.

2. $\sim$ is an equivalence relation; the $\sim$-equivalence classes of $\mathcal{K}(L)_{fin}$ are isomorphism classes of finite $\mathcal{K}(L)$-structures. We will identify $\mathcal{K}(L)_{fin}$ with the

isomorphism classes of $\mathcal{K}(L)_{fin}$. With this identification, $\mathcal{K}(L)_{fin}$ equipped with $\hookrightarrow$ becomes an order.

Definition 1.19 *Let $\mathcal{C} \subseteq \mathcal{K}(L)_{fin}$. Define*

$$\mathcal{K}(\neg\mathcal{C}) = \{B \in \mathcal{K} : C \not\hookrightarrow B, \text{ for each } C \in \mathcal{C}\}.$$

*The members of $\mathcal{C}$ are called **(negative) constraints** of $\mathcal{K}(\neg\mathcal{C})$.*

Our first characterization of $\forall_1$ classes is probably due to Mal'cev and (independently) Tarski.

Theorem 1.20 *Let $\mathcal{K} \subseteq \mathcal{K}(L)$. The following are equivalent.*

1. *$\mathcal{K}$ is $\forall_1$.*
2. *For some $\hookrightarrow$-antichain $\mathcal{C} \subseteq \mathcal{K}(L)_{fin}$, $\mathcal{K} = \mathcal{K}(\neg\mathcal{C})$.*

Proof. $(1 \Rightarrow 2)$ Assume that $\mathcal{K}$ is axiomatized by $\forall_1$ sentences T. Let

$$\mathcal{C}' = \{A \in \mathcal{K}(L)_{fin} : A \not\models \phi, \text{ for some } \phi \in T\}.$$

It suffices to choose $\mathcal{C}$ to be the minimal (with respect to $\hookrightarrow$) members of $\mathcal{C}'$.

$(2 \Rightarrow 1)$ Let

$$T = \{\phi : \phi = \neg\exists\bar{x} qft_{C,\bar{c}}(\bar{x}), \text{ where } C \in \mathcal{C}, \ \bar{c} \text{ enumerates } C \text{ and } |\bar{x}| = |\bar{c}|\}.$$

Then T axiomatizes $\mathcal{K}$. □

Example 1.21 Let $L = \{E\}$, where E is binary.

1. $\mathcal{K}(L) = \mathcal{K}(\neg\emptyset)$.

2. The class of directed graphs $\mathcal{D}$ is axiomatized by $\forall x(\neg xEx)$. $\mathcal{D} = \mathcal{K}(\neg\mathcal{A}_1)$, where the element of $\mathcal{A}_1$ is depicted in Figure 1.3.

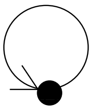

Figure 1.3: Digraph constraint, $\mathcal{A}_1$.

3. The class of graphs $\mathcal{G}$ is axiomatized by $\forall x(\neg xEx), \forall xy(xEy \to yEx)$. $\mathcal{G} = \mathcal{K}(\neg\mathcal{A}_2)$, where the elements of $\mathcal{A}_2$ are depicted in Figure 1.4.

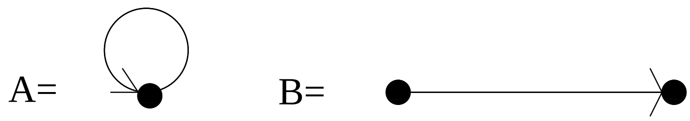

Figure 1.4: Graph constraints, $\mathcal{A}_2$.

4. The class of oriented (or asymmetric) graphs $\mathcal{O}$ is axiomatized by $\forall x(\neg xEx)$, $\forall xy(xEy \to \neg yEx)$. $\mathcal{O} = \mathcal{K}(\neg\mathcal{A}_3)$, where the elements of $\mathcal{A}_3$ are as in Figure 1.5.

5. The class of orders $\mathcal{P}$ is axiomatized by $\forall x(xEx)$, $\forall xy(xEy \wedge yEx \to x = y)$, $\forall xyz(xEy \wedge yEz \to xEz)$. $P = \mathcal{K}(\neg\mathcal{A}_4)$, where the elements of $\mathcal{A}_4$ are depicted in Figure 1.6.

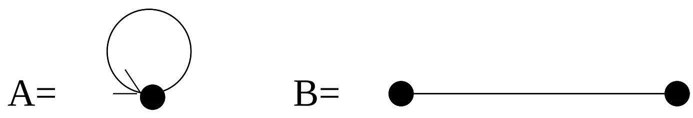

Figure 1.5: Oriented graph constraints, $\mathcal{A}_3$.

6. The class of tournaments is axiomatized by $\forall x(\neg xEx)$, $\forall xy(xEy \rightarrow \neg yEx)$, and $\forall xy((\neg x = y) \rightarrow xEy \vee yEx)$. $\mathcal{O} = \mathcal{K}(\neg\mathcal{A}_5)$, where the elements of $\mathcal{A}_5$ are depicted in Figure 1.7.

The second characterization of $\forall_1$ classes we describe is due to Łoś and Tarski. The definitions and basic facts about ultraproducts may be found in Ch. 9 of [23].

Definition 1.22 *Let $\mathcal{K} \subseteq \mathcal{K}(L)$.*

1. *$\mathcal{K}$ is **closed under** I if for every $A \in \mathcal{K}$ every isomorphic copy of A is again in $\mathcal{K}$.*

2. *$I(\mathcal{K})$ is the class of isomorphic copies of elements of $\mathcal{K}$.*

3. *$\mathcal{K}$ is **closed under** S if for every $A \in \mathcal{K}$ and every $B \leq A$, $B \in \mathcal{K}$.*

4. *$S(\mathcal{K})$ is the class of substructures of elements of $\mathcal{K}$.*

5. *$\mathcal{K}$ is **closed under** P_u if K is closed under the taking of ultraproducts. See Chapter 9 of [23] for more on ultraproducts.*

6. *$\mathcal{K}$ is of **finite character** if $A \in \mathcal{K}$ if and only if for all finite $B \leq A$ we have that $B \in \mathcal{K}$.*

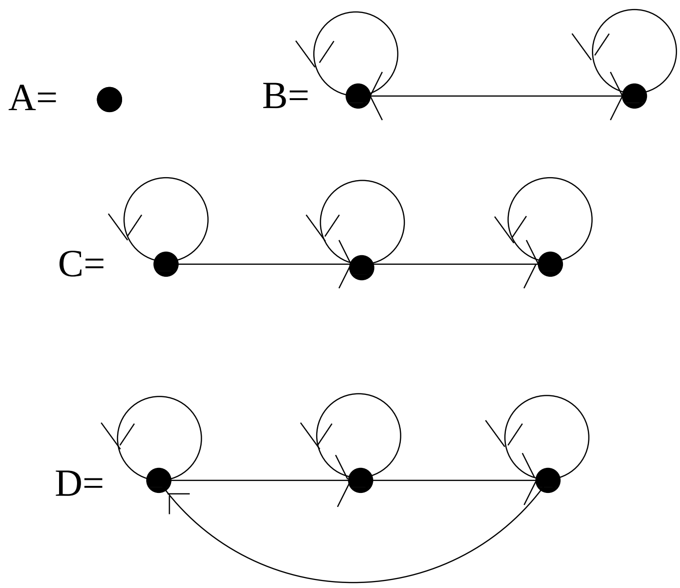

Figure 1.6: Order constraints, $\mathcal{A}_4$.

Unions of chains of structures are defined in Ch. 2 of [23].

Theorem 1.23 *Let $\mathcal{K} \subseteq \mathcal{K}(L)$. The following are equivalent.*

1. *$\mathcal{K}$ is $\forall_1$.*

2. *$\mathcal{K}$ is closed under I, S, and P_u.*

3. *$\mathcal{K}$ is closed under I, S, and unions of chains.*

4. *$\mathcal{K}$ is closed under I, S, and has finite character.*

PROOF. See Theorem 25.2 and 25.11 of [32]. □

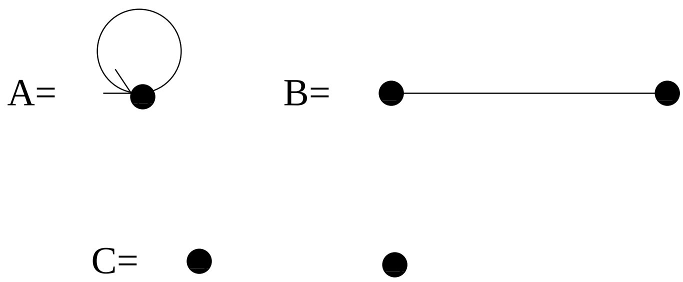

Figure 1.7: Tournament constraints, $\mathcal{A}_5$.

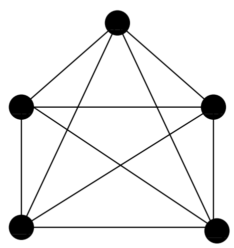

Figure 1.8: The complete graph on 5 *vertices,* K_5.

Example 1.24 Let $n > 1$. Define the graph K_n to be the complete graph with n vertices; that is, for $x \neq y$ in K_n, $xE^{K_n}y$.

Define the class $C(K_n)$ to be the class of all n-colourable graphs; that is, the class of graphs admitting a homomorphism into K_n. $C(K_n)$ is closed under I and S. A simple argument with the Compactness theorem (Theorem 1.10 above) shows that $C(K_n)$ has finite character. Hence, by Theorem 1.23, $C(K_n)$ is $\forall_1$; in particular, $C(K_n)$ may be realized as a class of graphs defined by excluding a class of finite graphs.

In the case $n = 2$, the constraints are well-known; namely, a graph is 2-colourable if and only if it has no induced subgraph isomorphic to an odd cycle.

The constraints in the general case are given by a theorem of Hajós (see [16]). We state this here.

A graph is Hajós-n-constructible if it can be obtained from the complete graph K_n by repeated applications of the following two operations. Recall that a set of vertices V in a graph G is *independent* if there are no edges in G between distinct elements of V.

1. *Hajós construction*: Let G_1 and G_2 be already obtained disjoint graphs with edges ab and cd. Remove ab and cd, identify a and c and add the edge bd.
2. *Identify sets of independent vertices.* If G_1 and G_2 are already obtained disjoint graphs with $S_1 \subseteq G_1$ an independent set of vertices listed as $\{a_1, \ldots, a_n\}$, and $S_2 \subseteq G_2$ an independent set of vertices listed as $\{b_1, \ldots, b_n\}$, form the graph G_3 with vertex set $G_1 \cup (G_2 - S_2)$ and edges defined as follows: $aE^{S_3}b$ if and only if
   (a) $a, b \in G_1$ and $aE^{G_1}b$;
   (b) $a, b \in G_2 - S_2$ and $aE^{G_2}b$;
   (c) there is some $i \in \{1, \ldots, n\}$ so that $a = a_i$ and $b \in G_2 - S_2$ and $b_i E^{G_2} b$.

Theorem 1.25 (Hajós) $G \in C(K_n)$ iff G contains no Hajós-$n+1$-constructible graph as an induced subgraph.

A problem that appears to be open is how Hajós' Theorem can be generalized (if at all) to the colour classes $C(G)$, for G a finite core graph (see Definition 3.6 below).

Definition 1.26 *Let $L \subseteq L'$ be first-order languages.*

1. *The **reduct** of $A \in \mathcal{K}(L')$ to L is the L-structure formed by forgetting the symbols in L' that are not in L, and is written $red_L(A)$.*

2. *If $\mathcal{K}$ is an L'-class, then $red_L(\mathcal{K}) = \{B \in \mathcal{K}(L) : B = red_L(A)$, for some $A \in \mathcal{K}\}$.*

Theorem 1.27 *Let $L \subseteq L'$ be first-order languages. Assume $\mathcal{K} \subseteq \mathcal{K}(L')$ is elementary. If $red_L(\mathcal{K})$ is closed under S, then $red_L(\mathcal{K})$ is $\forall_1$.*

PROOF. The proof is completed by showing that $red_L(\mathcal{K})$ is closed under P_u. □

Example 1.28 Theorem 1.27 yields another proof that $C(K_n)$ is $\forall_1$ for $n \geq 2$. Let $L' = \{E\} \cup \{P_1, \ldots, P_n\}$, where the P_i are 1-ary, for $1 \leq i \leq n$.

Define the L' class $C(K_n)'$ to be the $\forall_1$ class defined by:

$$\forall x(\bigvee_{i=1}^{n} P_i(x))$$

$$\forall x(\bigwedge_{1 \leq i < j \leq n} \neg(P_i(x) \wedge P_j(x)))$$

$$\forall xy((P_i(x) \wedge P_i(y)) \rightarrow \neg Exy).$$

$C(K_n)'$ is the class of K_n-**coloured** graphs. $C(K_n)$ is precisely the class of $\{E\}$-reducts of $C(K_n)'$.

1.2 Existentially closed structures and model companions in $\forall_1$ classes

The notions of existentially closed structures and model companions play a prominent role in the sequel. For this reason, we outline some of the essential facts about them.

1.2.1 Definitions and folklore

Definition 1.29 *Let $\mathcal{K}$ be an $\forall_2$ class of L-structures.*

1. *$A \in \mathcal{K}$ is **existentially closed in** $\mathcal{K}$ if for every B in $\mathcal{K}$, if $A \leq B$ and for $\bar{a} \subseteq A$ a finite tuple, $\theta(\bar{x}, \bar{y})$ a quantifier-free L-formula with $|\bar{y}| = |\bar{a}|$, if B satisfies $\exists \bar{x} \theta(\bar{x}, \bar{a})$ then so does A.*

   *A is said to be **e.c. in** $\mathcal{K}$.*

2. *$\mathcal{K}^{ec} = \{A \in \mathcal{K} : A$ is e.c. in $\mathcal{K}\}$.*

3. *$\mathcal{K}$ is **mutually model consistent** with a class $\mathcal{K}_1 \subseteq \mathcal{K}$ if $\mathcal{K} \subseteq IS(\mathcal{K}_1)$; that is, every member of $\mathcal{K}$ is isomorphic to a substructure of a member of $\mathcal{K}_1$.*

4. *An elementary class $\mathcal{K}_1$ is **model complete** if $\mathcal{K}_1 = \mathcal{K}_1^{ec}$.*

5. *$\mathcal{K}$ has a **model companion** if there is a model complete mutually model consistent elementary class $\mathcal{K}_1 \subseteq \mathcal{K}$. We write $\mathcal{K}_1 = \mathcal{K}^{mc}$.*

6. *An $\forall_2$ L-theory T has a **model companion** if the class of models of T has a model companion.*

Remark 1.30 An equivalent definition that $\mathcal{K}$ is model complete is that for every pair of $\mathcal{K}$-structures A, B, if $A \leq B$ then $A \preceq B$ (recall Definition 1.11). See Theorem 8.3.1 of [23].

In practice we will usually check that an $\forall_2$ class has a model companion by tacitly appealing to the following crucial theorem, that was first proven in [13].

Theorem 1.31 *Let $\mathcal{K}$ be an $\forall_2$ class of L-structures. $\mathcal{K}$ has a model companion if and only if $\mathcal{K}^{ec}$ is elementary.*

We collect some more folklore about existentially closed structures and model companions in the following lemma, the proof of each part of which the reader can find in Chapter 8 of [23].

Lemma 1.32 *Let $\mathcal{K}$ be an $\forall_2$ class of L-structures.*

1. *A is e.c. in $\mathcal{K}$ if and only if it is e.c. in $IS(\mathcal{K})$.*
2. *$\mathcal{K}^{ec}$ is closed under elementary substructures.*
3. *$\mathcal{K}$ is mutually model consistent with $\mathcal{K}^{ec}$; in particular, $\mathcal{K}^{ec} \neq \emptyset$. A fortiori, if $A \in IS(\mathcal{K})$ is infinite, then there is a $B \in (\mathcal{K}^{ec})_{\max(|L|,|A|)}$ so that A embeds in B.*

1.2.2 Saturated models

We will have occasion to use countably saturated models in the sequel. For more details the reader should see Chapter 10 of [23] or Chapter 2 of [5].

Definition 1.33 *1. Let $A \in \mathcal{K}(L)$ and $B \subseteq A$ be a subset of A. If we add constants $\{b : b \in B\}$ to L naming the elements of B in A, the resulting structure over the expanded language is written $(A, b)_{b \in B}$.*

2. *Let $A \in \mathcal{K}(L)$, and let $\bar{a}$ be a finite tuple from A. The **type of** $\bar{a}$, written $tp_A(\bar{a})$, is the set of all L-formulas $\theta(\bar{x})$ so that $A \models \theta(\bar{a})$ (up to equivalence). If $B \subseteq A$, $tp_A(\bar{a}/B)$ is the type of $\bar{a}$ in $(A, b)_{b \in B}$.*

3. *Let T be a complete L-theory. A **type over** T is a set of formulas that can be realized as $tp_A(\bar{a})$ for some finite tuple $\bar{a}$ from $A \models T$. A **type over** A is a type over $Th(A)$.*

4. *A type $p(\bar{x})$ over A is **realized** in A if there is some finite tuple $\bar{a}$ from A so that $p(\bar{x}) = tp_A(\bar{a})$.*

5. *A countable $A \in \mathcal{K}(L)$ is **countably-saturated** if for every $B \subseteq A$ with $|B| < \aleph_0$, $(A, b)_{b \in B}$ realizes every type over $(A, b)_{b \in B}$.*

Remark 1.34 A countably saturated model of a complete theory T is e.c. in the class of models of T.

A complete theory may not have a countably saturated model. However, there is a simple criterion which ensures the existence of a countably saturated model.

Proposition 1.35 *A complete theory T with only countably many countable models has a countably saturated model.*

Proof. A countable model of T realizes only countably many distinct types. Now use Proposition 2.2.1 of [5]. □

1.2.3 A problem

We are primarily interested in the case when $\mathcal{K}$ is an $\forall_1$ class. We pose the following question, which is really a research program.

Definition 1.36 *Let L be a relational language. Define*

$$MC(L) = \{\mathcal{K} : \mathcal{K} \subseteq \mathcal{K}(L) \textit{ is } \forall_1 \textit{ and } \mathcal{K}^{mc} \textit{exists}\}.$$

Model Companion Problem: Given L a relational language, classify those $\forall_1$ L-classes that are in $MC(L)$.

The model companion problem is prompted by how little understood $MC(L)$ is in general. It is quite possible there is no real "dividing line" between $\forall_1$ classes with and without a model companion.

1.2.4 Results on companionability

While $\mathcal{K}^{ec}$ may fail to have a first-order axiomatization, it always is infinitarily axiomatizable.

Theorem 1.37 (Simmons) *[39] Let $\mathcal{K} \subseteq \mathcal{K}(L)$ be an $\forall_2$ class where L is countable. Then $\mathcal{K}^{ec}$ is axiomatizable in $L_{\omega_1\omega}$.*

Proof. Let T axiomatize $\mathcal{K}$. Let $T_\forall$ be the set of $\forall_1$ consequences of T. For each $\forall_1$ formula $\theta(\bar{x})$ over L, let

$$S_\theta = \{\varphi(\bar{x}) : \varphi(\bar{x}) \text{ is } \exists_1 \text{ and } T_\forall \vdash \forall \bar{x}(\varphi(\bar{x}) \rightarrow \theta(\bar{x}))\}.$$

Let Φ_θ be the infinitary sentence

$$\forall \bar{x}(\theta(\bar{x}) \leftrightarrow \bigvee_{\varphi \in S_\theta} \varphi(\bar{x}));$$

as $|L| \leq \aleph_0$, $\Phi_\theta \in L_{\omega_1\omega}$.

Let $T' = T_\forall \cup \{\Phi_\theta : \theta \text{ is an } \forall_1\ L\text{-formula}\}$.

It can be shown that T' is an axiomatization of $\mathcal{K}^{ec}$; see *[39]*. □

A survey of the literature reveals few model companion existence theorems of a general nature. We describe some of what is known next.

Definition 1.38 *Let $\mathcal{K} \subseteq \mathcal{K}(L)$. $\mathcal{K}$ satisfies the* ***amalgamation property (AP)*** *if for all A, B, C in $\mathcal{K}$, and embeddings $f : A \to B$, $g : A \to C$, there is a D in $\mathcal{K}$ and embeddings $h : B \to D$ and $j : C \to D$ so that $hf = jg$. D is called an* ***amalgam*** *of B and C over A (relative to f and g).*

We assume throughout that L is countable.

Theorem 1.39 (Lipparini) *[29] Let $\mathcal{K} \subseteq \mathcal{K}(L)$ be an $\forall_1$ class satisfying AP. Then $\mathcal{K}^{mc}$ exists.*

Definition 1.40 *$\mathcal{K}$ is* ***finitely generated universal Horn*** *if there is a finite set $\mathcal{C}$ of finite $\mathcal{K}$-structures so that $\mathcal{K}$ is the class of isomorphic images of substructures of products of elements of $\mathcal{C}$.*

Theorem 1.41 (Burris) *[6] Let $\mathcal{K}$ be a finitely generated universal Horn class. Then $\mathcal{K}^{mc}$ exists.*

1.2.5 Examples and non-examples

We close this section with a few examples of companionable and non-companionable classes of relational structures. We also include a few classes where the model companion problem remains unsettled.

Example 1.42 *Classes with a model companion by Theorem 1.39.*

1. Sets.

2. Orders, linear orders, equivalence relations.

3. Directed graphs, graphs, oriented graphs, tournaments, K_n-free graphs.

Example 1.43 *Classes with a model companion by Theorem 1.41.* n-colourable graphs, for $n \geq 2$ (first proven by [40] without Theorem 1.41).

Example 1.44 *Sporadic examples (satisfying neither the hypotheses of Theorem 1.39 nor 1.41).* Width-two orders [4]; trees (as ordered sets) [35]; N-free graphs [11].

Example 1.45 *An example of an $\forall_1$ class without a model companion.*

Let $\mathcal{K}$ be the class of acyclic graphs; that is, graphs excluding each finite cycle. Each $A \in \mathcal{K}^{ec}$ is connected. A simple application of the Compactness theorem shows that $\mathcal{K}^{ec}$ cannot be elementary.

Problem: Do the classes of width n-orders, for $n > 2$, have model companions?

The same question for graphs omitting a single cycle of length > 3.

1.3 Amalgamation classes and Fraïssé's theorem

One of the most beautiful theorems in model theory is Fraïssé's theorem. At its heart, the theorem is an existence and uniqueness theorem, which, given as input an amalgamation class $\mathcal{K}$ of finite structures, produces an $\aleph_0$-categorical homogeneous structure $F(\mathcal{K})$ whose age is $\mathcal{K}$. When $\mathcal{K}$ is the class of finite members of an $\forall_1$ theory, $F(\mathcal{K})$ is the unique countable model of the model companion of $\mathcal{K}$.

1.3.1 Definitions, results, and examples

Throughout, we will assume that L is finite unless otherwise stated.

Definition 1.46 *Let $A \in \mathcal{K}(L)$ be countable.*

1. *The **age of** A is the class of L-structures which are isomorphic to some finite substructure of A, and is written $Age(A)$.*
2. *A class of finite structures $\mathcal{K}$ is an **age** if it is the age of some countable L-structure.*

Remark 1.47 Our assumption that L is finite implies that an age has only countably many non-isomorphic members.

Definition 1.48 *Let $\mathcal{K}$ be a class of L-structures. Recall that we assume that $\mathcal{K}$ is closed under isomorphism.*

1. *$\mathcal{K}$ satisfies the **hereditary property (HP)** or is **downward closed** if $A \leq B \in \mathcal{K}$ implies that $A \in \mathcal{K}$.*

2. *$\mathcal{K}$ satisfies the* ***joint embedding property (JEP)*** *or is* ***upward directed*** *if for all A, B in $\mathcal{K}$, there is a C in $\mathcal{K}$ and embeddings $f : A \to C$ and $g : B \to C$.*

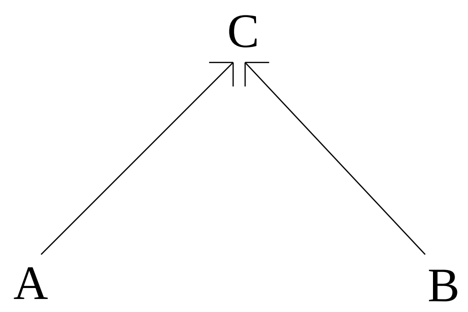

Figure 1.9: The joint embedding property (JEP).

3. *$\mathcal{K}$ satisfies the* ***amalgamation property (AP)*** *if for all A, B, C in $\mathcal{K}$, and embeddings $f : A \to B$, $g : A \to C$, there is a D in $\mathcal{K}$ and embeddings $h : B \to D$ and $j : C \to D$ so that $hf = jg$. D is called an* ***amalgam*** *of B, C over A (relative to f and g).*

4. *$\mathcal{K}$ is an* ***amalgamation class*** *if it satisfies HP, JEP, and AP.*

Remark 1.49 *1.* If $\mathcal{K} \subseteq \mathcal{K}(L)_{fin}$ satisfies HP and JEP, then $\mathcal{K}$ is an age (see Theorem 7.1.1 of [23]).

2. An $\forall_1$ class $\mathcal{K}$ of L-structures (with L finite) is an amalgamation class if and only if $\mathcal{K}_{fin}$ satisfies JEP and AP. The proof of the nontrivial direction follows from the Compactness theorem.

Definition 1.50 *Let $A \in \mathcal{K}(L)$ be countable.*

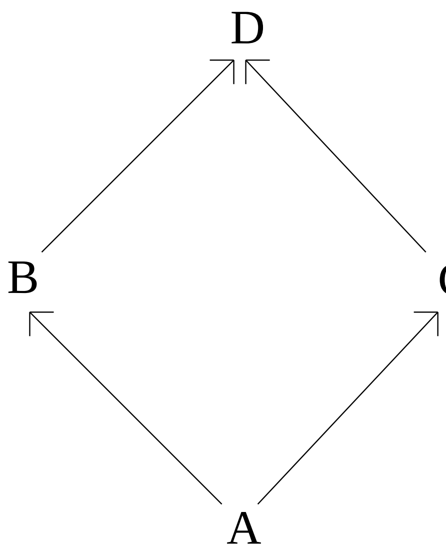

Figure 1.10: The amalgamation property (AP).

1. *A is* ***homogeneous*** *if every isomorphism between finite substructures extends to an automorphism of* A.

2. *A is* ***weakly-homogeneous****, if for all* $C, D \in Age(A)$ *with* $C \leq D$*, if* $f : C \to A$ *is an embedding then there is an embedding* $g : D \to A$ *which extends* f.

Remark 1.51 Homogeneity is equivalent to weak-homogeneity (by a standard back-and-forth argument; see Chapter 7 of *[23]*).

Remark 1.52 A is weakly-homogeneous if and only if C and D in Definition 1.50 may be chosen with $|D| = |C| + 1$.

The following theorem was first proven by Fraïssé in [15]. We assume here that L is at most countable. We omit the proof, which will follow as a corollary of

Theorem 1.63 below.

Theorem 1.53 *Let $\mathcal{K}$ be a class of finite L-structures. The following are equivalent.*

1. *$\mathcal{K}$ is an amalgamation class.*

2. *There is a unique (up to isomorphism) countable homogeneous L-structure $F(\mathcal{K})$ (the* ***Fraïssé limit*** *of $\mathcal{K}$) with age $\mathcal{K}$.*

PROOF. See Theorems 7.1.2 and 7.1.7 of [23]. □

Example 1.54 *1.* The class of finite graphs is an amalgamation class. $F(\mathcal{K})$ is called the random graph, R (see Chapter 7.4 of [23]).

2. Let L be a finite relational language, then the class of finite L-structures is an amalgamation class. The Fraïssé limit is the random L-structure (see Chapter 7.4 of [23]).

1.3.2 Further results

We now summarize some of the model theoretic properties of the Fraïssé limit.

Definition 1.55 *Let $\mathcal{K}$ be an amalgamation class of finite structures over a finite language L.*

1. *Let Θ consist of the $\exists_1$ L-sentences:*

$$\exists \bar{x} qft_{A,\bar{a}}(\bar{x}),$$

where $|\bar{x}| = |\bar{a}| = n$, $A \in \mathcal{K}_n$, and $\bar{a}$ enumerates A, for each $n \geq 1$.

2. *Define* $\Sigma_1 = \Theta \cup \{\varphi_n : n \in \omega^*\}$, *where the* φ_n *is the* $\forall_2$ L*-sentence:*

$$\bigwedge \forall \bar{x} \exists y (qft_{A,\bar{a}}(\bar{x}) \rightarrow qft_{B,\bar{a}b}(\bar{x}y)),$$

where the conjunction ranges over all sets of the form $(A, B, \bar{a}, b)$ *where*

   (a) A, B *are isomorphism types of structures in* $\mathcal{K}$ *with* $A \leq B$, $|A| \leq n$, $|B| \leq n + 1$;

   (b) $\bar{a}$ *is a set of distinct elements from* A *so that* $A = \bar{a}$ *and* $|\bar{x}| = |\bar{a}|$;

   (c) $b \in B$ *is such that* $B = \bar{a}b$.

Σ_1 *is the set of* $\mathcal{K}$-***extension axioms***.

3. *Define* $\Sigma_2 = \{\psi_n : n \in \omega^*\}$, *where the* ψ_n *is the* $\forall_1$ L*-sentence:*

$$\forall \bar{x} \bigvee qft_{A,\bar{a}}(\bar{x})$$

where the disjunction ranges over all sets of the form $(A, \bar{a})$ *where*

   (a) A *is an isomorphism type of a structure in* $\mathcal{K}$ *with* $|A| = n$;

   (b) $\bar{a}$ *is a tuple from* A *with* $A = \bar{a}$, *and* $|\bar{x}| = |\bar{a}| = n$.

4. *Define* $\Sigma = \Sigma_1 \bigcup \Sigma_2$.

Remark 1.56 *Let* $\mathcal{K}$ *be an amalgamation class of finite structures over a finite language* L.

1. *The reader may verify that each of φ_n and ψ_n, for $n \in \omega^*$, are indeed first-order sentences.*

2. *If $S \in \mathcal{K}(L)$ satisfies Σ_1 then S satisfies the following "algebraic" condition: if $A \leq S$ with $A \in \mathcal{K}$ and $A \leq B \in \mathcal{K}$ with $|B| = |A| + 1$, then there is a $C \leq S$ with $C \in \mathcal{K}$ and an isomorphism $f : B \to C$ so that $f \restriction A$ is the identity map.*

3. *If $S \in \mathcal{K}(L)$ satisfies Σ_2 then S satisfies the following condition: each finite substructure of S is in $\mathcal{K}$.*

Theorem 1.57 *Let $\mathcal{K}$ be an amalgamation class of finite structures over a finite language L. Then $F(\mathcal{K})$ is $\aleph_0$-categorical.*

The theory of $F(\mathcal{K})$ is axiomatized by Σ.

Proof. By definition, $F(\mathcal{K})$ satisfies Σ. A back-and-forth argument demonstrates that a countable model of Σ is isomorphic to $F(\mathcal{K})$. See Theorem 7.4.1 of [23]. □

The next theorem completely resolves the model companion problem for $\forall_1$ amalgamation classes.

Theorem 1.58 (Lipparini) *[29] Let $\mathcal{K}$ be an $\forall_1$ amalgamation class over a finite language. Then $\mathcal{K}^{mc}$ exists and $Th(\mathcal{K}^{mc}) = Th(F(\mathcal{K}_{fin}))$. In particular, the model companion is complete and $\aleph_0$-categorical.*

Proof. By Theorem 1.57 $F(\mathcal{K}_{fin})$ has an $\forall_2$ axiomatization Σ. $F(\mathcal{K}_{fin})$ embeds every countable $\mathcal{K}$-structure (see Lemma 7.1.3 of [23]).

Let $A \in \mathcal{K}^{ec}$, and let $B \preceq A$ be countable (using Theorem 1.12); then B itself is e.c. by Lemma 1.32. As Σ is $\forall_2$ and B embeds in $F(\mathcal{K}_{fin})$, $B \models \Sigma$. Hence, every e.c. must satisfy Σ.

We leave it to the reader to verify that $F(\mathcal{K}_{fin})$ is e.c. in $\mathcal{K}$. If $A \models \Sigma$, then A is back-and-forth equivalent to $F(\mathcal{K}_{fin})$; hence, $A \equiv_{\infty\omega} F(\mathcal{K}_{fin})$ and so A must be e.c. by Theorem 1.37. $\square$

Remark 1.59 If $\mathcal{K}$ is an $\forall_1$ amalgamation class over a finite language, then Σ_2 is a set of axioms for $\mathcal{K}$. So $\Sigma_1 \cup Th(\mathcal{K})$ axiomatizes $Th(\mathcal{K}^{mc})$.

Example 1.60 By Theorem 1.57 the random graph R is the unique countable graph satisfying Σ. In the class of graphs, Σ is equivalent to the following property (see Theorem 7.4.4 of [23]):

($\clubsuit$) For every $n, m \geq 1$, if $x_1, \ldots, x_n$ and $y_1, \ldots, y_m$ are distinct vertices of R, then there is a vertex $x \in R$ adjacent to the x_i and to none of the y_j.

The model companion of graphs is axiomatized by ($\clubsuit$) and the graph axioms.

1.4 Generic structures: an extension of Fraïssé's theorem

One of the main tools we will use in the sequel is the notion of a generic structure. The setting is similar to that of Theorem 1.53, but we no longer demand either that our class satisfy HP or that we work with the usual embeddings of structures. The result is a construction that produces many new examples; however, in contrast to Theorem 1.57 the examples constructed need not have an $\aleph_0$-categorical theory.

1.4.1 Definitions and results

We follow the exposition of [26], which also contains historical remarks. Throughout, L is a countable relational language. Whenever we consider a class $\mathcal{K}$ of L-structures, recall that we assume that $\mathcal{K}$ is closed under isomorphism.

Definition 1.61 *1. A class $\mathcal{K}$ of finite L-structures together with a relation $\leq^*$ on $\mathcal{K} \times \mathcal{K}$ is called* ***smooth*** *if*

(a) $\leq^*$ is transitive;

(b) $A \leq^* B$ implies $A \leq B$;

(c) for each $A \in \mathcal{K}$, there is a collection of universal formulas $p^A(\bar{x})$ with $|\bar{x}| = |A|$ such that for every $B \in \mathcal{K}$ so that $A \leq B$, $A \leq^* B$ if and only if $B \models \phi(\bar{a})$ for any $\bar{a}$ enumerating A, for all $\phi \in p^A$;

(d) $p^B = p^C$ if $B \cong C$.

*2. Let $(\mathcal{K}, \leq^*)$ be smooth. An L-structure A is a $(\mathcal{K}, \leq^*)$-**union** if there are $A_i \in \mathcal{K}$ with $A = \bigcup_{i<\omega} A_i$ and $A_i \leq^* A_{i+1}$ for $i < \omega$.*

3. If A is a $(\mathcal{K}, \leq^*)$-union and $B \in \mathcal{K}$ with $B \leq A$, we write $B \leq^* A$ if $A \models \phi(\bar{a})$ for any $\bar{a}$ enumerating B, for all $\phi \in p^B$.

*4. Let $(\mathcal{K}, \leq^*)$ be smooth. An L-structure A is $(\mathcal{K}, \leq^*)$-**generic** if*

(a) *A is a $(\mathcal{K}, \leq^*)$-union.*

(b) *For each $B \in \mathcal{K}$, there is $B' \leq^* A$ with $B \cong B'$.*

(c) *If $B \leq^* A$, $B \leq^* C$ for $B, C \in \mathcal{K}$, there is $C' \leq^* A$ and an isomorphism $f : C \to C'$ so that $f \restriction B = id_B$.*

5. *Let $(\mathcal{K}, \leq^*)$ be smooth.*

   (a) *$(\mathcal{K}, \leq^*)$ satisfies **(JEP')** if for all A, B in $\mathcal{K}$, there is a C in $\mathcal{K}$ and embeddings $f : A \to C$ and $g : B \to C$ so that $f(A) \leq^* C$, and $g(B) \leq^* C$.*

   (b) *$(\mathcal{K}, \leq^*)$ satisfies **(AP')** if for all A, B, C in $\mathcal{K}$ with $A \leq^* B$ and $A \leq^* C$ there is a D in $\mathcal{K}$ and embeddings $f : B \to D$ and $g : C \to D$ so that $f(B) \leq^* D$, and $g(C) \leq^* D$ with $f \restriction A = g \restriction A$.*

Remark 1.62 If $\mathcal{K}$ is a class of finite structures with $\leq^*$ simply the substructure relation, then $(\mathcal{K}, \leq^*)$ is smooth: for each $B \in \mathcal{K}$, p^B is the quantifier-free diagram of any $\bar{b}$ enumerating B.

The following theorem (Theorem 1.5 of [26]) is a useful analogue of Fraïssé's construction when, for example, the class under consideration is not closed under substructures.

Theorem 1.63 *Let $(\mathcal{K}, \leq^*)$ be smooth.*

1. *There is a $(\mathcal{K}, \leq^*)$-generic if and only if $\mathcal{K}$ contains only countably many isomorphism types and $(\mathcal{K}, \leq^*)$ satisfies JEP' and AP'.*

2. *Any two $(\mathcal{K}, \leq^*)$-generics are isomorphic.*

Example 1.64 Let $\mathcal{K}$ be the class of finite trees; that is, finite connected acyclic graphs. With $\leq^*=\leq$, $\mathcal{K}$ satisfies JEP' and AP', and hence, there is a $(\mathcal{K}, \leq^*)$-generic M, the $\aleph_0$-regular tree. Note that the class of trees is not elementary and does not have HP.

1.4.2 Generics when $\leq^*=\leq$

In this section, we assume that $\leq^*=\leq$, so that JEP'=JEP and AP'=AP.

Definition 1.65 *Let $\mathcal{K}_1 \subseteq \mathcal{K}(L)_{fin}$, $\mathcal{K}_2 \subseteq \mathcal{K}(L)$.*

1. *$\mathcal{K}_1$ is **cofinal** in $\mathcal{K}_2$ if for every finite $B \in \mathcal{K}_2$ there is a $C \in \mathcal{K}_1$ so that $B \hookrightarrow C$.*

2. *$\mathcal{K}_1$ is **large** in $\mathcal{K}_2$ if for every $B \in (\mathcal{K}_2)_{\aleph_0}$, B embeds in a union of a chain of $\mathcal{K}_1$-structures.*

Lemma 1.66 *(Lemma 2.4 of [26]) Let $\mathcal{K}_2$ be an $\forall_1$ class over a finite language, and suppose $\mathcal{K}_1 \subseteq (\mathcal{K}_2)_{fin}$ is cofinal in $(\mathcal{K}_2)_{fin}$. If A is a $\mathcal{K}_1$-generic, then $A \in \mathcal{K}_2^{ec}$.*

Proposition 1.67 *(Theorem 2.5 of [26]) Let $\mathcal{K} \subseteq \mathcal{K}(L)_{fin}$. Assume A is $\mathcal{K}$-generic and $T = Th(A)$. The following are equivalent.*

1. *A is a countably saturated model of T.*

2. *$\mathcal{K}$is a large subclass of the models of T and T is model complete.*

1.4.3 $\aleph_0$-categorical generics

We now supply a useful criterion to determine when (in certain cases) the generic is $\aleph_0$-categorical. Note that we drop the assumption of the previous Section that $\leq^* = \leq$, but assume that L is finite.

Theorem 1.68 *(Theorem 3.5 of [26]) Suppose that L is finite, and that $(\mathcal{K}, \leq^*)$ is smooth and satisfies JEP' and AP'. Let A be the $(\mathcal{K}, \leq^*)$-generic. If for each $B \in \mathcal{K}$, p^B consists of a single $\forall_1$ formula, then the following are equivalent.*

(a) $Th(A)$ is $\aleph_0$-categorical.

(b) A is a countably saturated model of $Th(A)$.

(c) (uniform boundedness) There is a function $f : \omega \to \omega$ so that if $S \leq A$ with S finite, there is $B \in \mathcal{K}$ with $S \leq B$, and $B \leq^* A$ with $|B| \leq f(|S|)$.

Remark 1.69 If $\mathcal{K}$ is $\forall_1$ and satisfies JEP and AP, Theorem 1.68 generalizes Theorem 1.57, with $\leq^* = \subseteq$.

Example 1.70 Continuing Example 1.64, by Theorem 1.68, M is not $\aleph_0$-categorical as uniform boundedness fails. To see this, consider B to be a finite path in M, S the leaves (vertices of degree 1) of B. Then the only possible extension of S to a subtree of M is B.

Chapter 2

Free Amalgamation in $\forall_1$ classes

In this Chapter, we will investigate the properties of $\forall_1$ free amalgamation classes. As we will see in Sections 2.2.2 and 2.2.3, if an $\forall_1$ class is a free amalgamation class and has edges (see Definition 2.25) then it has a "highly" non-finitely axiomatizable model companion: more precisely, the model companion is non-finitely axiomatizable modulo axioms asserting "I embed all finite structures in the class"; see Theorem 2.27 below. The results presented here set the stage for more delicate constructions in later chapters. The simplest and most direct applications of these results are to the class of graphs. For this reason, to aid the reader we suggest viewing the general results presented here specialized to the graph case.

2.1 Characterizations of free amalgamation classes: structural and syntactic

In this section, we characterize $\forall_1$ free amalgamation classes (over a finite relational language) by:

1. structural properties of their constraints; in particular, a class $\mathcal{K}$ is a free amalgamation class if and only if each of the constraints has a complete graph (Proposition 2.10 below).

2. syntactic properties of their axiomatization: by virtue of item (1), we prove a preservation theorem on classes over finite relational language that are closed under I, S, and unions (Proposition 2.18 below).

Both of these characterizations, while not essential to the understanding of the results on model companions of free amalgamation classes (Section 2.2) are interesting in their own right.

2.1.1 Unions of structures and the FAP

In this section, we develop machinery to work with free amalgams.

Definition 2.1 *Let A, B be L-structures, so that A and B* ***agree****, that is,*

$$A \restriction A \cap B = B \restriction A \cap B$$

if $A \cap B \neq \emptyset$. The ***union*** *of A and B, $A \cup B$, is the L-structure with universe*

$A \cup B$, and with relations $R^{A\cup B} = R^A \cup R^B$, for $R \in L$. We also call $A \cup B$ the ***free amalgam*** *of A and B over $A \cap B$.*

Remark 2.2 *1.* Let A,B,C be L-structures so that $C \leq A$ and $C \leq B$. By taking isomorphic copies we can assume that $A \cap B = C$. In this way, it makes sense to discuss the *free amalgam of A and B over C.*

2. Let $\mathcal{K} \subseteq \mathcal{K}(L)$. If $B, C \in \mathcal{K}$ agree, it does not necessarily follow that $B \cup C \in \mathcal{K}$. For example, let $\mathcal{K}$ be the class of orders, and let B and C be two-element chains that agree over the least element of B and the greatest element of C. However, $B \cup C$ fails to be transitive. See Figure 2.1.

Definition 2.3 *Let $\mathcal{K}$ be a class of L-structures.*

1. *$\mathcal{K}$ is* ***closed under disjoint union*** *if for $A, B \in \mathcal{K}$ with $A \cap B = \emptyset$,*

$$A \cup B \in \mathcal{K}.$$

In this case we write $A \uplus B$ for $A \cup B$.

2. *$\mathcal{K}$ has the* ***free amalgamation property (FAP)*** *if for A, $B \in \mathcal{K}$ with $A \cap B \neq \emptyset$ and so A and B agree then*

$$A \cup B \in \mathcal{K}.$$

3. *$\mathcal{K}$ is* ***closed under unions*** *or is a* ***free amalgamation class*** *if $\mathcal{K}$ is closed under disjoint union and has the free amalgamation property.*

Remark 2.4 As the reader can verify an $\forall_1$ class $\mathcal{K}$ has AP if and only if we can find an amalgam for any A, B, and C, as in Definition 1.48, with $A \leq B, C$ and $A = B \cap C$ and so that f, g are inclusion maps. In particular, every $\forall_1$ class with FAP has AP.

The converse is of course false. As we have seen in Remark 2.2 (2), orders do not have FAP, but they do have AP.

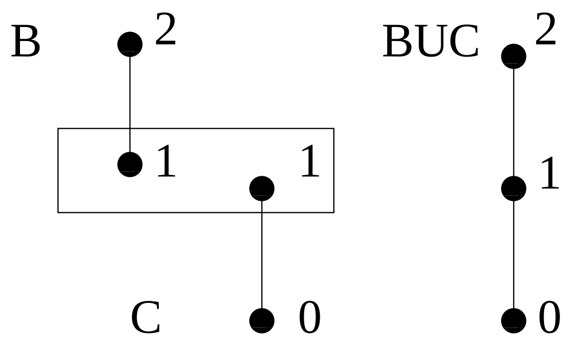

Figure 2.1: The unions of two chains may not be an order: $0 \not< 2$ in $B \cup C$.

Remark 2.5 Closure under disjoint union and FAP are independent notions. For example, the class of trees (connected acyclic graphs) has FAP but is not closed under disjoint union; the class of orders is closed under disjoint union but does not have FAP.

We now introduce the notion of unions of several structures.

Definition 2.6 *Let $\{A_i : 1 \leq i \leq n\} \subseteq K(L)$. For $n \geq 2$, we define $A = \bigcup_{i=1}^{n} A_i$ inductively.*

1. *For $n = 2$, if A_1 and A_2 agree, let $A = A_1 \cup A_2$.*

2. *For $k \geq 2$, assume $\bigcup_{i=1}^{k} A_i$ has been defined, so that $\bigcup_{i=1}^{k} A_i$ and A_{k+1} agree. Define*

$$A = \bigcup_{i=1}^{k+1} A_i = (\bigcup_{i=1}^{k} A_i) \cup A_{k+1}.$$

Remark 2.7 *1.* Let $\mathcal{K}$ be a free amalgamation class. If $\bigcup_{i=1}^{k+1} A_i$ in (2) of Definition 2.6 is defined, then it is in $\mathcal{K}$.

2. Let A_i, C be L-structures for $1 \leq i \leq n$, so that $C \leq A_i$, for all $i \in \{1, \ldots, n\}$. By taking isomorphic copies assume that for each $k \geq 1$,

$$\bigcup_{i=1}^{k} A_i \cap A_{k+1} = C.$$

In this way we may define inductively the *free amalgam of* $\{A_i : 1 \leq i \leq n\}$ *over* C as in Definition 2.6.

A crucial tool for this and later chapters is the notion of the graph of a structure.

Definition 2.8 *Let A be an L-structure. Define the* ***graph*** *of A, denoted by $G(A)$, to be the graph with vertices A, and edges $\{(x, y) : x, y \in A$ so that $x \neq y$ and there exists $R \in L$ and $\bar{a} \subseteq A$ so that $x, y \in \bar{a}$ and $\bar{a} \in R^A\}$.*

Example 2.9 *1.* If A itself is a graph, then $G(A) = A$.

2. If A is a digraph, then $G(A)$ results by forming the symmetric closure of the directed edges of A.

3. If A is an order, then $G(A)$ is the comparability graph of A : $xE^{G(A)}y$ iff $x < y$ or $y < x$.

Using the graph of a relational structure, we may classify relational structures via properties of their graphs. For example, we say that a structure is complete if and only if its graph is complete; a structure is connected if and only if its graph is connected, and so on.

2.1.2 Structural characterization of free amalgamation classes

The main theorem of this section is the following.

Proposition 2.10 *Let $\mathcal{K}$ be an $\forall_1$ L-class, with $\mathcal{K} = \mathcal{K}(\neg\{M_i : i \in I\})$ so that the M_i are minimal (see the proof of Theorem 1.20). Then the following are equivalent.*

1. *$\mathcal{K}$ is closed under unions.*

2. *For each $i \in I$, M_i is complete.*

Applications and Non-applications of Proposition 2.10

Exploiting Proposition 2.10, we list some examples of classes closed under union, and a few classes where FAP fails. The reader will note that while many classes are not closed under union, many familiar classes do have FAP.

Classes closed under unions

1. Sets: there are no minimal contraints.

2. $\mathcal{K}(L)$ for L an arbitrary finite relational language; (1) is a special case of this with $L = \{\emptyset\}$. Again, in this case, there are no minimal constraints.

3. $\forall_1$ classes of graphs closed under unions:

   (a) The class of all graphs: the minimal constraints have graphs K_1 and K_2 (see Chapter 1).

   (b) For $n \geq 3$, the classes of K_n-free graphs: the minimal constraints are the minimal constraints of graphs and K_n.

   By Proposition 2.10 there are no other $\forall_1$ classes of graphs closed under union.

4. $\forall_1$ classes of directed graphs closed under unions.

   (a) The class of all directed graphs: the minimal constraint has graph K_1 (see Chapter 1).

   (b) The class of all oriented graphs: the minimal constraints have graphs K_1 and K_2 (see Chapter 1).

   (c) A complete oriented graph is precisely a tournament. Hence, by Proposition 2.10, an $\forall_1$ class of oriented digraphs closed under union must have constraints which are tournaments.

   (d) The Henson classes of digraphs, defined by excluding a countable set of pairwise non-embeddable tournaments; see [20] (this gives $2^{\aleph_0}$ many examples).

$\forall_1$ classes not closed under unions.

(a) Orders: as in Example 1.21 above, there is a minimal constraint of order 3 whose graph is the two path, P_2. See Figure 2.2.

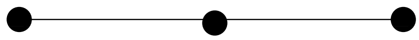

Figure 2.2: P_2.

(b) Tournaments: $\overline{K_2}$ is the graph of one of the minimal constraints. See Figure 2.3.

Figure 2.3: $\overline{K_2}$

(c) Equivalence relations: same reason as for orders.

Proof. (of Proposition 2.10) ($1 \Rightarrow 2$)

Claim 1: For each $i \in I$, M_i is connected.

Fix $i \in I$. If M_i is not connected, then $G(M_i) = A \uplus B$. Then $M_i = (M_i \restriction A) \uplus (M_i \restriction B)$: for $R \in L$, $\bar{a} \in R^M$ iff $\bar{a} \in R^{M_i \restriction A}$ or $\bar{a} \in R^{M_i \restriction B}$. However, M_i is minimal so $M_i \restriction A$ and $M_i \restriction B$ are in $\mathcal{K}$. This is a contradiction as $\mathcal{K}$ is closed under disjoint unions. This proves Claim 1.

Fix $i \in I$ so that $M = M_i$ is not complete. Let $x, y \in M$ so that $\neg x E^{G(M_i)} y$ with x and y distinct.

Define $A = M \restriction M - \{x, y\}$, $B = M \restriction M - \{y\}$, and $C = M \restriction M - \{x\}$.

Claim 2: B and C agree.

As $B \cap C = A$, it is enough to check that for each $R \in L$, $R^{B \restriction A} = R^{C \restriction A}$. But this is immediate as $B, C \leq M$.

Claim 3: $M = B \cup C$.

For $R \in L$, $\bar{a} \in R^M$ iff $\bar{a} \in R^B$ and $y \notin \bar{a}$, or $\bar{a} \in R^C$ and $x \notin \bar{a}$. Since $\neg x E^{G(M)} y$ this is in turn equivalent to $\bar{a} \in R^{B \cup C}$.

Finally, as M is connected by Claim 1, $A \neq \emptyset : x$ and y are not adjacent and so there must be some path connecting x to y. Hence, by Claims 2 and 3, we may realize M as a free amalgam of proper substructures A, B, and C, all of which are in $\mathcal{K}$ by the minimality of M. This is a contradiction, as $\mathcal{K}$ is closed under FAP.

($2 \Rightarrow 1$) We first show $\mathcal{K}$ is closed under disjoint union.

Let B, $C \in \mathcal{K}$ so that $B \uplus C \notin \mathcal{K}$. Then for some $i \in I$, M_i embeds in $B \uplus C$; without loss of generality, we assume that $M_i \leq B \uplus C$. As B, C are in $\mathcal{K}$, $M_i \cap B \neq \emptyset$ and $M_i \cap C \neq \emptyset$.

Let $M_B = M_i \restriction M_i \cap B$ and $M_C = M_i \restriction M_i \cap C$. Then $M_i = M_B \uplus M_C$. But then $G(M_i)$ is disconnected, contradicting that $G(M_i)$ is complete.

We next show that $\mathcal{K}$ is closed under FAP. If not then there are $A, B, C \in \mathcal{K}$ so that $A = B \cap C$ and B and C agree, and there is some minimal constraint M_i of $\mathcal{K}$ that embeds in $B \cup C$.

Since B, $C \in \mathcal{K}$, $M_i \cap B$ and $M_i \cap C$ are nonempty.

Case i) $M_i \cap A = \emptyset$.

Then as above, $M_i = M_B \uplus M_C$, with the same contradiction as before.

Case ii) $M_i \cap A \neq \emptyset$.

Let $M_A = M_i \restriction M_i \cap A$; M_B and M_C are as above. Then $M_\triangle \leq \triangle$, for $\triangle \in \{A, B, C\}$.

We show that M_i is a free amalgam of M_B and M_C over M_A.

Claim 4: M_B and M_C agree over M_A.

This follows as $M_B, M_C \leq M_i$.

Claim 5: $M_i = M_B \cup M_C$.

We use the fact that $M_i \leq B \cup C$ and that for $R \in L$, $R^{B \cup C} = R^B \cup R^C$.

For $R \in L$, $\bar{a} \in R^{M_i}$ iff $\bar{a}$ is from $M_i \cap B$ and $\bar{a} \in R^{M_i}$ or $\bar{a}$ is from $M_i \cap C$ and $\bar{a} \in R^{M_i}$. In turn, this is equivalent to $\bar{a} \in R^{M_B}$ or $\bar{a} \in R^{M_C}$, which itself is equivalent to $\bar{a} \in R^{M_B \cup M_C}$.

From Claims 4 and 5, we can realize M_i as a free amalgam of M_B and M_C over M_A. But then in $G(M_i)$ there is no edge from some element of $M_B - M_A$ to any element of $M_C - M_A$, contradicting that $G(M_i)$ is complete. $\square$

If $\mathcal{K}$ is closed under unions another consequence is that we may "delete edges" from $\mathcal{K}$-structures and remain in $\mathcal{K}$.

Definition 2.11 *Let $A \in \mathcal{K}(L)$, with $|A| \geq 2$. Let $x, y \in A$. Define A_{-xy} to be the L-structure with domain A and for $R \in L$,*

$$R^{A_{-xy}} = \{\bar{a} : \bar{a} \in R^A \text{ and } \{x, y\} \not\subseteq \bar{a}\}.$$

Lemma 2.12 *Let $\mathcal{K} \subseteq \mathcal{K}(L)$ be an $\forall_1$ free amalgamation class, and let $A \in \mathcal{K}$. Then for all $x, y \in A$, $A_{-xy} \in \mathcal{K}$.*

Proof. Let $B = A \restriction A - \{y\}$, $C = A \restriction A - \{x\}$. Then B and C agree over $A \restriction A - \{x, y\}$. Further, $B \cup C = A_{-xy}$, as $B \cup C$ contains all of the relations of A except those involving x, y. $\square$

A further consequence of closure under union is that, under certain restrictions on $\mathcal{K}$, the class of graphs of $\mathcal{K}$-structures contains all triangle free-graphs.

Definition 2.13 *Let $\mathcal{K} \subseteq \mathcal{K}(L)$. Define*

$$G(\mathcal{K}) = \{G(A) : A \in \mathcal{K}\}.$$

Definition 2.14 *Let $\mathcal{K} \subseteq \mathcal{K}(L)$*

1. *$A \in \mathcal{K}$ is a 2-**edge** if A is a two-element $\mathcal{K}$-structure with graph K_2.*
2. *$\mathcal{K}$ **has a 2-edge** if there is some 2-edge $A \in \mathcal{K}$.*

Let $\mathcal{C}$ be the class of all triangle-free graphs.

Lemma 2.15 *Let $\emptyset \neq \mathcal{K} \subseteq \mathcal{K}(L)$ be an $\forall_1$ free amalgamation class with 2-edge A and assume that there is a unique isomorphism type of one-element structure in $\mathcal{K}$. Then $\mathcal{C}_{fin} \subseteq G(\mathcal{K})$.*

Proof. We proceed by induction; the induction hypothesis is that $\mathcal{C}_n \subseteq \mathcal{K}$, for $n \geq 1$.

$\mathcal{C}_1 = \{K_1\}$ can be realized as the graph of a one-element substructure of A. $\mathcal{C}_2 = \{\overline{K_2}, K_2\}$. $\overline{K_2}$ can be realized as the disjoint union of a structure in $\mathcal{K}$ realizing K_1 with itself; K_2 can be realized by A itself.

Let $B \in \mathcal{C}_{n+1}$. Then B is a 1-element extension of a $\mathcal{C}_n$-structure B', which, by induction, is realized as the graph of a $\mathcal{K}$-structure C'. Let $a \in B - B'$.

B is determined by the vertices X that a is adjacent to in B'. Note that if $X \neq \emptyset$, then X is independent: an edge in X will result in a triangle in B. Let $|X| = m \geq 1$.

Let $A = \{x, y\}$. Using closure under union in $\mathcal{K}$ we can form a $\mathcal{K}$-structure S_X, with domain $X \cup \{x\}$, so that $G(S_X)$ is the following rooted tree with m-leaves:

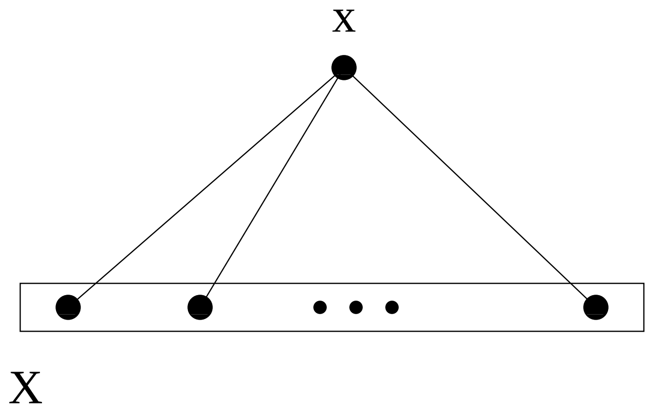

Figure 2.4: The graph of the structure S_X.

Let $A_1, \ldots, A_n$ be n copies of A.

Let $S_X(1) = A_1$.

Assume $S_X(k) \in \mathcal{K}$ has been defined for some $1 \leq k < n$.

Assume, by taking isomorphic copies, that $G(S_X(k))$ is a rooted tree with root x, and that

$$S_X(k) \cap A_{k+1} = \{x\}.$$

Define

$$S_X(k+1) = S_X(k) \cup A_{k+1}.$$

Define

$$S_X = S_X(n).$$

As there is only one isomorphism type of one-element structures in $\mathcal{K}$, S_X and C' agree; hence, we can form the free amalgam of S_X and C' over $S_X \restriction X = C' \restriction X$. Then

$$G(S_X \cup C') = B.$$

□

2.1.3 Syntactic characterization of free amalgamation classes

Let L be finite relational.

Definition 2.16 *1. An L-sentence θ is of type 1 if it is equivalent to an L-sentence θ' of the form:*

$$\forall\bar{x}(\bigvee R_i(\overline{x_i}) \vee \bigvee \neg R_j(\overline{x_j})),$$

with $\overline{x_i}, \overline{x_j} \subseteq \bar{x}$, $R_i, R_j \in L \cup \{=\}$ for all i, j, and either $|\bar{x}| = 1$ or for each pair x, y of distinct variables from $\bar{x}$, there is an $R_i \in L$ so that either

$$\neg R_i(-x-y-) \text{ or } \neg R_i(-y-x-)$$

appears in the negated atomic part of the matrix of θ'.

2. *An L-sentence θ is of type 2 if it is equivalent to a (finite) conjunction of type 1 sentences.*

Example 2.17 *1.* An axiomatization of k-uniform hypergraphs, for $k \geq 2$, consisting of type 2 sentences is:

$$\forall x_1 \dots x_k (\neg R(x_1 \dots x_k) \vee \bigwedge_{\sigma \in S_k} R(x_{\sigma(1)} \dots x_{\sigma(k)}),$$
$$\forall x_1 \dots x_k (\neg R(x_1 \dots x_k) \vee \bigwedge_{1 \leq i < j \leq n} \neg x_i = x_j)$$

where S_k is the symmetric group of order k. When $k = 2$, we recover an axiomatization for the class of graphs; so graphs also have a type 2 axiomatization.

2. The class of K_n-free graphs is axiomatized by the following type 2 sentences:

$$\forall x (\neg x E x)$$
$$\forall xy (\neg x E y \vee y E x)$$
$$\forall x_1 \dots x_n (\bigvee_{1 \leq i < j \leq n} \neg x_i E x_j).$$

3. The usual axiomatization for the class of orders is not type 2. Transitivity fails to be type 2:

$$\theta = \forall xyz (\neg x \leq y \vee \neg y \leq z \vee x \leq z).$$

In θ, there is no negated atomic formula containing both x and z.

We now prove a preservation theorem.

Proposition 2.18 *Let Φ be a set of L-sentences. The following are equivalent.*

1. *Φ is equivalent to a set of type 2 sentences.*

2. *The class of models of Φ is closed under I, S, and unions.*

PROOF. ($1 \Rightarrow 2$). Closure under I, S follow as type 2 sentences are $\forall_1$. By the definition of type 2 sentences, the corresponding minimal constraints of the class of models of Φ have complete graphs. Hence, by Proposition 2.10, the models of Φ are closed under union.

($2 \Rightarrow 1$) By the Theorem 1.20, and Proposition 2.10, there is a set of minimal constraints $\{M_i : i \in I\} \subseteq \mathcal{K}(L)_{fin}$ with complete graphs so that

$$\mathcal{K} = \mathcal{K}(\neg\{M_i : i \in I\}).$$

For each $i \in I$, let $\overline{a_i}$ be a tuple from M_i enumerating the elements of M_i.

Let

$$\Phi' = \{\neg\exists\bar{x}qft_{M_i,\bar{a}_i}(\bar{x}) : i \in I\}.$$

Then Φ' is equivalent to Φ, so it is enough to show that Φ' is type 2.

Each $\neg\exists\bar{x}qft_{M_i,\bar{a}_i}(\bar{x})$ is equivalent to a universally quantified disjunction of atomic and negated atomic fomulas

$$\forall\bar{y}(\bigvee R_i(\overline{y_i}) \vee \bigvee \neg R_j(\overline{y_j})) \tag{2.1}$$

with $\overline{y_i}, \overline{y_j} \subseteq \bar{y}$, $R_i, R_j \in L \cup \{=\}$, for all i, j. As M_i is complete, either $|\bar{x}| = 1$ or for each pair x, y of distinct variables from $\bar{y}$, there is $R_i \in L$ so that either

$$\neg R_i(-x - y-) \text{ or } \neg R_i(-y - x-)$$

appears in the negated atomic part of the matrix of (2.1).

But this implies that each element of Φ' is of type 1, so that Φ' is type 2. □

Example 2.19 By Proposition 2.18, the class of orders (for which we noticed in Example 2.17 (3) that the usual axiomatization is not type 2) has *no* possible type 2 axiomatization.

2.2 The model companions of free amalgamation classes: **nqfa**(1)

By Theorem 1.58 an $\forall_1$ amalgamation class has a model companion axiomatized by the theory of the Fraïssé limit of the finite members of the class. In this section, we prove that if $\mathcal{K}$ is a $\forall_1$ free amalgamation class satisfying a certain non-triviality assumption (see Definition 2.25 below), then $\mathcal{K}^{mc}$ has a non-finitely axiomatizable theory. In fact, even if we work modulo sentences asserting "I embed every finite $\mathcal{K}$-structure" then $\mathcal{K}^{mc}$ is non-finitely axiomatizable.

2.2.1 Finite axiomatizability modulo a set of sentences

Definition 2.20 *Let Θ be a set of L-sentences, and let T be a set of L-sentences so that $T \vdash \Theta$.*

1. *T is **finitely axiomatizable modulo** Θ if there is an L-sentence φ so that*

$$\Theta \vdash \bigwedge T \leftrightarrow \varphi.$$

2. *T is **non-finitely axiomatizable modulo** Θ if it is not finitely axiomatizable modulo Θ.*

Remark 2.21 Finite axiomatizability is the same as finite axiomatizability modulo $\emptyset$. Hence, finite axiomatizability implies finite axiomatizability modulo a set of sentences Θ (so that non-finitely axiomatizability modulo Θ implies non-finitely axiomatizability).

A notion that has been used in the study of totally categorical theories (see [24]) is quasi-finite axiomatizability: take Θ in Definition 2.20 to be axioms asserting "I am infinite"; namely,

$$\exists x_1 \dots x_n \bigwedge_{1 \leq i < j \leq n} x_i \neq x_j,$$

for $n \geq 1$. We abbreviate quasi-finite axiomatizability by *qfa*; non-quasi-finite axiomatizabililty by *nqfa*.

We introduce a stronger notion than nqfa, nqfa(1) that was alluded to above.

Definition 2.22 *Let T be an L-theory, whose class of models is $\mathcal{K}$. Let Γ consist of $Th(S(\mathcal{K}))$ and the axioms: "My age contains $S(\mathcal{K})_{fin}$":*

$$\exists\bar{x} qft_{A,\bar{a}}(\bar{x}),$$

where $|\bar{x}| = |\bar{a}| = n$, $A \in S(\mathcal{K})_n$, *and* $\bar{a}$ *enumerates* A, *for each* $n \geq 1$.

We write ***qfa***(1) *if* T *is finitely axiomatizable modulo* Γ, *and* ***nqfa***(1) *otherwise.*

Remark 2.23 If $A \in \mathcal{K}(L)$ satisfies Γ, then for every $B \in \mathcal{K}_{fin}$, B embeds in A. If $\mathcal{K}_{fin}$ is infinite then furthermore, $|A| \geq \aleph_0$.

Remark 2.24 Nqfa(1) implies nqfa. The converse is false.

Let $L = \{P\}$, P 1-ary, and let T be the theory asserting: "Both P and the complement of P are infinite". Note that T is the theory of the model companion of $\mathcal{K}(L)$.

Then Γ implies T, so that T is qfa(1) (we can choose φ to be simply $\forall x(x = x)$). However, if T were qfa, by Compactness, T would be axiomatized by T' consisting of "I am infinite" and "Both P and the complement of P have cardinality at least n", for some $n \geq 1$. This is a contradiction: the L-structure A with P^A of cardinality n and the complement of P^A of cardinality $\aleph_0$ is a model of T' but not a model of T.

2.2.2 Nqfa(1) of the model companion

If $\mathcal{K}$ is the class of sets over the empty language then $\mathcal{K}^{mc}$ is the class of infinite sets, whose theory is qfa, and hence, qfa(1). Even worse, if $L = \{P\}$ where P is 1-ary, if we let $\mathcal{K} = \mathcal{K}(L)$, the model companion is nqfa but qfa(1) (see Remark

2.24 above). One distinguishing property of these two classes is that there is no structure in them whose graph has edges.

Definition 2.25 *Let $\mathcal{K} \subseteq \mathcal{K}(L)$ be $\forall_1$.*

1. *$\mathcal{K}$ **has edges** if there is some $A \in \mathcal{K}$ so that $G(A)$ has edges.*
2. *If $\mathcal{K}$ has edges, suppose $A \in \mathcal{K}$ is such that $G(A)$ has edges. For some $n \geq 2$, assume $\bar{a} \subseteq A$ is so that $\bar{a}$ contains at least two distinct elements and there is some $R \in L$ with $\bar{a} \in R^A$. Then $A \restriction \bar{a} \in \mathcal{K}$ is called an **edge**.*

Example 2.26
1. The classes of graphs has edges: for example, choose A to be K_2.
2. The class $\mathcal{K}(L)$ for $L = \{P\}$ where P is 1-ary does not have edges. Note that the graph of a structure is insensitive to unary relations.

The main theorem of this Chapter and a template for later work is the following.

Theorem 2.27 *Let $\mathcal{K} \subseteq \mathcal{K}(L)$ be an $\forall_1$ free amalgamation class with edges. Then $Th(\mathcal{K}^{mc})$ is nqfa(1).*

The basic strategy in the proof of Theorem 2.27 is the following idea, which will also be exploited in later Chapters when we discuss the model companions of colour classes (see the proof of Theorem 3.51 below). If $\mathcal{K} \subseteq \mathcal{K}(L)$ is an $\forall_1$ free amalgamation class, by Theorem 1.57 we have that $Th(\mathcal{K}^{mc})$ is axiomatized by the axioms of $\mathcal{K}$ and $T = \{\varphi_n : n \in \omega\}$, the $\mathcal{K}_{fin}$-extension axioms defined in Definition 1.55.

We search for models $M_n \in \mathcal{K}$ that:

1. are infinite $\mathcal{K}$-structures,

2. embed all $\mathcal{K}_{fin}$-structures,

3. satisfy φ_n,

   but

4. do not satisfy all of T.

Lemma 2.28 *Let $\mathcal{K} \subseteq \mathcal{K}(L)$ be an $\forall_1$ amalgamation class axiomatized by Φ so that $\mathcal{K}$ has infinite models. Assume that there exists a family of countable $\mathcal{K}$-structures $\{M_n : n \geq 1\}$ so that for each $n \in \omega^*$,*

1. *$M_n \models \varphi_n$;*

2. *$M_n \models \Pi = \Phi \cup \Gamma$;*

3. *there exists a function $f : \omega \to \omega$ so that for all $n \in \omega^*$, $f(n) > n$ and*

$$M_n \models \neg\varphi_{f(n)}.$$

Then $Th(\mathcal{K}^{mc})$ is nqfa(1).

Proof. Recall by the proof of Theorem 1.58 that $T = \Theta \cup \{\varphi_n : n \in \omega\} \cup \Phi$ axiomatizes $\mathcal{K}^{mc}$, and that φ_{n+1} logically implies φ_n for all $n \geq 1$.

To obtain a contradiction, assume that $Th(\mathcal{K}^{mc})$ is qfa(1).

Then there is an L-sentence φ so that

$$\Pi \vdash \bigwedge T \leftrightarrow \varphi.$$

By the Compactness theorem, and as $\Gamma \vdash \Theta$, there is an $n \in \omega^*$ so that

$$\Pi \vdash \varphi_n \leftrightarrow \varphi.$$

In particular, $\Pi \cup \{\varphi_n\}$ axiomatizes $\mathcal{K}^{mc}$.

But by hypothesis, there is a $\mathcal{K}$-structure

$$M_n \models \{\varphi_n\} \cup \Pi$$

so that

$$M_n \models \neg\varphi_{f(n)}. \tag{2.2}$$

It follows that

$$M_n \vDash \{\varphi_n\} \cup \Pi$$

and yet by (2.2)

$$M_n \nvDash T,$$

contradiction. □

2.2.3 Proof of Theorem 2.27

In the following proof we make use of the following observation about free amalgams that we call "freeness".

Freeness: If A and B agree over $A \cap B \neq \emptyset$, then in $G(A \cup B)$, there is no edge between a vertex of $A - A \cap B$ and a vertex of $B - A \cap B$. See Figure 2.5.

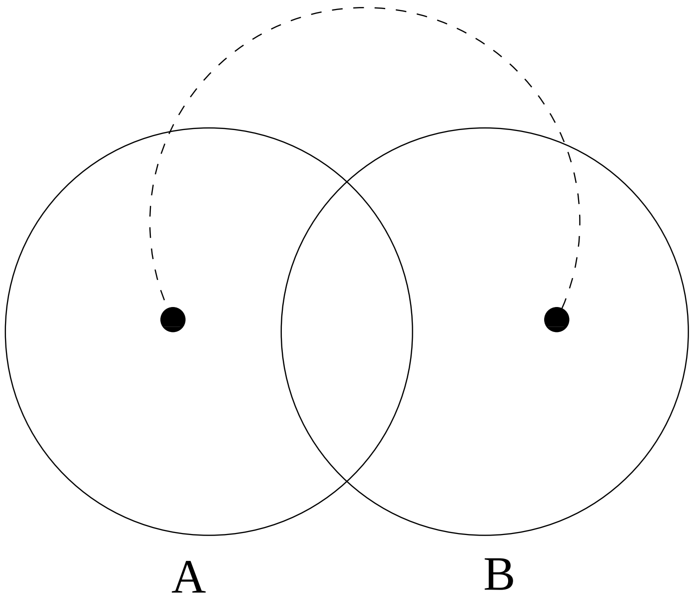

Figure 2.5: "Freeness."

The strategy of the proof of Theorem 2.27 is to construct models M_n satisfying the hypotheses of Lemma 2.28. In particular, M_n will be a countable $\mathcal{K}$-structure, embedding all finite K-structures, and satisfying only finitely many of the $\mathcal{K}_{fin}$-extension axioms, φ_n. The current strategy will be used with a few modifications in later chapters.

By hypothesis, we can choose an "edge" $A \in \mathcal{K}$ (see Definition 2.25).

Fix $a \in A$.

Define

$$1_a = A \restriction a \in \mathcal{K}.$$

Inductively define

$$(n+1)_a = (n)_a \uplus 1_a.$$

As $\mathcal{K}$ is closed under unions, $(n)_a \in \mathcal{K}$, for all $n \geq 1$.

Note that $(n)_a$ is an n-element structure so that

$$G((n)_a) = \overline{K_n}.$$

We define M_n, $n \geq 1$, as the union of a chain of finite $\mathcal{K}$-structures M_n^k, $k \in \omega$ (and hence, $M_n \in \mathcal{K}$ as $\mathcal{K}$ is $\forall_1$).

Let

$$M_n^0 = (n+1)_a.$$

Assume that for $k \geq 0$, $M_n^k \in \mathcal{K}_{fin}$ with M_n^0 a substructure of M_n^k.

Define M_n^{k+1} as follows. List the substructures of M_n^k of order $\leq n$ as $S_1, \ldots, S_i$ (there are only finitely many). For each $1 \leq r \leq i$, list the isomorphism types of extensions of S_r to a $\mathcal{K}_{fin}$-structure of order $\leq n+1$ as $T_1, \ldots, T_j$ (there are only finitely many as L is finite relational).

Freely amalgamate $T_1, \ldots, T_j$ and M_n^k over S_r to obtain $M_{n,r}^k \in \mathcal{K}_{fin}$. Freely amalgamate $M_{n,1}^k, \ldots, M_{n,i}^k$ over M_n^k to obtain $(M_n^{k+1})' \in \mathcal{K}_{fin}$.

Form $M_n^{k+1} \in \mathcal{K}_{fin}$ by taking the disjoint union of $(M_n^{k+1})'$ with all isomorphism types of $\mathcal{K}_{k+1}$-structures (there are only finitely many).

Define

$$M_n = \bigcup_{k \in \omega} M_n^k.$$

By construction, $M_n \models \varphi_n$ (every one-element extension of a $\mathcal{K}_n$-structure that embeds in M_n is realized in M_n) and M_n embeds all $\mathcal{K}_{fin}$-structures (at the kth step of the construction, we add all the elements of $\mathcal{K}_k$).

We now form an extension $C \in \mathcal{K}_{fin}$ of M_n^0 that is not realized in M_n. Of course, by "C not realized in M_n" we mean that there is no isomorphism β from C onto a substructure of M_n so that $\beta \restriction M_n^0 = id_{M_n^0}$.

Let $A_1, \ldots, A_{n+1}$ be $n+1$ copies of A.

Let $C \in \mathcal{K}_{fin}$ be the free amalgam of $A_1, \ldots, A_{n+1}$ over $A \restriction A - \{a\}$.

It can be arranged that $C \geq M_n^0$ (see Figure 2.6).

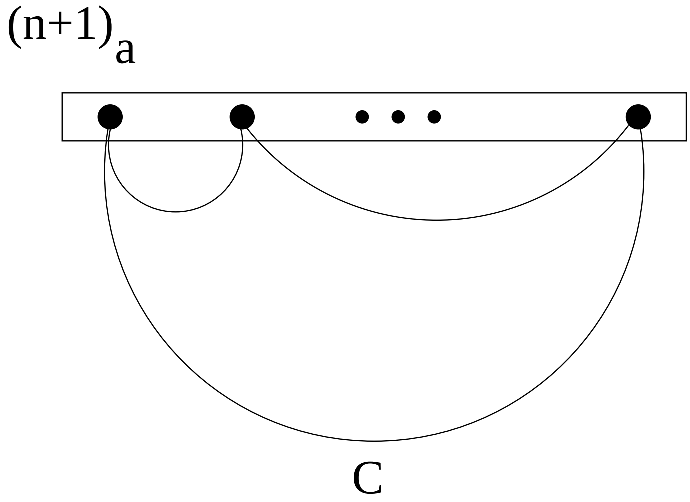

Figure 2.6: The extension C of M_n^0.

We show that C is not realized in M_n by showing that C is not realized in M_n^k for all $k \in \omega$.

C is not realized in M_n^0 as

$$|C| > |M_n^0|.$$

Assume C is not realized in M_n^k, for $k \geq 0$.

C is not realized in $(M_n^{k+1})' - M_n^k$ as no element of $(M_n^{k+1})' - M_n^k$ is adjacent (in the graph of $(M_n^{k+1})'$) to $n+1$-elements of M_n^k by freeness of the amalgamations. C is not realized in $M_n^{k+1} - (M_n^{k+1})'$ as no element of $M_n^{k+1} - (M_n^{k+1})'$ is adjacent (in the graph of M_n^{k+1}) to an element of M_n^0.

As C is not realized in M_n, M_n does not satisfy all of the $\mathcal{K}_{fin}$-extension axioms; for example, it does not satisfy $\varphi_{|A|+n-1}$: note that

$$|C| = |A| - 1 + n + 1 = |A| + n.$$

By Lemma 2.28 with $f : \omega \to \omega$ defined by

$$f(n) = |A| + n - 1,$$

we have that $Th(\mathcal{K}^{mc})$ is nqfa(1).

This completes the proof of Theorem 2.27.

2.2.4 Examples

We list some applications of Theorem 2.27.

1. The following classes each have nqfa(1) model companions. Here, f (as in Lemma 2.28) is simply the successor function:

$$f(n) = n + 1.$$

   (a) Digraphs;

(b) Graphs (in particular, the theory of the random graph is nqfa(1));

(c) Oriented graphs;

(d) K_n-free graphs, $n \geq 3$;

(e) n-coloured graphs, $n \geq 2$;

(f) Classes of oriented graphs determined by excluding the Henson digraphs;

(g) $\mathcal{K}(L)$, where L has at least one symbol of arity ≥ 2.

2. Let $L = \{R\}$ with R m-ary, $m \geq 3$. The classes of m-uniform hypergraphs each have nqfa(1) model companions. f is now taken to be

$$f(n) = m + n - 1.$$

The results of Theorem 2.27 are in stark contrast to some other known model companion axiomatization results for $\forall_1$ amalgamation classes where the amalgamation is not free amalgamation. For example, the model companion of the class of orders is finitely axiomatizable (see [1]).

Chapter 3

Colouring of Structures

3.1 Introduction

In this chapter we study classes of structures admitting a homomorphism into a fixed finite structure. General colour classes are inspired by the well-known classes $C(K_n)$ of n-colourable graphs for $n \geq 2$. Recall that $C(K_n)$ is the class of all graphs whose vertices have partitions into n independent sets (that is, vertices with no edges in common), or equivalently, admit a homomorphism into K_n. $C(K_n)$ has an $\aleph_0$- categorical model companion, as first proven by Wheeler in [42]. As thorough as Wheeler's analysis is of the model companion of $C(K_n)$, [42] contains no discussion of the "generic-structure" of the unique countable e.c. n-colourable graph. We intend to address this issue in this chapter.

Fix an $\forall_1$ class $\mathcal{K}$ over a finite relational language L, and let $A \in \mathcal{K}_{fin}$. Define $C_{\mathcal{K}}(A)$ to be the class of $\mathcal{K}$-structures that admit a homomorphism into A. $C_{\mathcal{K}}(A)$ is the **colour class** determined by A.

We will prove the following results on colour classes under certain assumptions on $\mathcal{K}$ and A (see the assumption before Theorem 3.51).

1. The model companion exists; the unique countable e.c. structure has the structure of a generic (see Definition 1.61): see Theorems 3.47 and 3.43.

2. The model companion is nqfa(2) (see Definition 3.49): see Theorem 3.51.

The additional assumptions on $\mathcal{K}$ are too lengthy to be discussed in this introduction, but we require that $\mathcal{K}$ is an $\forall_1$ free amalgamation class with edges, that L have no unary symbols, and that "fixations exist." That $\mathcal{K}$ satisfy the first two conditions are essential to items (1) and (2); at this point it is uncertain what role the final condition plays.

3.2 Preliminaries

3.2.1 Homomorphisms

Throughout, $\mathcal{K}$ will be an $\forall_1$ class over a finite relational language L.

Definition 3.1 *Let $A, B \in \mathcal{K}$.*

1. $\hom(A, B)$ *is the set of homomorphisms from A to B.*

2. *end*$(A) = \hom(A, A)$ *is the set of endomorphisms of A.*

3. *ends*(A) *is the set of surjective endomorphisms of A.*

4. *aut*(A) *is the set of embeddings of A onto itself.*

We collect some basic folklore about the objects defined in Definition 3.1.

Lemma 3.2 *Let $A, B \in \mathcal{K}$.*

1. *$ends(A) = aut(A)$ if A is finite. Equality may fail if A is infinite.*
2. *$end(A)$ forms a monoid under the operation of composition; $aut(A)$ forms a group under the operation of composition.*
3. *$aut(B)$ acts on $\hom(A, B)$ by left composition.*

Definition 3.3 *Let $A, B \in \mathcal{K}$.*

1. *$A \rightarrow B$ if $\hom(A, B) \neq \emptyset$.*
2. *$A \leftrightarrow B$ if $A \rightarrow B$ and $B \rightarrow A$.*

The $\rightarrow$-relation gives rise to a pre-order on $\mathcal{K}$; that is, a reflexive, transitive binary relation on $\mathcal{K}$. We can extract an order from $\rightarrow$ in the usual way.

Lemma 3.4 *Let $A, B \in \mathcal{K}$.*

1. *$A \leftrightarrow B$ is an equivalence relation on $\mathcal{K}$.*
2. *$A \rightarrow B$ defines an order on the $\leftrightarrow$-equivalence classes of $\mathcal{K}$.*
3. *Consider $\leftrightarrow$ restricted to $\mathcal{K}_{fin}$. Assume $\mathcal{K}$ is closed under disjoint unions and products.*

Then the order induced by $\rightarrow$ on the $\leftrightarrow$-equivalence classes of $\mathcal{K}$ is a distributive lattice $L(\mathcal{K})$, with

$$A \vee B = A \uplus B,$$

$$A \wedge B = A \times B.$$

PROOF. (1) and (2) are immediate.

(3) $A \uplus B$ is an upper bound of A and B. If $A, B \rightarrow C$, form the union of the maps to obtain $A \uplus B \rightarrow C$. That $A \times B$ is a greatest lower bound of A and B is similar.

Distributivity follows as we have "left distributivity":

$$A \times (B \uplus C) = (A \times B) \uplus (A \times C).$$

□

Remark 3.5 *The lattice $L(\mathcal{K})$, in the special case when $\mathcal{K}$ is the class of graphs, is dense above K_2, by Theorem 5.1 of [40].*

Definition 3.6 *$A \in \mathcal{K}_{fin}$ is a* ***core*** *if*

$$end(A) = aut(A).$$

Remark 3.7 According to Lemma 3.2, $A \in \mathcal{K}_{fin}$ is a core if every endomorphism of A is onto; we will find that this characterization will be useful in practice.

Example 3.8 The reader may verify the following.

1. Each complete graph is a core graph.
2. Each odd cycle is a core graph.
3. The only core reflexive order is the one-element order.

The following lemma simplifies our discussion of colour classes considerably.

Lemma 3.9 *Let $A \in \mathcal{K}_{fin}$. Then there is a unique (up to isomorphism) substructure $Core(A)$ of A so that $Core(A)$ is a core and $Core(A) \leftrightarrow A$.*

Proof. Define $Core(A)$ to be an element of

$$\{f(A) : f \in end(A)\}$$

with the least cardinality. Note that $Core(A) \in \mathcal{K}$ as $\mathcal{K}$ is $\forall_1$. Then $Core(A)$ is core and $Core(A) \leftrightarrow A$.

If C is another core so that $C \leftrightarrow A$, then there are homomorphisms $g : C \rightarrow Core(A)$ and $h : Core(A) \rightarrow C$. As C and $Core(A)$ are cores, $g \circ h$ and $h \circ g$ are isomorphisms, so that g and h are isomorphisms. □

3.2.2 Colour classes defined

We now introduce our main objects of study.

Definition 3.10 *Let $A \in \mathcal{K}_{fin}$. Define*

$$C_{\mathcal{K}}(A) = \{B \in \mathcal{K} : B \rightarrow A\}.$$

$C_{\mathcal{K}}(A)$ *is the* ***colour class in*** $\mathcal{K}$ ***determined by*** A *(when* $\mathcal{K}$ *is clear from context we call* $C_{\mathcal{K}}(A)$ *simply the* ***colour class determined by*** A, *and write* $C(A)$*).*

Lemma 3.11 *Let* $A, B \in \mathcal{K}_{fin}$.

1. $A \rightarrow B$ *if and only if* $C_{\mathcal{K}}(A) \subseteq C_{\mathcal{K}}(B)$.

2. $A \leftrightarrow B$ *if and only if* $C_{\mathcal{K}}(A) = C_{\mathcal{K}}(B)$.

PROOF. (1) Let $A \rightarrow B$. If $D \in C_{\mathcal{K}}(A)$, then

$$D \rightarrow A \rightarrow B$$

so $D \rightarrow B$ and $D \in C_{\mathcal{K}}(B)$. Conversely, if

$$C_{\mathcal{K}}(A) \subseteq C_{\mathcal{K}}(B),$$

then as $A \in C_{\mathcal{K}}(A)$ we have that $A \in C_{\mathcal{K}}(B)$, and hence $A \rightarrow B$.

(2) is immediate from (1). □

Lemmas 3.9 and 3.11 give us the following useful observation.

Key Fact: When discussing a colour class $C_{\mathcal{K}}(A)$, we may assume, without loss of generality, that A is a core.

Proposition 3.12 *Let* $A \in \mathcal{K}_{fin}$ *be a core. Then* $C_{\mathcal{K}}(A)$ *is* $\forall_1$.

PROOF. We show that $C_{\mathcal{K}}(A)$ is closed under isomorphism, substructures, and is the class of reducts of some elementary class (this is sufficient by Theorem 1.27).

Closure under isomorphism is immediate; for closure under substructure, let $B \leq C \in C_{\mathcal{K}}(A)$. Then

$$B \hookrightarrow C \to A,$$

and so $B \in C_{\mathcal{K}}(A)$.

Let

$$L' = L \cup \{P_1, \ldots, P_n\},$$

with $A = \{1, \ldots, n\}$ and P_i 1-ary for $1 \leq i \leq n$.

Define $C_{\mathcal{K}}(A)'$ to be the $\forall_1$ L'-class axiomatized by axioms for $\mathcal{K}$ and the following sentences.

For $1 \leq i \leq n$,

$$\forall x(\bigvee_{1 \leq i \leq n} P_i x \wedge \bigwedge_{1 \leq i < j \leq n} \neg(P_i x \wedge P_j x)); \tag{3.1}$$

for each $\bar{a} = (a_1, \ldots, a_m) \notin R^A$, for $R \in L$ of arity m,

$$\forall x_1 \ldots x_m ((\bigwedge_{1 \leq i \leq m} P_{a_i} x_i) \to \neg R x_1 \ldots x_m). \tag{3.2}$$

Claim: $C_{\mathcal{K}}(A) = red_L\,(C_{\mathcal{K}}(A)')$.

Let $B \in C_{\mathcal{K}}(A)$ with $f : B \to A$ a homomorphism. Define an L'-structure C by interpreting P_i as

$$\{b \in B : f(b) = i\}.$$

Then C satisfies (3.1) and (3.2).

Conversely, if

$$B \in red_L\,(C_{\mathcal{K}}(A)'),$$

let $C \in C_K(A)'$ so that $B = red_L(C)$. Define a function $f : B \to A$ as follows: for $1 \leq i \leq n$,

$$f(x) = i$$

for $x \in P_i^C$. That f is a homomorphism now follows as the formulas (3.1) and (3.2) are satisfied in C. $\square$

Remark 3.13 *1.* The preimages of f as in the above proof of Proposition 3.12 are called the *colour blocks* of B relative to f. If we define

$$\ker(f) = \{(a, b) \in B^2 : f(a) = f(b)\},$$

then $\ker(f)$ is an equivalence relation whose equivalence classes are the colour blocks of B relative to f.

2. The class $\{B \in \mathcal{K} : B \to A\}$ with A infinite may not even be elementary. For example, let $\mathcal{K}$ be the class of graphs, $A = K_{\aleph_0}$. If $C_{\mathcal{K}}(A)$ were elementary, as $K_{\aleph_0} \in C_{\mathcal{K}}(A)$, any ultrapower of $K_{\aleph_0}$ would be in $C_{\mathcal{K}}(A)$. But an ultrapower of $K_{\aleph_0}$ is complete: all of the factors satisfy

$$\forall xy(x \neq y \to xEy).$$

This means that the complete graphs, K_λ, with $\lambda \geq \aleph_0$ are in $C_{\mathcal{K}}(A)$, which is a contradiction (any homomorphism from a complete graph is an embedding).

3.2.3 An application: G-colourable graphs

For this section, we relativize the discussion to the case when $\mathcal{K}$ is the class of graphs. In this case, the classes $C_{\mathcal{K}}(G)$ are the classes of G-colourable graphs studied in [18], [19], and [40]. The case when $G = K_n$ forms the "classical" theory of n-colourable graphs.

As an illustrative example, we prove that the graphs C_{2n+1}, for $n \geq 1$ form a descending class of cores in the lattice L.

Definition 3.14 *1. Define the **chromatic number** of a finite graph G to be least n so that $G \in C_{\mathcal{K}}(K_n)$, and is written $\chi(G)$.*

*2. A graph G is **(point) critical** if for every vertex $x \in G$,*

$$\chi(G \restriction G - \{x\}) < \chi(G).$$

Example 3.15 The complete graphs and odd cycle graphs are critical.

Lemma 3.16 *1. Every critical graph is a core. In particular, each C_{2n+1} for $n \geq 2$ is a core.*

2. The homomorphic image of a connected graph is connected.

Proof. (1) Let G be critical, with $\chi(G) = n$. If G is not a core, then there is a homomorphism $f : G \to H$, where H is a proper induced subgraph of G. By hypothesis, $\chi(H) < n$, so that $\chi(G) < n$, which is a contradiction.

(2) Let A, B be graphs, with A connected and let $f : A \to B$ be a homomorphism with $f(A)$ disconnected. Suppose that the image decomposes as $A_1 \uplus A_2$. Then

there cannot be an edge between any vertex in $f^{-1}(A_1)$ and a vertex in $f^{-1}(A_2)$. Contradiction. $\square$

Proposition 3.17 *The set* $\{C_{2n+1} : n \geq 1\}$ *forms a strictly decreasing chain of* 3*-chromatic cores in* L.

PROOF. Fix $n \geq 1$.

We argue that

$$C_{2n+3} \to C_{2n+1}$$

but that

$$C_{2n+1} \not\to C_{2n+3}.$$

i) To see that

$$C_{2n+3} \to C_{2n+1}$$

we exhibit an explicit colouring. Label the vertices of C_{2n+1} and C_{2n+3} by $\{1, \ldots, 2n+1\}$ and $\{1, \ldots, 2n+3\}$, respectively. Colour the first $2n+1$ vertices of C_{2n+1} by the colours $\{1, \ldots, 2n+1\}$ in succession, then colour the remaining vertices as $\{2n, 2n+1\}$. The reader can check that this assignment is a C_{2n+1}-colouring.

ii) To obtain a contradiction, assume

$$C_{2n+1} \to C_{2n+3}.$$

As C_{2n+1} is connected, by Lemma 3.16, the image of C_{2n+1} is either all of C_{2n+3} or a path. Both conclusions are impossible. $\square$

Example 3.18 Not every core graph is critical; for example, the Peterson graph (see Figure 1.1) is a core that is not critical.

3.3 Uniquely colourable structures

A uniquely n-colourable graph G is an n-colourable graph so that every homomorphism from G to K_n is onto, and every homomorphism from G to K_n induces the same kernel. At the time of writing, the extension of this notion to "unique G-colourings" for G a graph that is not a complete graph has apparently not been discussed in the literature.

One of the key observations of this chapter is the importance of uniquely colourable structures within general colour classes. When L has no unary symbols, and under certain restrictions on $\mathcal{K}$, the uniquely A-colourable structures help prove the existence of the model companion of $C_{\mathcal{K}}(A)$, and give an explicit axiomatization of the model companion, which in turn can be used to show that the theory of the model companion is nqfa(2).

In this section, we will achieve the following.

1. Introduce uniquely colourable structures.

2. Introduce fixations and give a sufficient condition for fixations to exist in $\forall_1$ classes. Fixations are used to prove cofinality of the class of uniquely A-colourable structures in $C_{\mathcal{K}}(A)$. We provide an example where cofinality fails when the language contains unary predicates.

3.3.1 Definitions

Definition 3.19 *Let $A \in \mathcal{K}_{fin}$ be a core. B is **uniquely** A-**colourable** if*

1. $B \to A$;
2. *every* $f \in \hom(B, A)$ *is onto;*
3. $aut(A)$ *acts transitively on* $\hom(B, A)$; *in other words, for all* $f, h \in \hom(B, A)$, *there is a* $g \in aut(A)$ *so that* $f = gh$.

The class of uniquely A-colourable structures in $\mathcal{K}$ is written $C!_{\mathcal{K}}(A)$.

Remark 3.20 1. Item (2) implies the action in (3) is *faithful*; that is, the permutation representation

$$\varphi : aut(A) \to S_{\hom(B,A)}$$

is injective. To see this, let $g, h \in aut(A)$ so that $\varphi(g) = \varphi(h)$. Fix $x \in A$, $f \in \hom(B, A)$, and $y \in B$ so that $f(y) = x$.

Then

$$\begin{aligned} g(x) &= g(f(y)) \\ &= h(f(y)) = h(x). \end{aligned}$$

2. The class $C!_{\mathcal{K}}(A)$ need not be elementary. For example, if $\mathcal{K}$ is the class of graphs, and A is K_n for $n \geq 2$, then each uniquely A-colourable graph is connected (see Theorem 12.16 in [17]).

Lemma 3.21 *Every core A is uniquely A-colourable; in particular, $C!_{\mathcal{K}}(A) \neq \emptyset$.*

Proof. As A is core, $\hom(A, A) = aut(A)$. □

Definition 3.22 *Let $A \in \mathcal{K}$. $a \in A$ is* ***totally isolated*** *if*

1. *a is an isolated point in $G(A)$;*

2. *the relations of $A \restriction \{a\}$ are all empty.*

Definition 3.23 *$A \in \mathcal{K}$ is* ***non-unit*** *if for some $R \in L$, $R^A \neq \emptyset$; otherwise, A is* ***unit****.*

We note the following observation.

Lemma 3.24 *No non-unit core A has a totally isolated element.*

Proof. We may assume $|A| > 1$.

Let A have a totally isolated element a. Let $b \in A - \{a\}$. Define a map $f : A \to A$ so that $f(x) = \begin{cases} b & \text{if } x = a \\ x & \text{else} \end{cases}$. Then f is an endomorphism of A that is not onto. □

3.3.2 A different definition?

The usual definition of a "uniquely n-colourable graph" makes no mention of automorphisms; indeed, the usual definition replaces item (3) of Definition 3.19 with "the kernel of each $f \in \hom(B, A)$ induces the same partition of B" (see p. 137 of [17]). In this section, we show that the two different definitions give rise to two distinct classes. The class $C!_{\mathcal{K}}(A)$ is the "right" class for our investigations, as the set of strong amalgamation bases in $C_{\mathcal{K}}(A)$ is precisely $C!_{\mathcal{K}}(A)$ (see Proposition 3.39 below).

Definition 3.25 *Let $A \in \mathcal{K}_{fin}$ be a core. B is* ***uniquely**** A-***colourable*** *if*

1. $B \to A$;
2. *every $f \in \hom(B, A)$ is onto;*
3. *the kernel of each $f \in \hom(B, A)$ induces the same partition of B.*

The class of uniquely A-colourable structures in $\mathcal{K}$ is written $C *_{\mathcal{K}} (A)$.*

When $\mathcal{K}$ is the class of graphs, and $A = K_n$ for $n \geq 1$, then

$$C!_{\mathcal{K}}(A) = C *_{\mathcal{K}} (A).$$

The reason for this is that $aut(A) = S_n$, the symmetric group of order n.

In general, however,

$$C!_{\mathcal{K}}(A) \subsetneqq C *_{\mathcal{K}} (A).$$

For example, let $\mathcal{K}$ be the class of graphs, and let G and H be the graphs in Figures 3.1 and 3.2.

As H is a subgraph of G, $H \to G$. As $\chi(H) = 4$ and G is critical and 4-chromatic, every homomorphism of H is onto G. In particular, as $|H| = |G|$ every $f \in \hom(H, G)$ induces the singleton partition of H. Hence, $H \in C *_{\mathcal{K}} (A)$.

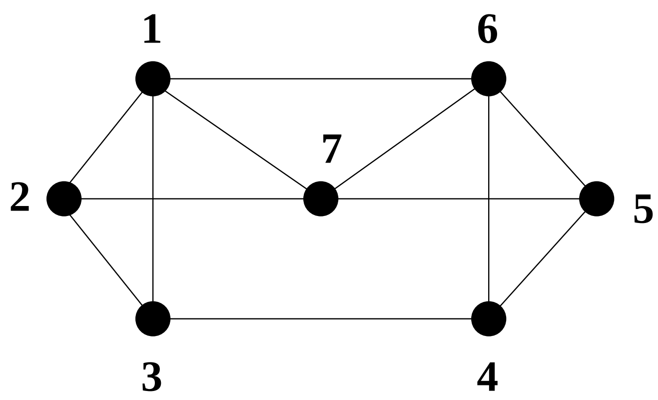

Figure 3.1: The graph G.

Let f be the identity map from H to G, and h be the map $(12) \in S_7$. Then h is a homomorphism from H to G. To see this use the fact that 1 and 2 have the same neighbours in H.

Claim: There does not exist $g \in aut(G)$ so that $f = gh$.

To see this, note that as $h(2) = 1$,

$$\begin{aligned} g(1) &= g(h(2)) \\ &= f(2) = 2. \end{aligned}$$

Further, as $h(1) = 2$,

$$\begin{aligned} g(2) &= g(h(1)) \\ &= f(1) = 1. \end{aligned}$$

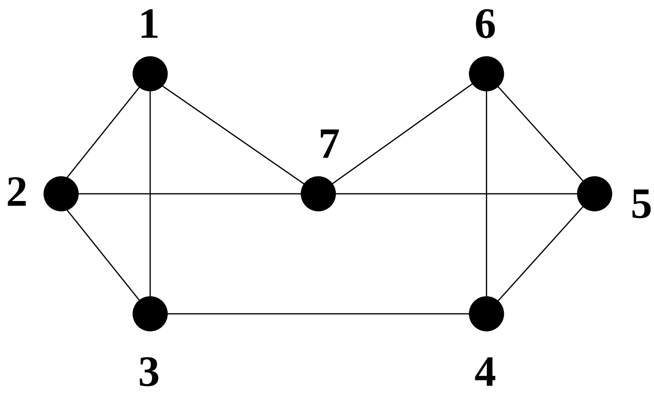

Figure 3.2: The graph H.

If $x \notin \{1,2\}$ then $h(x) = x$ so that

$$\begin{aligned} g(x) &= g(h(x)) \\ &= f(x) = x. \end{aligned}$$

However, g is not even an endomorphism of G: $1E^G6$ but $\neg 2E^G6$.

Problem: Given $\mathcal{K} \subseteq \mathcal{K}(L)$ an $\forall_1$ class, classify those cores $A \in \mathcal{K}_{fin}$ so that $C!_{\mathcal{K}}(A) = C *_{\mathcal{K}} (A)$.

We have a partial answer to this problem for graphs.

Lemma 3.26 *Let $\mathcal{K}$ be the class of all graphs, and G be a core graph with the property that for every proper subgraph G' of G,*

$$|Core(G')| < |G|.$$

Then $C!_{\mathcal{K}}(G) = C *_{\mathcal{K}} (G)$.

PROOF. Let $H \in C *_{\mathcal{K}} (G)$, with $f \in \hom(H, G)$.

Claim 1: If $xE^G y$ then there exists $a \in f^{-1}(x)$ and $b \in f^{-1}(y)$ so that $aE^H b$.

Otherwise, let G' be the subgraph of G formed by removing the edge xy. Then H maps onto G' which maps onto $Core(G')$. But then by hypothesis, H maps homomorphically onto a proper subgraph of G, contradicting that $H \in C *_{\mathcal{K}} (G)$.

Claim 2: $H \in C!_{\mathcal{K}}(G)$.

Let $f, h \in \hom(H, G)$ so that f, h are onto.

For $a \in G$, let $b \in f^{-1}(a)$. Define

$$g(a) = h(b).$$

Then g is a well-defined map as $H \in C *_{\mathcal{K}} (G)$ (f and h share the same kernel).

Claim 3: $g \in aut(G)$.

As G is core, it is enough to show that g is a homomorphism.

Let $xE^G y$. By Claim 1, there are $a \in f^{-1}(x)$ and $b \in f^{-1}(y)$ so that $aE^H b$. But then $h(a)E^G h(b)$ so that $g(x)E^G g(y)$.

Claim 3 establishes Claim 1.

□

Example 3.27 By Lemma 3.26, the following graphs satisfy

$$C!_{\mathcal{K}}(G) = C *_{\mathcal{K}} (G),$$

with $\mathcal{K}$ equal to the class of all graphs.

1. K_n, $n \geq 2$.

2. C_{2n+1}, $n \geq 1$.

3. For graphs G,H, define $G + H$ to have vertices $G \uplus H$ and edges

$$E^G \cup E^H \cup \{(x, y), (y, x) : x \in G, y \in H\}.$$

$G + H$ is sometimes called the *join* of G and H.

Then $C_{2n+1} + K_m$, $n, m \geq 1$ satisfy the conditions of Lemma 3.26.

3.3.3 Fixations

We introduce a construction that will be useful in the sequel.

Definition 3.28 *Let $B \in \mathcal{K}(L)$, $A \in \mathcal{K}(L)_{fin}$ with A a core. Let $f : B \to A$ be a homomorphism. The A-**fixation of** (B, f) is the following L-structure $B(f)$.*

1. *The domain of $B(f)$ is $B \uplus A$.*

2. *Define a tuple $\bar{a}$ from $B(f)$ to be **mixed** if*

$$\bar{a} \cap B, \bar{a} \cap A \neq \emptyset.$$

For a fixed $R \in L$,

$$R^{B(f)} = R^B \cup R^A \cup \{\bar{a} \in B(f) : \bar{a} \textit{ is mixed and } (f \cup id_A)(\bar{a}) \in R^A\}.$$

Remark 3.29 *1.* With A, B, f as in Definition 3.28, $A, B \leq B(f)$ and $f \cup id_A : B(f) \to A$ is a homomorphism.

2. If $B \leq C$ and $f \in \hom(C, A)$ then $B(f \restriction B) \leq C(f)$.

Definition 3.30 *Let* $\mathcal{K} \subseteq \mathcal{K}(L)$*.* $\mathcal{K}$ ***has fixations*** *if for each* $A \in \mathcal{K}_{fin}$*,* $B \in C_{\mathcal{K}}(A)$*, and* $f \in \hom(B, A)$*, we have* $B(f) \in \mathcal{K}$*.*

An A-fixation of B may not be in $\mathcal{K}$, as the following example illustrates.

Example 3.31 Let $\mathcal{K}$ be the class of tournaments, let $A = \{1, 2\}$ be the directed edge (oriented from 1 to 2), $B = K_1 = \{3\}$. Then $B \in C_{\mathcal{K}}(A)$ with $f : B \to A$ defined by $f(3) = 1$.

But $B(f)$ has no edge between 1 and 3 and so is not a tournament. See Figure 3.31.

Definition 3.32 *Let* L *be a relational language.*

1. *An* L*-sentence is of* ***type*** 3 *if it is of the form:*

$$\forall x_1 \dots x_n (\bigvee \bigwedge \neg R_i(\overline{x_i}) \vee \bigvee \bigwedge R_j(y_{j_1} \dots y_{j_m})), \tag{3.3}$$

with $\{x_1, \dots, x_n\} = \{y_{j_1}, \dots, y_{j_m}\}$*,* $R_i \in L \cup \{=\}$ *and* $R_j \in L$*.*

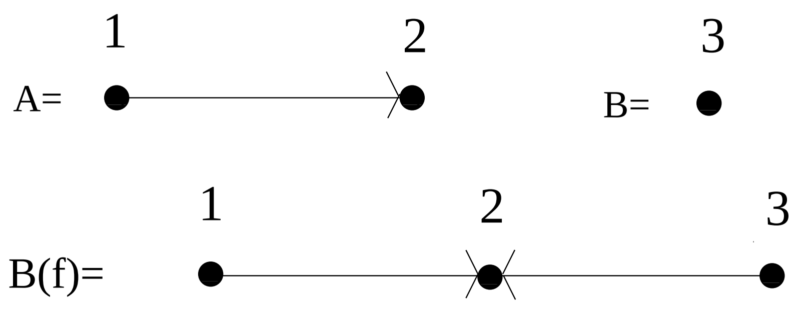

Figure 3.3: $B(f)$ may not be in $\mathcal{K}$.

2. A set Σ of L-sentences is ***type*** 3 *if each $\theta \in \Sigma$ is type 3.*

Example 3.33 The usual axiomatization of tournaments is not type 3 as there is the non-type 3 sentence:

$$\forall xy(x = y \vee Rxy \vee Ryx).$$

Theorem 3.34 *Let L be a finite relational language, and let $\mathcal{K} \subseteq \mathcal{K}(L)$ be $\forall_1$ with a type 3 axiomatization. Then fixations exist in $\mathcal{K}$.*

Proof. Let Σ be a type 3 axiomatization of $\mathcal{K}$.

Let $A, B \in \mathcal{K}$ with A finite and $B \in C_{\mathcal{K}}(A)$. Fix $f : B \to A$ a homomorphism with fixation $B(f)$.

If $B(f) \notin \mathcal{K}$, then there is some $\theta \in \Sigma$ so that

$$B(f) \not\models \theta.$$

Let $\{M_i : i \in I\}$ be the set of minimal constraints (see Theorem 1.20) for the $\forall_1$ class axiomatized by θ. Choose $M \in \{M_i : i \in I\}$ so that $M \leq B(f)$ and

$$M \nvDash \theta.$$

As $A, B \in \mathcal{K}$, and $A, B \leq B(f)$, it must follow that $M \cap A$, $M \cap B \neq \emptyset$.

Let $g = f \cup id_A : B(f) \to A$.

Claim 1: $g \upharpoonright M$ preserves $\neg\theta$; that is,

$$A \upharpoonright Im(g \upharpoonright M) \vDash \neg\theta.$$

Once Claim 1 is proven, we will be finished as we will obtain the contradiction that $A \vDash \neg\theta$ (as $\neg\theta$ is $\exists_1$).

Assume θ is of the form

$$\forall x_1 \ldots x_n (\bigvee \bigwedge \neg R_i(\overline{x_i}) \vee \bigvee \bigwedge R_j(y_{j_1} \ldots y_{j_m}))$$

with $\{x_1, \ldots, x_n\} = \{y_{j_1}, \ldots, y_{j_m}\}$ and $R_j \in L$; then $\neg\theta$ is equivalent to a sentence of the form

$$\exists x_1 \ldots x_n (\bigwedge \bigvee R_i(\overline{x_i}) \wedge \bigwedge \bigvee \neg R_j(y_{j_1} \ldots y_{j_m}))$$

with $\{x_1, \ldots, x_n\} = \{y_{j_1}, \ldots, y_{j_m}\}$ and $R_j \in L$.

As

$$M \vDash \neg\theta, \tag{3.4}$$

let $\bar{a} \subseteq M$ be a witness in $B(f)$ to (3.4), with $\overline{a_i} \subseteq \bar{a}$ so that

$$B(f) \models \bigwedge \bigvee R_i(\overline{a_i}),$$

and $\{b_{j_1}, \ldots, b_{j_m}\} = \{a_1, \ldots, a_n\}$ so that

$$B(f) \models \bigwedge \bigvee \neg R_j(\overline{b_j}).$$

Note that in fact $\bar{a} = M$ by the minimality of M.

Claim 2: $A \restriction Im(g \restriction M) \models \bigwedge \bigvee R_i((g \restriction M)(\overline{a_i}))$.

Claim 2 is immediate as g and hence $g \restriction M$ is a homomorphism.

Claim 3: $A \restriction Im(g \restriction M) \models \bigwedge \bigvee \neg R_j((g \restriction M)(\overline{b_j}))$.

By the definition of $B(f)$, for $\bar{b}$ mixed from $B(f)$,

$$\bar{b} \in R^{B(f)} \text{ iff } g(\bar{b}) \in R^A. \tag{3.5}$$

We note that key fact that $\overline{b_j}$ is mixed; the reason for this is that $\{x_1, \ldots, x_n\} = \{y_{j_1}, \ldots, y_{j_m}\}$ and so $\overline{b_j} = \bar{a} = M$ (and recall $M \cap A, M \cap B \neq \emptyset$).

Hence, as

$$B(f) \models \bigwedge \bigvee \neg R_j(\overline{b_j})$$

and as $R_j \in L$, by (3.5) we have that

$$A \restriction Im(g \restriction M) \models \bigwedge \bigvee \neg R_j((g \restriction M)(\overline{b_j}))$$

as desired.

Claim 2 and Claim 3 together prove Claim 1 which finishes the proof. □

Problem: Find a necessary and sufficient condition (syntactic or via constraints) characterizing $\forall_1$ classes with fixations.

3.3.4 Cofinality

Recall the following definition from Chapter 1.

Definition 3.35 *Let $\mathcal{K} \subseteq \mathcal{K}(L)$. A class $\mathcal{K}' \subseteq \mathcal{K}_{fin}$ is **cofinal** in $\mathcal{K}_{fin}$ if for each $A \in \mathcal{K}$ there is a $B \in \mathcal{K}'$ so that $A \leq B$.*

This section is devoted to proving the following theorem.

Theorem 3.36 *Let L be a finite relational language without unary predicates, and let $\mathcal{K} \subseteq \mathcal{K}(L)$ be $\forall_1$ with fixations.*

If $A \in \mathcal{K}_{fin}$ is a core, then $C!_{\mathcal{K}}(A)$ is cofinal in $C_{\mathcal{K}}(A)$.

In particular, for all $B \in C_{\mathcal{K}}(A)$ there is a $B' \in C!_{\mathcal{K}}(A)$ with $B \hookrightarrow B'$ and $|B'| \leq |B| + |A|$.

The condition that L contain no unary predicates cannot be weakened as the following example illustrates.

Let $L = \{P, Q, E\}$, with P, Q unary and E binary. Let $\mathcal{K} \subseteq \mathcal{K}(L)$ be the class of L-structures with graph $\{E\}$-reduct.

Let A be the L-structure depicted in Figure 3.4.

Claim 1: A is a core and is rigid (that is, $aut(A) = 1$).

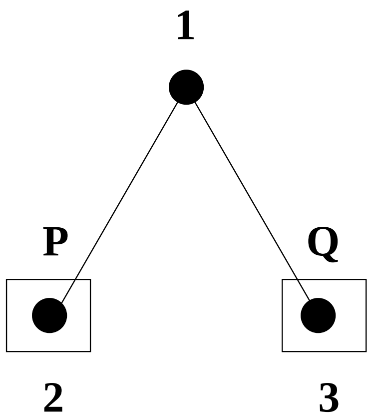

Figure 3.4: The structure A.

Let $f : A \to A$ be a homomorphism; then $f(2) = 2$ and $f(3) = 3$. If $f(1) \neq 1$, without loss of generality, let $f(1) = 2$. Then $1E^A2$ implies $f(1)E^Af(2)$, so that $2E^A2$, contradiction. The same problem arises if $f(1) = 3$.

Claim 2: $B \in C!_{\mathcal{K}}(A)$ if and only if $|B| \geq |A|$ and $|\hom(B, A)| = 1$.

Claim 2 follows as A is rigid by Claim 1.

Let B be the L-structure depicted in Figure 3.5.

$B \in C_{\mathcal{K}}(A)$ as the map sending $4, 6, 7$ to $1, 2, 3$ respectively, and 5 to 2 is a homomorphism.

Claim 3: B has no extension to any $C \in C!_{\mathcal{K}}(A)$.

Assume otherwise, with $B \leq C$ and $f : C \to A$ a homomorphism.

Then $f(6) = 2$ and $f(7) = 3$. We must have that $f(4) = 1$ (as the f-image of 4 must be adjacent to both the f-images of 6 and 7) and so $f(5) \in \{2, 3\}$.

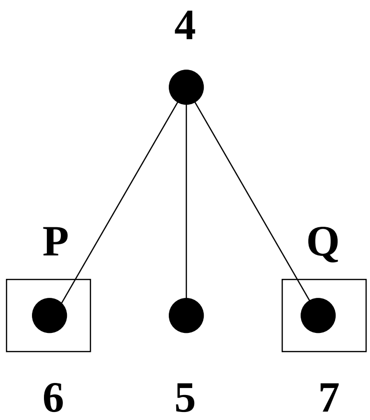

Figure 3.5: The structure B.

Assume $f(5) = 2$.

Define $f' : C \to A$ so that $f'(x) = \begin{cases} 3 & \text{if } x = 5 \\ f(x) & \text{else} \end{cases}$.

Claim 4: $f' \in \hom(C, A)$.

f' preserves P, Q and $xE^C y$ when $x, y \notin \{5\}$.

Suppose that $5E^C y$ (so $y \neq 5$). Then $f(5)E^A f(y)$, so that $2E^A f(y)$. But then $f(y) = 1$ and so $f'(y) = 1$.

But now $f'(5) = 3$, $f'(y) = 1$, and $3E^A 1$ implies that f' is a homomorphism.

As $f' \neq f$, by Claim 2, $C \notin C!_{\mathcal{K}}(A)$.

The proof of Claim 4 completes the proof of Claim 3.

Because of the previous example, we tacitly make the following assumption throughout the rest of the chapter.

Main Assumption: L contains no unary predicates.

Proof of Theorem 3.36

Let $f : B \to A$ be a homomorphism.

The desired B' in the conclusion of the theorem will be the fixation $B(f)$.

Claim 1: $B(f) \in C_{\mathcal{K}}(A)$.

The map $f' = f \cup id_A$ is a homomorphism.

Claim 2: Any $g \in \hom(B(f), A)$ is onto.

This follows as $A \leq B(f)$ and A is a core.

Claim 3: $B(f) \in C!_{\mathcal{K}}(A)$.

Let $f'' : B(f) \to A$ be a homomorphism. As A is a core, $f'' \upharpoonright A \in aut(A)$, so if we let $g = (f'' \upharpoonright A)^{-1}$ then gf'' is the identity on A in $B(f)$.

We show that $f' = gf''$. To obtain a contradiction, assume that

$$gf'' \upharpoonright B \neq f.$$

Without loss of generality, there are $i, k \in A$ and $x \in (f)^{-1}(k)$ so that

$$gf''(x) = i \neq k.$$

If there exist tuples $\bar{a}, \bar{b} \subseteq A$ so that for $R \in L$ fixed,

$$(\bar{a}k\bar{b}) \in R^A$$

then

$$(\bar{a}x\bar{b}) \in R^{B(f)},$$

by the definition of $B(f)$.

As gf'' is a homomorphism,

$$(\bar{a}i\bar{b}) \in R^A.$$

Hence,

$$(\bar{a}k\bar{b}) \in R^A \text{ implies } (\bar{a}i\bar{b}) \in R^A. \tag{3.6}$$

Define $f^3 : A \to A$ by

$$f^3(z) = \begin{cases} i & \text{if } z = k \\ z & \text{else} \end{cases}.$$

Claim 4: f^3 is a homomorphism.

As f^3 is the identity off of k, it is enough to show that f^3 preserves relations of the form

$$\bar{a} = (\bar{a}_1 k \bar{a}_2 \ldots \bar{a}_{n-1} k \bar{a}_n) \in R^A, \tag{3.7}$$

with $n \geq 1$, $R \in L$, and for $1 \leq i \leq n$, $\bar{a}_i$ a tuple from $A - \{k\}$ that is possibly empty.

Then $\bar{a} \in R^A$ implies that

$$\bar{b} = (\bar{a}_1 x \bar{a}_2 k \ldots \bar{a}_{n-1} k \bar{a}_n) \in R^{B(f)} \tag{3.8}$$

as $\bar{b}$ is mixed and by the definition of $B(f)$.

Note that $\bar{b}$ is chosen so as to replace only one "k" in $\bar{a}$.

As gf'' is a homomorphism, we have that

$$(\bar{a}_1 i \bar{a}_2 k \ldots \bar{a}_{n-1} k \bar{a}_n) \in R^{B(f)}, \tag{3.9}$$

by (3.6).

Proceeding inductively, we obtain that

$$(\bar{a}_1 i \bar{a}_2 i \ldots \bar{a}_{n-1} i \bar{a}_n) \in R^{B(f)},$$

which proves Claim 4 (note that the above procedure works even for tuples of the form $(k, \ldots, k)$).

By Claim 4, f^3 is a homomorphism. f^3 is not surjective, contradicting that A is a core in $\mathcal{K}$. □

Remark 3.37 *The procedure of "replacing k's by x's" in the proof of Theorem 3.36 breaks down if $ar(R) = 1$.*

3.4 Uniquely colourables have free amalgamation

In this section, we find that while $C_{\mathcal{K}}(A)$ may not have AP in general, $C!_{\mathcal{K}}(A)$ has FAP.

Definition 3.38 *1. $B \in \mathcal{K}$ is a* ***strong amalgamation base*** *in $\mathcal{K}$ if for all embeddings $e : B \to C$ and $f : B \to D$ of B into $C, D \in \mathcal{K}$, there is*

$E \in \mathcal{K}$ and embeddings $g : C \to E$ and $h : D \to E$ so that $ge = hf$ and $g(C) \cap h(D) = ge(B)$.

2. *$B \in \mathcal{K}$ is a **free amalgamation base** in $\mathcal{K}$ if for all $C, D \in \mathcal{K}$ so that C and D agree and $C \cap D = B$, we have that $C \cup D \in \mathcal{K}$.*

Proposition 3.39 *Let $\mathcal{K}$ be an $\forall_1$ free amalgamation class with fixations, and let $A \in \mathcal{K}_{fin}$ be a core in $\mathcal{K}$. Let $B \in C_{\mathcal{K}}(A)$. The following are equivalent.*

1. *B is a strong amalgamation base for $C_{\mathcal{K}}(A)$.*
2. *B is a free amalgamation base for $C_{\mathcal{K}}(A)$.*
3. *$B \in C!_{\mathcal{K}}(A)$.*

Proof. Clearly $(2 \Rightarrow 1)$.

$(1 \Rightarrow 2)$ Let A' be a strong amalgamation base for $C_{\mathcal{K}}(A)$. Let $B, C \in C_{\mathcal{K}}(A)$ so that $A' \leq B, C$, $A' = B \cap C$, and so B, C agree. Let $e : A' \to B$ and $f : A' \to C$ be the inclusion maps; by hypothesis there is $D \in C_{\mathcal{K}}(A)$ and embeddings $g : B \to D$ and $h : C \to D$ so that $ge = hf$ and $g(B) \cap h(C) = ge(A')$. Then $B \cup C \to D \to A$ implies $B \cup C \to A$, so that $B \cup C \in C_{\mathcal{K}}(A)$.

$(2 \Rightarrow 3)$ We show that if $A' \notin C!_{\mathcal{K}}(A)$ then some "free amalgamation problem" over A' fails in $C_{\mathcal{K}}(A)$. If $A' \notin C!_{\mathcal{K}}(A)$ then there are $f, h \in \hom(A', A)$ so that there does not exist $g \in aut(A)$ so that $f = g \circ h$ (in particular, $f \neq h$, otherwise, we may choose $g = id_A$).

Let $B = (A')(f)$ and $C = (A')(h)$ with $B \cap C = A'$.

Claim: $B \cup C \notin C_{\mathcal{K}}(A)$.

Otherwise, let $j \in \hom(B \cup C, A)$.

Then $j \restriction B$, and $j \restriction C$ are homomorphisms.

As $B, C \in C!_{\mathcal{K}}(A)$, by Claim 3 in the proof of Theorem 3.36, there are $\alpha, \beta \in aut(A)$ so that

$$\begin{aligned} j &\restriction & B = \alpha \circ (f \cup id_A), \\ j &\restriction & C = \beta \circ (h \cup id_A). \end{aligned}$$

Hence,

$$\alpha \circ (f \cup id_A) \restriction A' = \beta \circ (h \cup id_A) \restriction A'.$$

Now, for $x \in A'$,

$$\begin{aligned} \alpha \circ f(x) &= \alpha \circ (f \cup id_A)(x) \\ &= \beta \circ (h \cup id_A)(x) \\ &= \beta \circ h(x), \end{aligned}$$

so that

$$\alpha \circ f = \beta \circ h.$$

But then

$$\beta^{-1}\alpha \circ f = h. \tag{3.10}$$

As $\beta^{-1}\alpha \in aut(A)$, we have contradicted our hypothesis.

(3 ⇒ 2) Let $A' \in C!_{\mathcal{K}}(A)$ with $B, C \in C_{\mathcal{K}}(A)$ so that $A' \leq B, C$, $A' = B \cap C$, and so B, C agree.

Let $f : B \to A$ and $h : C \to A$ be homomorphisms. As $f \upharpoonright A'$ and $h \upharpoonright A'$ are homomorphisms, and as $A' \in C!_{\mathcal{K}}(A)$, there is $g \in aut(A)$ so that $f \upharpoonright A' = g \circ h \upharpoonright A'$.

Define $h' = g \circ h$; then $h' \in \hom(C, A)$.

By construction, f and h' agree on A' so that $f \cup h' \in \hom(B \cup C, A)$, and hence, $B \cup C \in C_{\mathcal{K}}(A)$. □

Corollary 3.40 *With $\mathcal{K}$ and A as in Proposition 3.39, $C_{\mathcal{K}}(A)^{ec} \subseteq C!_{\mathcal{K}}(A)$.* □

Proof. Each $B \in C_{\mathcal{K}}(A)^{ec}$ is a strong amalgamation base in $C_{\mathcal{K}}(A)$ (see Corollary 8.6.2 of [23]); now apply Proposition 3.39. □

Proposition 3.41 *If $\mathcal{K}$ and A are as in Proposition 3.39 then $C!_{\mathcal{K}}(A)$ and $C!_{\mathcal{K}}(A)_{fin}$ both satisfy JEP and FAP.*

Proof. Let $B, C \in C!_{\mathcal{K}}(A)$. Then

$$B \uplus C \in C_{\mathcal{K}}(A);$$

let $f : B \uplus C \to A$ be a homomorphism. Then

$$(B \uplus C)(f) \in C!_{K}(A),$$

by Claim 3 in the proof of Theorem 3.36. $(B \uplus C)(f)$ embeds B and C, and hence, $C!_{\mathcal{K}}(A)$ has JEP. The same argument works for $C!_{\mathcal{K}}(A)_{fin}$.

Now, let $A', B, C \in C!_{\mathcal{K}}(A)$ with $A' \leq B, C$, $A' = B \cap C$, and so B, C agree over A'.

By the proof of Proposition 3.39, $B \cup C \in C_{\mathcal{K}}(A)$. If $f : B \cup C \to A$ is a homomorphism, then $f \restriction A'$ is onto, and hence, f is onto. To show $B \cup C \in C!_{\mathcal{K}}(A)$ it remains to show item (3) of Definition 3.19.

Now, let $f, h \in \hom(B \cup C, A)$.

Define $f_1 = f \restriction B$, $f_2 = f \restriction C$, $h_1 = h \restriction B$, $h_2 = h \restriction C$.

The following hold.

1. f_i, h_i are homomorphisms, $i \in \{1, 2\}$;
2. $f_1 \restriction A' = f_2 \restriction A'$ and $h_1 \restriction A' = h_2 \restriction A'$;
3. $f = f_1 \cup f_2$ and $h = h_1 \cup h_2$.

As $B, C \in C!_{\mathcal{K}}(A)$, there are g_1, $g_2 \in aut(A)$ so that

$$f_1 = g_1 \circ h_1 \text{ and } f_2 = g_2 \circ h_2. \tag{3.11}$$

Claim 2: $g_1 = g_2$.

Let $x \in A$ be fixed. As $A' \in C!_{\mathcal{K}}(A)$ there is $y \in A'$ so that $h_1(y) = x$ (as every A-colouring of A' is onto).

Then

$$\begin{aligned} g_1(x) &= g_1(h_1(y)) \\ &= f_1(y) \\ &= f_2(y) \\ &= g_2(h_2(y)) \\ &= g_2(x), \end{aligned}$$

the second equality holding by (3.11), the third by item (2) above, the fourth by (3.11), and the fifth by (2) again. This proves Claim 2.

Let $g = g_1 = g_2$.

Hence, by item (3), (3.11), and Claim 2,

$$\begin{aligned} f &= f_1 \cup f_2 = g_1 \circ h_1 \cup g_2 \circ h_2 \\ &= g \circ h_1 \cup g \circ h_2 \\ &= g \circ (h_1 \cup h_2) = g \circ h. \end{aligned}$$

The same argument works for $C!_{\mathcal{K}}(A)_{fin}$.

□

Corollary 3.42 $C!_{\mathcal{K}}(A)$ *is closed under unions of chains.*

Proof. The proof follows by Proposition 3.39, and by the fact that the subclass of strong amalgamation bases of the class of models of an $\forall_1$ class are closed under unions of chains (see Ch. 8.6, Exercise 7, [23]). □

3.5 The generic $M(A)$

Throughout this section, $\mathcal{K}$ is an $\forall_1$ free amalgamation class with a type 3 axiomatization, and $A \in \mathcal{K}_{fin}$ is a core in $\mathcal{K}$. Each class $C_{\mathcal{K}}(A)$ gives rise to a fascinating countable structure whose properties are crucial for our discussion.

Theorem 3.43 *There is an $\aleph_0$-categorical $C!_{\mathcal{K}}(A)_{fin}$-generic $M(A)$.*

Proof. By Proposition 3.41, $C!_{\mathcal{K}}(A)_{fin}$ satisfies JEP' and AP', with $\leq^* = \leq$. Further, as $|L|$ is finite, $C!_{\mathcal{K}}(A)_{fin}$ contains only countably many isomorphism types. By Theorem 1.63, there is a $(C!_{\mathcal{K}}(A)_{fin}, \leq)$-generic $M(A)$.

Let $S \leq M(A)$ with S finite. By item (4a) of Definition 1.61, there is $T \in C!_{\mathcal{K}}(A)_{fin}$ so that $S \leq T$. Fix $h : T \to A$ a homomorphism. By Theorem 3.34 fixations exist in $\mathcal{K}$, and so by Claim 3 in the proof of Theorem 3.36, $T(h) \in C!_{\mathcal{K}}(A)_{fin}$. As $T \leq T(h)$, by item (4c) of Definition 1.61, we can amalgamate $T(h)$ into $M(A)$ as B'. Define $B = S(h \restriction S) \in C!_{\mathcal{K}}(A)_{fin}$. By Remark 3.29 (2), $B \leq B'$. Define $f : \omega \to \omega$ by $f(0) = 0$ and $f(n) = n + |A|$. Then f verifies item (3) of Theorem 1.68. By Theorem 1.68, $M(A)$ is $\aleph_0$-categorical. $\square$

3.6 An axiomatization of $Th(M(A))$

Again, $\mathcal{K}$ is an $\forall_1$ free amalgamation class with a type 3 axiomatization, and $A \in \mathcal{K}_{fin}$ is a core in $\mathcal{K}$.

Definition 3.44 *Let $n \geq 1$.*

1. *Let* Θ *consist of the* $\exists_1$ *L-sentences*

$$\exists \bar{x} qft_{A',\bar{a}}(\bar{x}),$$

where $|\bar{x}| = |\bar{a}| = n$, $A' \in C!_{\mathcal{K}}(A)_n$, *and* $\bar{a}$ *enumerates* A', *for each* $n \geq 1$.

2. *Define*

$$\varphi_n = \bigwedge \forall \bar{x} \exists \bar{y} (qft_{A',\bar{a}}(\bar{x}) \to qft_{B,\bar{a}\bar{b}}(\bar{x}\bar{y})),$$

where the conjunction is over all sets of the form $(A', B, \bar{a}, \bar{b})$ *where*

   (a) A', B *are isomorphism types of structures in* $C!_{\mathcal{K}}(A)_{fin}$ *with* $A' \leq B$ *and* $|A'| \leq n$ *and* $|B| \leq n+1$;

   (b) $\bar{a}$ *is a set of distinct elements from* A' *so that* $A' = \bar{a}$, $|\bar{a}| = |\bar{x}|$;

   (c) $\bar{b}$ *is a set of distinct elements from* B *so that* $B = \bar{a}\bar{b}$, $|\bar{b}| = |\bar{y}|$, *and* $|\bar{x}| + |\bar{y}| \leq n+1$.

Define $\Sigma_0 = \Theta \cup \{\varphi_n : n \in \omega^*\}$; Σ_0 *is the set of* $C!_{\mathcal{K}}(A)_{fin}$-***extension axioms***.

3. *Define*

$$\psi_n = \bigwedge \forall \bar{x} (qft_{A',\bar{a}}(\bar{x}) \to \bigvee \exists \bar{y} qft_{B,\bar{a}\bar{b}}(\bar{x}\bar{y})),$$

where the conjunction is over all sets of the form $(A', \bar{a})$ *so that*

   (a) A' *is an isomorphism type of structure from* $C_{\mathcal{K}}(A)_{fin}$ *so that* $|A'| \leq n$;

   (b) $\bar{a}$ *is a set of distinct elements from* A' *so that* $A' = \bar{a}$, $|\bar{a}| = |\bar{x}|$;

and the disjunction is over all sets of the form $(B, \bar{b})$ so that

(a) *B is an isomorphism type of structure from $C!_{\mathcal{K}}(A)_{fin}$ so that $A' \leq B$, and $|B| \leq |A'| + |A|$;*

(b) *$\bar{b}$ is a set of distinct elements from B so that $B = \bar{a}\bar{b}$, and $|\bar{b}| = |\bar{y}|$.*

*Define $\Sigma_1 = \{\psi_n : n \in \omega^*\}$; Σ_1 is the set of axioms enforcing **uniform boundedness**.*

4. *Define $\Phi = Th(C_{\mathcal{K}}(A)) \cup \Sigma_0 \cup \Sigma_1$.*

Remark 3.45 1. *The reader may verify that each of φ_n and ψ_n, for $n \in \omega^*$, are first-order sentences.*

2. *If $S \in \mathcal{K}$ satisfies Σ_0 then S satisfies the following "algebraic" condition: if $A' \leq S$ with $A' \in C!_{\mathcal{K}}(A)_{fin}$ and $A' \leq B \in C!_{\mathcal{K}}(A)_{fin}$, then there is a $C \leq S$ with $C \in C!_{\mathcal{K}}(A)_{fin}$ and an isomorphism $f : B \to C$ so that $f \upharpoonright A'$ is the identity map.*

3. *If $S \in \mathcal{K}$ satisfies Σ_1 then S satisfies the following condition: each $A' \leq S$ with $A' \in C_{\mathcal{K}}(A)_{fin}$ is contained in $B \leq S$ so that $B \in C!_{\mathcal{K}}(A)_{fin}$ and $|B| \leq |A'| + |A|$.*

Proposition 3.46 *Φ axiomatizes $Th(M(A))$.*

PROOF. By genericity and by the proof of Theorem 3.43,

$$M(A) \models \Phi.$$

Let $B \models \Phi$; by the Löwenheim-Skolem Theorem (see Theorem 1.12) we may assume B is countable.

Claim: B satisfies item (4a), (4b), and (4c) of Definition 1.61.

As $B \models \Sigma_0$,

$$Age(B) \supseteq C!_{\mathcal{K}}(A)_{fin},$$

so B satisfies (4b).

As $B \models \Sigma_1$, B is a union of a chain of $C!_{\mathcal{K}}(A)_{fin}$-structures. To see this, let $B = \{b_i : i \in \omega\}$. Using Σ_1, embed b_0 in a $C!_{\mathcal{K}}(A)_{fin}$-structure B_0 inside B. In general, if $B_n \leq B$ is defined so that $B_n \in C!_{\mathcal{K}}(A)_{fin}$ and B_n contains b_n, define B_{n+1} to be a $C!_{\mathcal{K}}(A)_{fin}$-structure in B extending $\{b_{n+1}\} \cup B_n$ (again using Σ_1). Then

$$\{B_n : n \in \omega\}$$

is the desired chain. Hence, B satisfies (4a).

The fact that B satisfies (4c) follows by the $C!_{\mathcal{K}}(A)_{fin}$-extension axioms.

From the Claim and Theorem 1.63 (2), $B \cong M(A)$, so that

$$B \models Th(M(A)).$$

□

3.7 The model companion: an explicit axiomatization

Recall that $\mathcal{K}$ is an $\forall_1$ free amalgamation class with a type 3 axiomatization, and $A \in \mathcal{K}_{fin}$ is a core in $\mathcal{K}$. $M(A)$ is the $(C!_{\mathcal{K}}(A)_{fin}, \leq)$-generic, where $M(A)$ is as in Theorem 3.43.

Theorem 3.47 *$C_{\mathcal{K}}(A)^{mc}$ exists, and*

$$Th(C_{\mathcal{K}}(A)^{mc}) = Th(M(A)),$$

where $M(A)$ is as in Theorem 3.43.

Proof. We show that $B \in C_{\mathcal{K}}(A)^{ec}$ if and only if $B \models \Sigma_0 \cup \Sigma_1 \cup Th(C_{\mathcal{K}}(A))$.

Let $B \in C_{\mathcal{K}}(A)^{ec}$. Let $S, T \in C!_{\mathcal{K}}(A)_{fin}$ with $S \leq T$ and $S \leq B$. Let $S = \bar{a}$ and $T = \bar{a}\bar{b}$.

Amalgamate T and B over S to obtain $B \leq D \in C_{\mathcal{K}}(A)$. Then

$$D \models \exists \bar{x} qft_{T,\bar{a}\bar{b}}(\bar{a}, \bar{x});$$

as B is e.c. in $C_{\mathcal{K}}(A)$,

$$B \models \exists \bar{x} qft_{T,\bar{a}\bar{b}}(\bar{a}, \bar{x}). \tag{3.12}$$

The solution of (3.12) in B is a realization of T in B extending S.

Hence, $B \models \Sigma_0$.

Now, let S be a finite subset of B; we show that S is contained in a uniquely A-colourable substructure T of B of cardinality $\leq |S| + |A|$.

As $S \in C_{\mathcal{K}}(A)$, we may fix $f : S \to A$ a homomorphism. Then $S \leq S(f) \in C!_{\mathcal{K}}(A)_{fin}$ by Claim 3 in the proof of Theorem 3.36, and $|S(f)| = |S| + |A|$. Using the same argument as above, we can realize $S(f)$ in B as the desired substructure T.

Hence, $B \models \Sigma_1$ and the forward direction follows.

Conversely, assume $B \models \Sigma_0 \cup \Sigma_1 \cup Th(C_{\mathcal{K}}(A))$.

Let $C \in C_{\mathcal{K}}(A)$ extend B so that

$$C \models \exists \bar{x} \theta(\bar{x}, \bar{a})$$

with $\theta(\bar{x}, \bar{a})$ quantifier-free and $\bar{a} \subseteq B$; let $\bar{b}$ witness $\exists \bar{x} \theta(\bar{x}, \bar{a})$ in C. Let $S \leq B$ so that $S \in C!_{\mathcal{K}}(A)_{fin}$ and $\bar{a} \subseteq S$ (this is possible as $B \models \Sigma_1$).

Let $D = C \upharpoonright S \cup \bar{b}$, so that $D \in C_{\mathcal{K}}(A)_{fin}$; let $E \in C!_{\mathcal{K}}(A)_{fin}$ extend D.

As $B \models \Sigma_0$ there is a $E' \leq B$ and $\beta : E \to E'$ an isomorphism that is the identity on S.

As

$$E \models \exists \bar{x} \theta(\bar{x}, \bar{a}),$$

$$E' \models \exists \bar{x} \theta(\bar{x}, \bar{a}),$$

so

$$B \models \exists \bar{x} \theta(\bar{x}, \bar{a}).$$

□

Corollary 3.48 *$C_{\mathcal{K}}(A)$ has a complete, $\aleph_0$-categorical model companion. In particular, there is a countable universal $C_{\mathcal{K}}(A)$-structure.*

PROOF. $Th(M(A))$ is complete, and $\aleph_0$-categorical by Theorem 3.43; now apply Theorem 3.47. $M(A)$ is countable universal, as $M(A)$ is the unique countable e.c. in $C_{\mathcal{K}}(A)$ and by Lemma 1.32 (3). □

3.8 Nqfa(2)

Throughout, we assume the following.

1. A is a core in $\mathcal{K}_{fin}$ that is not unit (recall Definition 3.23).
2. $\mathcal{K}$ is an $\forall_1$ free amalgamation class with edges.
3. $\mathcal{K}$ has fixations.

(1) is a weak assumption, as we effectively omit a single A in our construction, namely, the unit A.

We will see that the conditions of (2) cannot be dropped in Section 3.9 below. (3) is the defect in the result; at this point, we do not know of a necessary and sufficient condition ensuring uniform boundedness of uniquely colourable extensions.

Definition 3.49 *1. Σ is the set of L-sentences axiomatizing "my age $\supseteq C_{\mathcal{K}}(A)_{fin}$".*

2. T is **nqfa**(2) *if T is not finitely axiomatizable modulo $\Sigma \cup \Sigma_1 \cup Th(C_{\mathcal{K}}(A))$.*

Proposition 3.50 *If A is unit, then $C_{\mathcal{K}}(A)^{mc}$ is qfa.*

PROOF. $B \in C_{\mathcal{K}}(A)$ if and only if every element of B is totally isolated. The model companion then just consists of infinite $\mathcal{K}(L)$-structures D so that $R^D = \emptyset$, for all $R \in L$, which is qfa. □

Theorem 3.51 *$C_{\mathcal{K}}(A)^{mc}$ is nqfa(2).*

Problem: If $\mathcal{K}$ is an $\forall_1$ free amalgamation class with edges without a type 3 axiomatization, and $A \in \mathcal{K}_{fin}$ is a non-unit core, is $C_{\mathcal{K}}(A)^{mc}$ nqfa(2)? Is $C!_{\mathcal{K}}(A)_{fin}$ cofinal in $C_{\mathcal{K}}(A)_{fin}$?

As in Chapter 2, we construct countably infinite models M_n of $\Sigma \cup \Sigma_1 \cup Th(C_{\mathcal{K}}(A))$ that satisfy *only finitely many* of the φ_n.

As A is not-unit, we may fix $x \in A$ so that x is in a tuple $\bar{a} \in R^A$ with $|\bar{a}| \geq 2$ ($\bar{a}$ may be constant).

Let

$$1(x) = A \restriction x.$$

For $n \geq 1$, if $n(x)$ is defined, let

$$n + 1(x) = n(x) \uplus 1(x).$$

From our assumptions on $\mathcal{K}$, $n(x) \in \mathcal{K}$ for every $n \geq 1$.

3.8.1 Building M_n

Define $M_n^{0\prime} = n+1(x)$. Enumerate $M_n^{0\prime}$ as

$$\{1, \dots, n+1\}.$$

Then $M_n^{0\prime} \in C_{\mathcal{K}}(A)$: the constant map $f : M_n^{0\prime} \to A$, $y \mapsto x$ is a homomorphism.

Define M_n^0 to be $M_n^{0\prime}(f) \in C!_{\mathcal{K}}(A)$.

For $k \geq 0$, assume M_n^k has been defined, contains M_n^0, and is uniquely A-colourable.

List the substructures of M_n^k in $C!_{\mathcal{K}}(A)$ with cardinality at most n as $S_1, \dots, S_i$. For each $1 \leq r \leq i$, list the extensions of S_r to a $C!_{\mathcal{K}}(A)$-structure of at most size $n+1$ elements as $T_1, \dots, T_j$. Freely amalgamate $T_1, \dots, T_j$ and M_n^k over S_r to obtain $M_{n,r}^{k+1}$. Freely amalgamate $M_{n,1}^{k+1}, \dots, M_{n,i}^{k+1}$ over M_n^k to obtain $M_n^{k+1\prime\prime} \in C!_{\mathcal{K}}(A)$. Define $M_n^{k+1\prime}$ to be the disjoint union of $M_n^{k+1\prime\prime}$ and one copy of every isomorphism type of $C_{\mathcal{K}}(A)_{k+1}$-structure. Then $M_n^{k+1\prime} \in C_{\mathcal{K}}(A)_{fin}$.

Define M_n^{k+1} to be $M_n^{k+1\prime}(h)$ for some $h \in \hom(M_n^{k+1\prime}, A)$, and observe that $M_n^k \leq M_n^{k+1\prime} \leq M_n^{k+1}$ by Remark 3.29 (1).

Define

$$M_n = \bigcup_{k \in \omega} M_n^k.$$

As $C!_{\mathcal{K}}(A)$ is closed under unions of chains (see Corollary 3.42), $M_n \in C!_{\mathcal{K}}(A)$.

By construction, $M_n \models \Sigma \cup \{\varphi_n\}$.

We show that

$$M_n \models \Sigma_1. \tag{3.13}$$

For this, let $S \leq M_n$ with S finite. Then there is some $k \in \omega$ so that $S \leq M_n^k$. Since $M_n^k \leq M_n^{k+1\prime}$, we have that $S \leq M_n^{k+1\prime}$. Recall that $M_n^{k+1} = M_n^{k+1\prime}(h)$, where h is a fixed element of $\hom(M_n^{k+1\prime}, A)$. Define $T = S(h \restriction S) \in C!_{\mathcal{K}}(A)_{fin}$. By Remark 3.29 (2), $T \leq M_n^{k+1}$. As $|T| = |S| + |A|$, (3.13) follows.

3.8.2 M_n does not satisfy all φ_n

We find a finite $C!_{\mathcal{K}}(A)$-extension D of M_n^0 that is not realized in M_n.

Define D as follows.

Let D' be the $\mathcal{K}(L)$-structure obtained by forming the free amalgam of two copies $M_n^0(1)$ and $M_n^0(2)$ of M_n^0 over $M_n^{0\prime}$. By hypothesis, $D' \in \mathcal{K}$. Let A_1, A_2 be the two (disjoint) copies of A in $M_n^0(1)$, $M_n^0(2)$, respectively, with $A_1 = A$. List the elements of A_2 as $\{y' : y \in A\}$.

Claim 1: $D' \in C_{\mathcal{K}}(A)$.

Define $f' = f \cup id_A \cup g$, where $g : A_2 \to A$ is the isomorphism given by $y' \mapsto y$. Then $f' \in \hom(D', A)$.

Define $D = D'(f') \in C!_{\mathcal{K}}(A)$. Let $A_3 = D \restriction (D - D')$. Note that $A_3 \cong A$.

We recall the definition of an edge in a structure $S \in \mathcal{K}(L)$ (see Definition 2.25). Let $\bar{a}$ be a tuple from S that contains at least two distinct elements so that there is some $R \in L$ with $\bar{a} \in R^S$. Then $S \restriction \bar{a}$ is called an *edge*.

Claim 2: There is a $z' \in A_2$ satisfying the following properties:

1. for each $i \in \{1, \ldots, n+1\}$, there is an edge of D containing $\{z', i\}$;

2. there is no edge of D with domain $\bar{a}$ satisfying:

   (a) $z' \in \bar{a} \subseteq \{z'\} \cup A_1$;

   (b) $\bar{a} \cap A_1 \neq \emptyset$.

x is not totally isolated in A: there is some tuple $\bar{a}$ (possibly constant) containing x so that $\bar{a} \in R^A$ for some $R \in L$ with $ar(R) \geq 2$. But then, by the definition of fixations, there is a $z \in \bar{a}$ that is adjacent to each element of $\{1, \ldots, n+1\}$ in $G(M_n^0)$. Then z' is adjacent to each element of $\{1, \ldots, n+1\}$ in $G(D')$, and so item (1) is verified for z'.

By freeness, z' is not in an edge of D' with an element of A_1. But $D' \leq D$ by Remark 3.29 (1). By this and the definition of fixations, if z' is contained in an edge of D with domain $\bar{a}$ that contains an element of A_1, then $\bar{a}$ must be a mixed tuple from D with

$$\bar{a} \cap A_3 \neq \emptyset.$$

Hence, item (2) is verified for z'.

Fix $z' \in A_2$ with properties as described in Claim 2.

It can be arranged that D is an extension of M_n^0 if we identify $M_n^0(1)$ with M_n^0.

Claim 3: D is not realized in M_n^0.

Immediate as $|D| > |M_n^0|$.

Claim 4: Assuming D is not realized in M_n^k then D is not realized in $M_n^{k+1\prime\prime}$.

By freeness, each element of $M_n^{k+1\prime\prime} - M_n^k$ can only be adjacent in $G(M_n \restriction M_n^{k+1\prime\prime})$ to at most n elements of M_n^0. But then no element of $M_n^{k+1\prime\prime} - M_n^k$ can realize the isomorphism type of z' in D by Claim 2 (1).

Claim 5: D is not realized in $M_n^{k+1\prime}$.

There are no edges between elements of $M_n^{k+1\prime} - M_n^{k+1\prime\prime}$ and $M_n^k \supseteq M_n^0$. Again apply Claim 2 (1).

Claim 6: D is not realized in $M_n^{k+1} - M_n^{k+1\prime}$.

If D were realized in $M_n^{k+1} - M_n^{k+1\prime}$, to obtain a contradiction, fix $w \in M_n^{k+1} - M_n^{k+1\prime}$ satisfying the properties of z' described in Claim 2. Define

$$B = M_n \restriction M_n^{k+1} - M_n^{k+1\prime}.$$

As A is a non-unit core and $A \cong B$, by Lemma 3.24, w is not totally isolated in B; let $R \in L$ and $\bar{a} \in R^B$ so that $w \in \bar{a}$ (again, $\bar{a}$ is possibly constant).

Recall that $h : M_n^{k+1\prime} \to B$ is a homomorphism; as A is a core and $A = A_1$, $h \restriction A_1$ is an isomorphism. Hence,

$$(h \restriction A_1)^{-1}(w) \in (h \restriction A_1)^{-1}(\bar{a}) \in R^{A_1};$$

in particular, $(h \restriction A_1)^{-1}(\bar{a})$ is the domain of an edge in A_1.

Consider the mixed tuple $\bar{b}$ of M_n^{k+1} obtained by replacing one instance of $(h \restriction A_1)^{-1}(w)$ in $(h \restriction A_1)^{-1}(\bar{a})$ by w (here we use the fact that $ar(R) \geq 2$). By the definition of fixations, $\bar{b} \in R^{M_n^{k+1}}$, and so $\bar{b}$ is the domain of an edge in M_n^{k+1}

satisfying items (2a) and (2b) of Claim 2 (with w replacing z' and $\bar{b}$ replacing $\bar{a}$). This is a contradiction.

By Claims 3-6, D cannot be realized in M_n.

In particular,

$$M_n \not\models \varphi_{3|A|+n}.$$

This completes the proof of Theorem 3.51.

3.9 Counterexamples

As we stated earlier, for the results stated in the Introduction, some of the main assumptions on A and $\mathcal{K}$ cannot be dropped without compromising the results in this chapter. In this brief section, we consider examples of this nature.

1. If A is unit, then $C_{\mathcal{K}}(A)^{mc}$ is qfa by Proposition 3.50.

2. If $\mathcal{K}$ is not a free amalgamation class, then $C_{\mathcal{K}}(A)^{mc}$ may even be finitely axiomatizable. For example, let $\mathcal{K}$ be the class of (reflexive) orders, and let A be the one-element order. Then $C_{\mathcal{K}}(A) = \mathcal{K}$ and $\mathcal{K}^{mc}$ is finitely axiomatizable by [1].

3. If $\mathcal{K}$ is a free amalgamation class, but has no edges, then $C_{\mathcal{K}}(A)^{mc}$ may be qfa(1), even if A is non-unit. For example, let $\mathcal{K} = \mathcal{K}(L)$, where $L = \{P\}$, P unary, A the one element structure whose domain is coloured P. Then $C_{\mathcal{K}}(A) = \mathcal{K}(L)$ whose model companion is qfa(1), by Example 2.24 in Chapter 2.

Chapter 4

Isometric Universal Structures

Given a class $\mathcal{K}$ whose structures are equipped with a metric, a natural question is whether there is a countable structure in $\mathcal{K}$ isometrically embedding each countable structure in $\mathcal{K}$. As proven by Moss in [33], for graphs with the usual "least path" metric, there is a countable graph (distinct from the random graph) that isometrically embeds every countable graph. In this chapter, we extend this result to $\mathcal{K}$ an $\forall_1$ class closed under unions.

Our goal is to prove the following results: for $\mathcal{K}$ an $\forall_1$ class closed under unions over a finite relational language,

1. there is a countable $\mathcal{K}$-structure M isometrically embedding (with respect to the metric of Definition 4.1 below) every countable $\mathcal{K}$-structure.

2. The class of distanced $\mathcal{K}$-structures, $\mathcal{K}^+$ (see Definition 4.3), has a model companion axiomatized by $Th(M^+)$ where M^+ is the expansion of M to $\mathcal{K}^+$.

3. $(\mathcal{K}^+)^{mc}$ is nqfa(3) (see Theorem 4.34).

Although several of the techniques of this chapter are generalizations of those in Moss [33], there are some important differences. In item (2) above, a dichotomy emerges between classes whose distanced e.c.'s are all disconnected and those classes where there are connected distanced e.c.'s; this differs from the special case of the class of graphs, which have connected e.c. distanced structures. Item (3) was not treated in [33], although it was proven there that the model companion of distanced graphs is not finitely axiomatizable (see Section 8 of [33]).

4.1 Isometric universal structures defined

Definition 4.1 *1. Let G be a graph. Define d_G to be the usual least path metric on G, so that $d_G(x, y) \leq n$ if and only if there exists a path from x to y having $\leq n$ edges, with value ∞ on pairs of vertices in different connected components of G.*

2. Let $A \in \mathcal{K}(L)$. Define $d_A : A \times A \to \omega \cup \{\infty\}$ by

$$d_A(x, y) = d_{G(A)}(x, y).$$

Remark 4.2 *As $d_{G(A)}$ is a metric, so is d_A.*

Definition 4.3 *Let $\mathcal{K}$ be an $\forall_1$ over a finite relational signature L.*

1. For $A \in \mathcal{K}$, take d_A as defined in Definition 4.1. Define 2-ary predicates $\{d_n : n \in \omega\}$ with d_n^A interpreted as tuples (x, y) with $d_A(x, y) = n$ (we interpret d_0 as equality).

2. *Let* $L^+ = L \cup \{d_n : n \in \omega\}$. *For* $A \in \mathcal{K}$, *define*

$$A^+ = \langle A, (d_n^A)_{n\in\omega} \rangle.$$

Define $\mathcal{K}^+ = \{A^+ : A \in \mathcal{K}\}$. $\mathcal{K}^+$ *is the class of* ***distanced*** $\mathcal{K}$*-structures. For* $A \in \mathcal{K}^+$, A^- *denotes the reduct of* A *to* L.

3. *Let* $A, B \in \mathcal{K}$. *Then* A *is an* ***isometric substructure*** *of* B *if* $A^+ \leq B^+$; *we write this as* $A \leq^i B$.

4. $\mathcal{K}$ *admits an* ***isometric universal*** *countable structure if there is a countable* $M \in \mathcal{K}$ *so that for each countable* $A \in \mathcal{K}$, *there is an* $A' \leq^i M$ *so that* $A \cong A'$.

Remark 4.4 *1.* In general, $\mathcal{K}^+$ is not $\forall_1$, but it is at worst $\forall_2$. In particular, the predicates d_n are L-definable in $\mathcal{K}^+$ by $\forall_2$ sentences; this axiomatization may be found in Chapter 1.5 of [12]. For clarity, we repeat it here.

We define the predicates d_n be induction on n.

First define: $\forall xy(d_0(x,y) \leftrightarrow x = y)$

Fix $n \geq 0$. Assuming d_n has been defined, define:

$$\forall xy(d_{n+1}(x,y) \leftrightarrow \exists z(d_n(x,z) \wedge \bigvee_{R\in L} \exists x_1 \ldots x_{ar(R)}(R(x_1 \ldots x_{ar(R)}) \wedge \bigvee_{1\leq i,j\leq ar(R)} x_i = z \wedge x_j = y) \wedge \bigwedge_{0\leq i\leq n} \neg d_i(x,y)).$$

2. In general, $A \leq B$ does not imply $A \leq^i B$.

4.2 Main theorem

We supply a sufficient condition for an isometric universal structure to exist in a class.

Definition 4.5 *$\mathcal{K}$ is* ***good*** *if $\mathcal{K}^+_{fin}$ satisfies JEP and AP, and $\mathcal{K}^+_{fin}$ is large in $\mathcal{K}^+$; that is, every countable $\mathcal{K}^+$-structure is contained in a $(\mathcal{K}^+_{fin}, \leq)$-chain.*

Theorem 4.6 *If $\mathcal{K}$ is a good class then $\mathcal{K}$ contains a countable isometric universal structure M.*

Proof. $(\mathcal{K}^+_{fin}, \leq)$ is smooth and has only countably many isomorphism types of finite structures. As $\mathcal{K}^+_{fin}$ satisfies JEP and AP by hypothesis, by Theorem 1.63 there is a $(\mathcal{K}^+_{fin}, \leq)$ generic M^+.

We claim that $(M^+)^-$ is the desired isometric universal structure.

Let $A \in \mathcal{K}^+$ countable. By largeness, there is a $\leq$-chain of $\mathcal{K}^+_{fin}$ structures $\{A_i : i \in \omega\}$ so that

$$A \leq \bigcup_{i \in \omega} A_i.$$

By an inductive use of the genericity of M^+ there is an $A' \leq M^+$ so that

$$\bigcup_{i \in \omega} A_i \cong A'.$$

More explicitly, by genericity, there is an embedding $f_0 : A_0 \to M^+$. Assume that for $n \geq 0$ fixed, $f_n : A_n \to M^+$ is an embedding. As $A_n \leq A_{n+1}$, $A_n, A_{n+1} \in \mathcal{K}^+_{fin}$, by genericity, there is an embedding $f_{n+1} : A_{n+1} \to M^+$ so that

$$f_{n+1} \restriction A_n = f_n.$$

Then

$$f = \bigcup_{i\in\omega} f_i : \bigcup_{i\in\omega} A_i \to M^+$$

is an embedding; choose

$$A' = f(\bigcup_{i\in\omega} A_i).$$

□

4.2.1 Free amalgamation classes revisited

The aim of this section is to prove that a free amalgamation class has an isometric universal structure. To accomplish this, we modify a few constructions of Moss [33].

Our first task is to isolate the notion of a minimal path in a relational structure. As we will see, the generalization of minimal paths from graphs to relational structures brings with it some new features.

Definition 4.7 *Let $A \in \mathcal{K}$, with $a, b \in A$.*

1. *A **path of length** n from a to b is a substructure of A of the form $P = A \restriction \bigcup_{1\leq i\leq n} \overline{a_i}$, where $(\overline{a_i} : 1 \leq i \leq n)$ is a sequence of edges (see Definition 2.25) from A with the property that if $\overline{a_i}$ enumerates the subset S_i of A, then*

   (a) $a \in S_1$ and $b \in S_n$;

(b) *For* $1 \leq i < j \leq n$, $S_i \cap S_j \neq \emptyset$ *if and only if* $j = i + 1$.

2. *If* a *and* b *are in the same component of* A, *a* ***minimal path*** *from* a *to* b *is a path from* a *to* b *of length* $d_A(a, b)$.

Remark 4.8 *1.* With notation as in Definition 4.7, $d_P(a, b) = d_A(a, b)$. Minimal paths from a to b may not be unique, and they need not be isomorphic.

2. See Figure 4.1 for a pictorial representation of an n path from a to b.

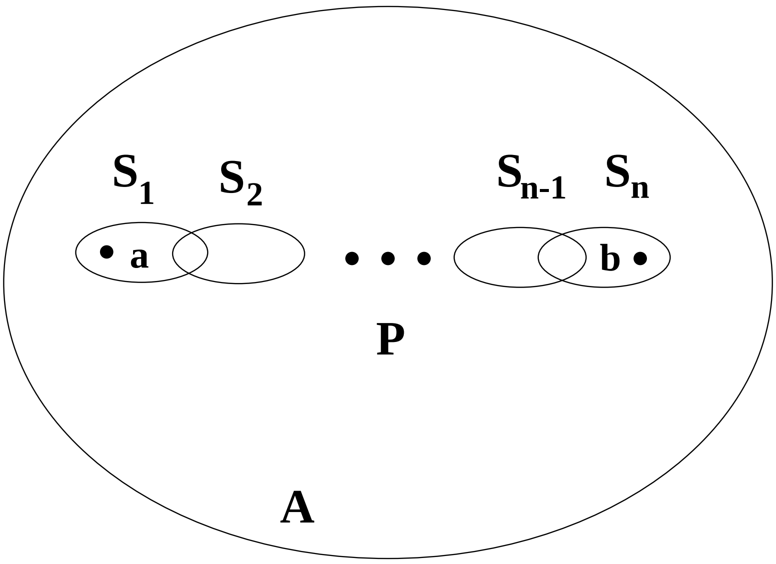

Figure 4.1: A minimal path P *between* a, b *in* A.

3. With notation as in Definition 4.7 and $c \in P$, then it may happen that

$$d_A(a, b) < d_A(a, c) + d_A(c, b),$$

even if P is a minimal path. For example, if $\mathcal{K}$ is the class of 3-uniform hypergraphs, A is the structure depicted in Figure 4.2 with circles representing hyperedges S_1, S_2, S_3 as shown, and our minimal path is $P = A$ then $d_A(a, b) = 3$ but $d_A(a, c) = d_A(c, b) = 2$.

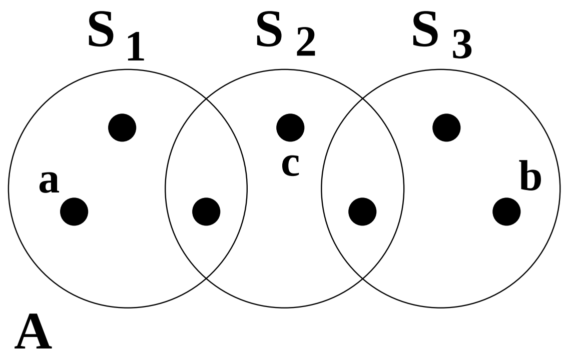

Figure 4.2: The 3*-uniform hypergraph* A*.*

Definition 4.9 *Let* $A \in \mathcal{K}$*, with* $a, b \in A$*, and let* P *be a minimal path in* A *connecting* a *to* b *with domain* $\{\overline{a_i} : 1 \leq i \leq n\}$*. Assume that for* $1 \leq i \leq n$*,* $\overline{a_i}$ *enumerates the subset* S_i *of* A*, and that* $\{\overline{a_i} : 1 \leq i \leq n\}$ *is ordered successively.*

$c \in P$ *is* ***special*** *if either* $c = a$ *or* $c = b$ *or* $c \in S_i \cap S_{i+1}$ *for some* $1 \leq i \leq n-1$*.*

Remark 4.10

1. *If* A *is a graph, then each vertex of a minimal path in* A *is special.*

2. *With notation as in Definition 4.9, choose a single special* $a_i \in S_i \cap S_{i+1}$ *for* $1 \leq i \leq n-1$*.* $\{a, a_1, \ldots, a_{n-1}, b\}$ *is called a* ***set of special representatives of*** P*. The choice of* a_i *may not be unique. The path* $aa_1 \ldots a_{n-1}b$ *witnesses* $d_A(a, b) = n$ *in* $G(A)$*.*

Lemma 4.11 *If A, a, b, P are as in Definition 4.9, and $c \in P$ is special, then*

$$d_A(a,b) = d_A(a,c) + d_A(c,b). \tag{4.1}$$

Proof. If $c = a$ or $c = b$ the conclusion is immediate.

Assume c is not equal to either a or b. Let $d_A(a,b) = n > 1$.

Let $\{a, a_1, \ldots, a_{n-1}, b\}$ be a set of special representatives of P; without loss of generality, we may assume that $c = a_i$ for some $1 \leq i \leq n-1$. As $aa_1 \ldots c$ is a path in $G(A)$ from a to c, $d_A(a,c) \leq i$. Similarly, $d_A(c,b) \leq n-i$.

If $d_A(a,c) < i$, let Q be a path between a and c witnessing this. Then Q adjoined with P restricted to the tuples of P connecting c to b is a path of length $< n$ from a to b in A, which is a contradiction. The same contradiction arises if $d_A(c,b) < n-i$.

Hence, $d_A(a,c) = i$ and $d_A(c,b) = n-i$ as desired. □

Definition 4.12 *Let $A \in \mathcal{K}^+_{fin}$ and $\bar{a}$ list n distinct elements from A for some $n \in \omega^*$.*

$qft'_{A,\bar{a}}(\bar{x})$ is the conjunction of all atomic and negated atomic $L \cup \{d_1, \ldots, d_N\}$-formulas satisfied by $\bar{a}$ in A, where N is the maximum distance between two elements in a component of A.

The *canonical structure* of A over a finite subset of B of A relative to a set of minimal paths of A is the structure built over $A \restriction B$ be freely adjoining the minimal paths representing the distances in A of tuples from B.

Definition 4.13 *Let $A \in \mathcal{K}$, with $B \subseteq A$ finite. Define an L-structure as follows.*

List the distinct pairs from B as $\{(a_i, b_i) : 1 \leq i \leq n\}$.

Define $can_A(B)_0 = A \restriction B$

Let $1 \leq k \leq n$. *Assume* $can_A(B)_{k-1}$ *has been defined, is finite, and contains* B.

Let P_k *be some minimal path in* A *between* a_k, b_k *of length* ≥ 1, *if* a_k, b_k *are connected or* $\{a_k, b_k\}$ *otherwise. Let* $B_k = A \restriction P_k$. *We may assume (by taking isomorphic copies if necessary) the elements of* $B_k - \{a_k, b_k\}$ *are neither in* A *nor* $can_A(B)_{k-1}$.

Define $can_A(B)_k$ *to be the free amalgam of* B_k *with* $can_A(B)_{k-1}$ *over* $\{a_k, b_k\}$.

Define $can_A(B)$ ***relative to*** $\{P_k : 1 \leq k \leq n\}$ *to be* $can_A(B)_n$.

Remark 4.14 *1.* When the choice of paths is irrelevant, we may abuse notation and write $can_A(B)$.

2. If $\mathcal{K}$ is $\forall_1$ and closed under unions, $can_A(B)$ is in $\mathcal{K}$ if A, B are as in Definition 4.13.

3. With notation as in Definition 4.13, $B \leq can_A(B)$.

Definition 4.15 *Let* $A \in \mathcal{K}$.

Let $B \subseteq C$ *in* A, *and define* $can_A(B)$ *relative to a set of paths* $\mathcal{P}$ *in* A. *Define* $can_A(C)$ *relative to a set of minimal paths from* a *to* b $\mathcal{P}'$ *so that for* a, b *in* B, *the minimal path in* $\mathcal{P}'$ *connecting* a *to* b *is the same as the minimal path in* $\mathcal{P}$ *connecting* a *to* b.

If $can_A(C)$ *is defined in this way, we say* $can_A(C)$ *is* ***built over*** $can_A(B)$.

Lemma 4.16 *With notation as in Definition 4.13 we have the following.*

1. For $B \subseteq A$, *if* $x, y \in B$, *then* $d_{can_A(B)}(x, y) = d_A(x, y)$.

2. Assume $can_A(C)$ is built over $can_A(B)$. Then $can_A(B) \leq^i can_A(C)$.

3. Let $B \leq^i A$. Then $B \leq^i can_A(B)$.

Proof. (1) In the inductive step of the definition of $can_A(B)$, minimal paths of elements not in A are added, with no relations added except relations on the minimal paths themselves.

(2) By our assumptions, we certainly have that $can_A(B) \leq can_A(C)$.

We must show that for all $x, y \in can_A(B)$,

$$d_{can_A(B)}(x, y) = d_{can_A(C)}(x, y). \tag{4.2}$$

Case 1) $d_{can_A(B)}(x, y) < \infty$.

Case i). $x, y \in B$. Immediate by (1).

Case ii). x, y are not both from B.

Let $B' = can_A(B)$, $C' = can_A(C)$.

Without loss of generality, assume $x \in B' - B$; hence, x is a member of a minimal path P added to B in B' connecting two elements of B.

If $d_{B'}(x, y) = 1$, then $d_{C'}(x, y) = 1$ as $B' \leq C'$.

We can assume therefore that $d_{B'}(x, y) > 1$.

To obtain a contradiction, assume that $n = d_{C'}(x, y) < d_{B'}(x, y)$. Let Q be a minimal path in C' witnessing $d_{C'}(x, y)$, with special representatives $\{x, a_1, \ldots, a_{n-1}, y\}$ (as $B' \leq C'$, $n > 1$).

Claim: $a_1 \in B' - \{y\}$.

As $d_{B'}(x, y) > 1$, $a_1 \neq y$. As P is freely adjoined to B in B', and C' is built over B', we have the crucial fact that x cannot be adjacent to an element of $C' - P$. Hence, $a_1 \in P \subseteq B'$. The claim follows.

Now,

$$\begin{aligned} d_{C'}(x, y) &= d_{C'}(x, a_1) + d_{C'}(a_1, y) \\ &= d_{B'}(x, a_1) + d_{B'}(a_1, y) \\ &\geq d_{B'}(x, y), \end{aligned}$$

the first equality follows by Lemma 4.11, the second equality by inductive hypothesis and by Case i) (depending on whether a_1 and y are in B or not), the inequality by the triangle inequality in B'. Contradiction.

Case 2). $d_{can_A(B)}(x, y) = \infty$.

Say $x \notin B$ and x connected in $can_A(B)$ to $x' \in B$. Then $d_{can_A(B)}(x', y) = \infty$.

If $d_{can_A(C)}(x, y) < \infty$, then $d_{can_A(C)}(x', y) < \infty$, so that $d_{can_A(B)}(x', y) < \infty$ by previous cases. Contradiction.

(3) Let $x, y \in B$.

Then

$$\begin{aligned} d_B(x, y) &= d_A(x, y) \\ &= d_{can_A(B)}(x, y), \end{aligned}$$

the first equality as $B \leq^i A$ (by hypothesis), the second equality by (1). But then $B \leq^i can_A(B)$. □

Theorem 4.17 *Let L be finite relational, and let $\mathcal{K}$ be an $\forall_1$ L-class that is closed under unions. Then $\mathcal{K}$ is a good class.*

Proof. We first prove the following Claim.

Claim 1: $\mathcal{K}^+_{fin}$ satisfies JEP and AP.

Proof. JEP follows as $\mathcal{K}^+_{fin}$ is closed under disjoint union. For $A, B, C \in \mathcal{K}^+_{fin}$ so that $A = B \cap C$, define

$$D = (B^- \cup C^-)^+.$$

The verification that $B, C \leq D$ is similar to the proof of Strong Amalgamation Lemma in [33].

Assume that $B \not\leq D$ (the case when $C \not\leq D$ is handled similarly). Let m be the least positive integer so that there exist $x, y \in B$ with

$$m = d_D(x, y) < d_B(x, y). \tag{4.3}$$

Note that $d_D(x, y) \leq d_B(x, y)$ for every $x, y \in B$.

Let $P \subseteq D$ be a minimal path witnessing (4.3) in D.

Case i) $P \subseteq B$. Then $d_D(x, y) = d_B(x, y)$, as B acquires no new relations in D. Contradiction.

Case ii) $P \subseteq C$. Then $x, y \in A = B \cap C$ and $d_C(x, y) = d_B(x, y)$, which is a contradiction.

Case iii) $P \subsetneq B$ and $P \subsetneq C$. Let $a \in P \cap (C - A)$, $b \in P \cap (B - A)$.

Let P_1 be the "subpath" of P connecting a to b. More precisely, assume $P = \bigcup\{\overline{a_i} : 1 \leq i \leq n\}$, and there are $1 \leq j \leq k \leq n$ so that $a \in \overline{a_j}$ and $b \in \overline{a_k}$. Define P_1 to be the path with domain $\bigcup\{\overline{a_i} : j \leq i \leq k\}$.

By the definition of $B \cup C$, $P_1 \cap A \neq \emptyset$. Let $w \in P_1 \cap A$, $w \neq x, y$. We can choose w to be special. Otherwise, each tuple $\bar{c}$ in P_1 containing some $w \in P_1 \cap A$ so that $w \neq x, y$ intersects tuples in P_1 not in A. But then such a $\bar{c}$ would contain elements from $B - A$ and $C - A$ contradicting the definition of the relations of $B \cup C$.

Then

$$\begin{aligned} d_D(x,y) &= d_D(x,w) + d_D(w,y) \\ &= d_B(x,w) + d_B(w,y) \\ &\geq d_B(x,y), \end{aligned}$$

the first equality holds by Lemma 4.11, the second equality follows by induction hypothesis, and the inequality follows by the triangle inequality in B. But this contradicts (4.3). □

Claim 2: For each $A \in \mathcal{K}^+$ countable, there is a $\leq$-chain of $\mathcal{K}^+_{fin}$ structures $\{A_i : i \in \omega\}$ so that

$$A \leq \bigcup_{i \in \omega} A_i.$$

Proof. To prove Claim 2, enumerate A as $\{a_i : i \in \omega\}$. Define A_i to be

$$can_{A^-}(\{a_0, \ldots, a_i\})^+,$$

built over $can_{A^-}(\{a_0, \dots, a_{i-1}\})^+$. $\{A_i : i \in I\}$ is a $\leq$-chain by Lemma 4.16 (2). $A \leq \bigcup_{i\in\omega} A_i$ by Remark 4.14 (3) and Lemma 4.16 (1). □

This finishes the proof of the theorem. □

Corollary 4.18 *Let $\mathcal{K}$ be as in Theorem 4.17. Then $\mathcal{K}$ admits an isometric universal countable structure.*

PROOF. By Theorem 4.6 and Theorem 4.17. □

4.3 The model companion of $\mathcal{K}^+$

Let L be finite relational, and let $\mathcal{K}$ be an $\forall_1$ free amalgamation class over L; let $M^+ \in \mathcal{K}^+$ be the associated generic structure. We now show that the model companion of $\mathcal{K}^+$ exists and $Th(M^+)$ axiomatizes $(\mathcal{K}^+)^{mc}$. The main complication that now arises is that L^+ is infinite.

To accomplish our objectives, we first introduce T', the extension axioms for the connected $\mathcal{K}^+_{fin}$-structures. Next, we introduce axioms T'' capturing: "each finite subset is contained in a finite isometric substructure." As we will see, M^+ models both T' and T'', and each pair of countable *connected* models of $T' \cup T''$ are isomorphic.

Definition 4.19 *Let $n \geq 1$.*

1. *Let Θ consist of the $\exists_1$ L^+-sentences:*

$$\exists \bar{x} qft'_{A',\bar{a}}(\bar{x}),$$

where $|\bar{x}| = |\bar{a}| = n$, $A' \in \mathcal{K}_n^+$, A' *is connected, and* $\bar{a}$ *enumerates* A', *for each* $n \geq 1$.

2. *Define* φ'_n *to be the* L^+*-sentence*

$$\bigwedge \forall \bar{x} \exists \bar{y} (qft'_{A,\bar{a}}(\bar{x}) \rightarrow qft'_{B,\bar{a}\bar{b}}(\bar{x}, \bar{y})),$$

where the conjunction ranges over all sets of the form $(A, B, \bar{a}, \bar{b})$ *so that*

   (a) A, B *are isomorphism types of structures in* $\mathcal{K}_{fin}^+$ *with* $A \leq B$, B *connected, with* $|A| \leq n$ *and* $|B| \leq n + 1$;

   (b) $\bar{a}$ *is a set of distinct elements from* A *so that* $A = \bar{a}$ *and* $|\bar{x}| = |\bar{a}|$;

   (c) $\bar{b}$ *is a set of distinct elements from* B *so that* $B = \bar{a}\bar{b}$ *and* $|\bar{x}| + |\bar{y}| \leq n+1$.

3. *Let* $T' = \Theta \cup \{\varphi'_n : n \in \omega^*\}$.

Remark 4.20 If $S \in \mathcal{K}^+$ satisfies T' then the following condition holds: if $A \leq S$ with $A \in \mathcal{K}_{fin}^+$ and $A \leq B \in \mathcal{K}_{fin}^+$ so that B is connected, then there is a $C \leq S$ with $C \in \mathcal{K}_{fin}^+$ and an isomorphism $f : B \rightarrow C$ so that $f \upharpoonright A$ is the identity map.

Given $A \in \mathcal{K}_{fin}$ and $S \subseteq A$, $|can_A(S)|$ is bounded above not by a function of $|S|$, but rather by a function of the maximum index of a symbol d_n appearing in $qft'_{A,S}(\bar{x})$.

Definition 4.21 *Let* $S \subseteq A \in \mathcal{K}_{fin}$.

1. *Let* $m(L)$ *be the maximum arity of a symbol from* L.

2. *Define* $\alpha(S)$ *to be the maximum index of a symbol* d_n *appearing in* $qft'_{A,S}(\bar{x})$, *and* ∞ *else.*

3. *Define* $\beta(S) = |S| + \binom{|S|}{2}\alpha(S)m(L)$.

The following Lemma is immediate from the definitions.

Lemma 4.22 *Let* $S \subseteq A \in \mathcal{K}$ *with* A *connected. Then*

$$|can_A(S)| \leq \beta(S).$$

Definition 4.23 *Let* $n \geq 1$.

1. *Define* φ''_n *to be the* L^+*-sentence*

$$\bigwedge \forall \bar{x}(qft'_{A,\bar{a}}(\bar{x}) \rightarrow \bigvee \exists \bar{y} qft'_{B,\bar{a}\bar{b}}(\bar{x}, \bar{y})),$$

where the conjunction ranges over all sets of the form $(A, \bar{a})$, *where*

(a) A *is an isomorphism type of connected structure in* $\mathcal{K}^+_{fin}$ *so that* $|A| \leq n$;

(b) $\bar{a}$ *is a set of distinct elements from* A, *so that* $|\bar{a}| \leq n$, *and* $|\bar{x}| = |\bar{a}|$;

and the disjunction ranges over all sets of the form $(B, \bar{b})$, *where*

(a) B *is an isomorphism type of connected structure in* $\mathcal{K}^+_{fin}$ *with* $|B| \leq \beta(\bar{a})$;

(b) $\bar{b}$ *is a set of distinct elements from* B, *so that* $B = \bar{a}\bar{b}$, $A \restriction \bar{a} = B \restriction \bar{a}$, *and* $|\bar{y}| = |\bar{b}|$.

2. Let T'' *be* $\{\varphi''_n : n \in \omega^*\}$.

Remark 4.24 *1.* For each $n \geq 1$, φ''_n is first-order.

2. Let C be a model of T''. Fix a finite subset S of C contained in a component of C. Let $\bar{a}$ list the elements of S. Then $A = can_A(S)$ is a connected structure in $S' \in \mathcal{K}^+_{fin}$, and by Lemma 4.16 (1),

$$A \models qft'_{C,\bar{a}}(\bar{a}).$$

As $C \models T''$, S is contained in $S' \leq C$ with $S' \in \mathcal{K}^+_{fin}$ connected and $|S'| \leq \beta(S)$.

Lemma 4.25 *Let* $A, B \in \mathcal{K}^+$ *be connected models of* $T' \cup T''$. *Then* $A \equiv_{\infty\omega} B$.

Proof. We first prove the following Claim.

Claim 1: Let C be a connected model of T''. Then each finite $S \subseteq C$ is contained in $S' \leq C$ with $S' \in \mathcal{K}^+_{fin}$ connected.

To see this apply Remark 4.24 (2), and the fact that C is connected.

We show that A and B are back-and-forth equivalent. To accomplish this, we first play a "modified" game whereby $\forall$ and $\exists$ make moves by choosing certain *subsets* of A and B. More precisely, $\forall$ still chooses elements, while $\exists$ chooses finite isometric substructures. We then use this game to show that $\exists$ can always win the "usual" game where moves consist of choices of *elements* of A and B.

We first prove the following Claim.

Claim 2: If $\forall$ and $\exists$ have chosen $\{a_1, \ldots, a_n\} \subseteq A$, $\{b_1, \ldots, b_n\} \subseteq B$ and connected $A_n, B_n \in \mathcal{K}^+_{fin}$ with $\{a_1, \ldots, a_n\} \subseteq A_n \leq A$, $\{b_1, \ldots, b_n\} \subseteq B_n \leq B$ and $A_n \cong B_n$, then

1. for each $a_{n+1} \in A$ there are $A_{n+1}, B_{n+1} \in \mathcal{K}^+_{fin}$ connected with $A_n \cup \{a_{n+1}\} \subseteq A_{n+1} \leq A$, $B_n \subseteq B_{n+1} \leq B$ and $A_{n+1} \cong B_{n+1}$.
2. for each $b_{n+1} \in B$ there are $A_{n+1}, B_{n+1} \in \mathcal{K}^+_{fin}$ connected with $A_n \subseteq A_{n+1} \leq A$, $B_n \cup \{b_{n+1}\} \subseteq B_{n+1} \leq B$ and $A_{n+1} \cong B_{n+1}$.

Let $\{a_1, \ldots, a_n\}$, $\{b_1, \ldots, b_n\}$, A_n, B_n, a_{n+1} be as in the hypothesis of Claim 2. By Claim 1, there is $A_{n+1} \leq A$ with $A_{n+1} \in \mathcal{K}^+_{fin}$ connected so that $A_n \cup \{a_{n+1}\} \subseteq A_{n+1}$. As B satisfies T', and A_{n+1} is isomorphic to a connected finite extension of B_n, B realizes a copy of A_{n+1} extending B_n; define B_{n+1} to be this copy of A_{n+1} in B.

Now, $\exists$ wins any usual game by using Claim 2. In the first step of play, no matter what element a $\forall$ chooses, as A and B satisfy Θ, $\exists$ can counter by choosing an element (in the other structure from where $\forall$ has chosen) with same isomorphism type as a. In general, if $\{a_1, \ldots, a_n\}$, $\{b_1, \ldots, b_n\}$, A_n, B_n, have been chosen, and $\forall$ chooses an element a_{n+1} in A (say), then $\exists$ can counter by using A_{n+1} and B_{n+1} as in Claim 2, and then choosing an appropriate element of B_{n+1}. $\square$

Lemma 4.26 *Let $B \in \mathcal{K}^+$ and let A be a connected component of B. Then $A \leq B$ and $A \in \mathcal{K}^+$.*

Proof. As $B \in \mathcal{K}^+$, B contains paths between elements in a single component; hence, so does A. Thus, $A \in \mathcal{K}^+$ as there is a minimal path in A connecting each pair of distinct elements.

Fix $a, b \in A$. If $n = d_B(a, b) < d_A(a, b)$, then a path witnessing n in B is already in A; contradiction. $\square$

Lemma 4.27 *Let $\mathcal{K}$ be as in Theorem 4.17. Then $M^+ \models T' \cup T''$.*

Proof.

By $(\mathcal{K}^+_{fin}, \leq)$-genericity, $M^+ \models T'$. First note that $M^+ \models \Theta$ by item 3(b) in Definition 1.61.

Now, let $A, B, \bar{a}, \bar{b}$ be as in the definition of φ'_n. Let $\bar{c}$ be a finite subset of M^+ so that $M^+ \models qft'_{A,\bar{a}}(\bar{c})$. Then $M^+ \restriction \bar{c} \cong A$. Now as $A, B \in \mathcal{K}^+_{fin}$, amalgamate B into M^+ over A; that is, use item (4c) in Definition 1.61 (with A, B, C in the definition replaced by M^+, A, B, respectively). This proves that

$$M^+ \models T'.$$

Further, $M^+ \models T''$. Given $A, \bar{a}$ be as in the definition of φ''_n, let $\bar{c}$ be a finite subset of M^+ so that $M^+ \models qft'_{A,\bar{a}}(\bar{c})$. Let $\bar{c} \subseteq C$, a finite isometric substructure of M^+ (we use the fact that M^+ is a $(\mathcal{K}^+_{fin}, \leq)$-union).

Claim: We can choose C connected.

As A is connected, $\bar{c} \subseteq M^+_i$ for some component M^+_i of M^+. Define $C_i = M^+_i \restriction C \cap M^+_i$.

But C_i is a connected component of C. By Lemma 4.26, $C_i \in \mathcal{K}^+$ and $C_i \leq C$. As $M^+_i \leq M^+$ by Lemma 4.26, $C_i \leq M^+$.

Thus we may let $C = C_i$.

Define $D = can_M(C)^+$ (relative to some set of minimal paths $\mathcal{P}$).

Let $\mathcal{P}_0$ be the minimal paths from $\mathcal{P}$ connecting elements of $\bar{c}$, and let $B = can_M(\bar{c})^+$ relative to $\mathcal{P}_0$. Then B is connected by Lemma 4.16 (1), $B \leq D$ by Lemma 4.16 (2), clearly $B \in \mathcal{K}^+_{fin}$, and $|B| \leq \beta(\bar{c}) = \beta(\bar{a})$ by Lemma 4.22. By Lemma 4.16 (3) we have $C \leq D$; so by the genericity of M^+ we can amalgamate D into M^+ over C to get $D \leq M^+$. Thus $\bar{c} \subseteq B \leq M^+$, verifying that

$$M^+ \models T''.$$

□

Theorem 4.28 *Let $\mathcal{K}$ be as in Theorem 4.17. Then $(\mathcal{K}^+)^{mc}$ exists and is axiomatized by $T = Th(M^+)$.*

Proof. We first note the following Claim.

Claim: Let $A \in \mathcal{K}^+$ so that $A \models T' \cup T''$. Then $A_i \models T' \cup T''$ for every component A_i of A.

This is immediate.

By the Claim, Lemma 4.25, and Theorem 1.16, every pair of components of a $\mathcal{K}^+$-structure that is a model of $T' \cup T''$ are $L^+_{\infty\omega}$ equivalent. In particular, by Lemma 4.27, and Theorem 1.16, the components of M^+ are pair-wise isomorphic; let the isomorphism type of any of these components by labelled N.

Let $A \in \mathcal{K}^+$ be countable so that $A \models T$. Then $A \models T' \cup T''$, and so

$$A = \bigcup_{1 \leq i \leq n} N_i,$$

with $1 \leq n \leq \omega$ and $N_i \cong N$, for $1 \leq i \leq n$. In particular, T has only countably many countable models. Hence, T is small (that is, T realizes only countably many types), so that there is a countable saturated model for T. M^+ must be the saturated model: a countable saturated model of T must have the maximum possible number of components as does M^+ (as M^+ embeds every countable $\mathcal{K}^+$-structure). But then by Proposition 1.67, T is model complete.

To show mutual model consistency, it is enough to show that every $B \in (\mathcal{K}^+)^{ec}$ is a model of T. Without loss, we may assume B is countable (by Lemma 1.32). But then $B \leq M^+$. As T is model complete, T is $\forall_2$ (see, for example, Theorem 8.3.3 of [23]). But then $B \models T$ as B is e.c.. □

4.4 Axiomatization of the model companion of $\mathcal{K}^+$

Recall that we have shown so far that for $\mathcal{K}$ an $\forall_1$ free amalgamation class over a finite language, and for the generic $M^+ \in \mathcal{K}^+$, the model companion of $\mathcal{K}$ exists and is axiomatized by $Th(M^+)$. We are now interested in presenting an explicit axiomatization of $Th(M^+)$. Two cases emerge, reflecting the nature of $\mathcal{K}$: either every e.c. in $\mathcal{K}^+$ has more than one component, or there are connected e.c.'s in $\mathcal{K}^+$. This appears to be a new phenomenon: the class of distanced graphs does have connected e.c.'s; we will see some examples where there are no connected e.c.'s in $\mathcal{K}^+$ (see Subsection 4.4.2).

Definition 4.29 *Let $A, B \in \mathcal{K}^+$ and fix $m \geq 1, n \geq 0$. Let $\bar{a} = (a_1, \ldots, a_m)$ enumerate a subset of A and $\bar{b} = (b_1, \ldots, b_m)$ enumerate a subset of B.*

1. *We write $(A, \bar{a}) \approx (B, \bar{b})$ if $A^- \models qft_{B^-, \bar{b}}(\bar{a})$ and $B^- \models qft_{A^-, \bar{a}}(\bar{b})$.*

2. *We write $(A, \bar{a}) \equiv_n (B, \bar{b})$ if $(A, \bar{a}) \approx (B, \bar{b})$ and for $1 \leq i, j \leq m$, $0 \leq k \leq n$, $A \models d_k(a_i, a_j)$ iff $B \models d_k(b_i, b_j)$.*

3. *If $A, B \in \mathcal{K}^+_{fin}$, $\bar{a} = A$ and $\bar{b} = B$, then we write $A \equiv_n B$ if $(A, \bar{a}) \equiv_n (B, \bar{b})$.*

Remark 4.30 1. *If $A, B \in \mathcal{K}^+_{fin}$ and $A \equiv_n B$ for each $n \geq 1$, then $A \cong B$.*

2. *If $A, B \in \mathcal{K}^+_{fin}$ are connected and $n = \max(\alpha(A), \alpha(B))$, then $A \equiv_n B$ implies $A \cong B$.*

Definition 4.31 *Define φ^*_n, for $n \in \omega^*$ fixed, to be the L^+-sentence:*

$$\bigwedge \forall \bar{x} \exists \bar{y} \ (qft'_{B,\bar{b}}(\bar{y}) \wedge \bigwedge_{x \in \bar{x}, y \in \bar{y}, 1 \leq j \leq n} \neg d_j(x, y)),$$

where the conjunction ranges over all sets of the form $(B, \bar{b})$ so that B is an isomorphism type of connected structure in $\mathcal{K}^+_{fin}$ with B connected, $B = \bar{b}$, $|\bar{y}| = |\bar{b}|$, $1 \leq |\bar{x}| \leq n$, $|\bar{x}| + |\bar{y}| \leq n + 1$.

Let T^* be $\{\varphi^*_n : n \in \omega^*\}$.

Lemma 4.32 1. *$M^+ \models T^*$.*

2. $(\mathcal{K}^+)^{ec} \models T^*$.

Proof. (1) Let $A \subseteq M^+$ be finite, and let $B \in \mathcal{K}^+_{fin}$ be connected (disjoint from A). By genericity, there is $A' \in \mathcal{K}^+_{fin}$ with $A \subseteq A' \leq M^+$.

By taking isomorphic copies if necessary, we may assume M^+ and B are disjoint; form $C = M^+ \uplus B$. Then $A' \leq A' \uplus B$ in C, and by genericity there is a $D \leq M^+$ and an isomorphism $f : A' \uplus B \to D$ so that f is the identity on A'. But then $B \cong f(B) \leq M^+$ is not in the same component of M^+ with A', and hence, A.

(2) follows from (1) and by Theorem 4.28. $\square$

4.4.1 An axiomatization

Theorem 4.33 *$Th(\mathcal{K}^+) \cup T^* \cup T' \cup T''$ axiomatizes $(\mathcal{K}^+)^{mc}$. If there is a connected e.c. in $\mathcal{K}^+$ then $Th(\mathcal{K}^+) \cup T' \cup T''$ axiomatizes $(\mathcal{K}^+)^{mc}$.*

Proof. For the first sentence of the theorem, because of Theorem 4.28 and Lemma 4.32 it is enough to show that for $A \in \mathcal{K}^+$, $A \models T^* \cup T' \cup T''$ implies that A is e.c. in $\mathcal{K}^+$.

Let $A \leq B \models \exists \bar{x}\theta(\bar{x}, \bar{a})$, for $\bar{a}$ in A, $B \in \mathcal{K}^+, \theta(\bar{x}, \bar{a})$ a quantifier-free L^+-formula; let $\bar{b}$ witness $\exists \bar{x}\theta(\bar{x}, \bar{a})$ in B. Without loss of generality, we may assume that B is e.c., and hence, by Lemma 4.32, B is a model of $T^* \cup T' \cup T''$.

Decompose $\bar{a}$ as $\overline{a_1} \uplus \ldots \uplus \overline{a_n}$ with $\overline{a_i}$ in a component $A_i \leq A$ for $1 \leq i \leq n$.

As $A_i \models T' \cup T''$, by the Claim in the proof of Lemma 4.25, there is a finite connected $C_i \leq A_i$ containing $\overline{a_i}$.

Let $\bar{b} = \overline{b_1} \cup \overline{b_2}$, with each element of $\overline{b_1}$ of finite distance to some element of $\bar{a}$ and no element of $\overline{b_2}$ of finite distance to any element of $\bar{a}$.

Let $\overline{b_1} = \overline{e_1} \uplus \ldots \uplus \overline{e_n}$ be a component decomposition of $\overline{b_1}$ in B, with each element $\overline{e_i}$ of finite distance to some element of $\overline{a_i}$, and $\overline{e_i}$ in a component $B_i \leq B$ for $1 \leq i \leq n$ (in particular, $A_i \leq B_i$ in B). As B_i models $T' \cup T''$ there is a finite connected $D_i \leq B_i$ containing $C_i \cup \overline{e_i}$.

As $A_i \models T'$ using the fact that $C_i \leq D_i$, realize D_i inside A_i as D_i'; let $\overline{e_i'}$ name the image of $\overline{e_i}$ inside D_i', and define $\overline{b_1}' = \overline{e_1}' \uplus \ldots \uplus \overline{e_n}'$. Note that $(A, \overline{b_1}') \equiv_m (B, \overline{b_1})$ for each $m \geq 0$.

Let s be the maximum index of the "d_s" in $\theta(\bar{x}, \bar{a})$.

Let $\overline{b_2} = \overline{f_1} \uplus \ldots \uplus \overline{f_r}$ be a component decomposition of $\overline{b_2}$ in B, with each $\overline{f_i}$ in a component $S_i \leq B$ for $1 \leq i \leq r$. As before, each $\overline{f_i}$ is contained in some finite connected $T_i \leq S_i$.

Using T^* with $C = A \restriction D_1' \uplus \ldots \uplus D_n' \leq A$ playing the role of $\bar{x}$, and T_1 playing the role of B, realize T_1 inside A as T_1' of distance $> s$ from elements of C. Proceeding inductively, assume $T_1, \ldots, T_k$ have been realized in A as $T_1', \ldots, T_k'$ of distance $> s$ from C and each other. Define $C_k = C \uplus \{T_1' \uplus \ldots \uplus T_k'\}$. Then there is some $C_k' \in \mathcal{K}_{fin}^+$ so that $C_k \subseteq C_k' \leq A$ (let $X_1 \uplus \ldots \uplus X_t$ be a component decomposition of C_k, with X_i in a component $E_i \leq A$. As before, we can find $Y_i \in \mathcal{K}_{fin}^+$ so that $X_i \subseteq Y_i \leq E_i$. Then $A \restriction Y_1 \uplus \ldots \uplus Y_t \leq A$.)

Using T^* with T_{k+1} and C_k' realize T_{k+1} in A as T_{k+1}' of distance $> s$ from C_k'.

Hence, we can realize the elements of $\{\overline{f_1}, \ldots, \overline{f_r}\}$ in A as elements of distance $> s$ to each element of $\bar{a}$ in A, and so that the realizations of $\overline{f_i}$ and $\overline{f_j}$ in A for $1 \leq i < j \leq r$ are of distance $> s$ to each other and to $\overline{b_1}'$ in A.

Label the realizations of $\{\overline{f_1}, \ldots, \overline{f_r}\}$ in A as $\overline{b_2}'$.

Then $(A, \overline{b_1}'\overline{b_2}'\bar{a}) \equiv_s (B, \overline{b_1 b_2}\bar{a})$. Hence,

$$A \models \theta(\overline{b_1}'\overline{b_2}', \bar{a})$$

so that

$$A \models \exists \bar{x}\theta(\bar{x}, \bar{a}).$$

This proves that A is e.c. in $\mathcal{K}^+$.

For the final statement of the theorem, let A be a connected e.c. in $\mathcal{K}^+$. Let $B \models Th(\mathcal{K}^+) \cup T' \cup T''$. We show B is e.c. in $\mathcal{K}^+$.

Let $B = \bigcup_{i \in I} B_i$ be a component decomposition; then $B_i \models Th(\mathcal{K}^+) \cup T' \cup T''$, for each $i \in I$.

By Lemma 4.25, $B_i \equiv_{\infty\omega} A$. But then B_i is e.c. by Theorem 1.37, and so

$$B_i \models T^* \cup T' \cup T''.$$

The argument that B is e.c. is now similar to the above argument. □

Problem: Find a necessary and sufficient condition for a free amalgamation class to have connected distanced e.c.'s.

4.4.2 Examples

Our results now apply to the following classes. Each of the classes listed in 1) and 2) below satisfy the conclusions of Corollary 4.18 and Theorem 4.28. Closure under union may be checked directly or with the aid of Proposition 2.10.

1) Let $L = \{E\}$, with E 2-ary. Let $\mathcal{K}$ be one of the L-classes of graphs (recovering Corollary 5.2 of Moss in [33]), digraphs, oriented graphs, K_n-free graphs, $n > 2$, the Henson digraphs ($\forall_1$ classes of oriented digraphs defined by excluding a set of finite mutually non-embeddable tournaments).

Each of the above classes has a connected e.c. and so the model companion of $\mathcal{K}^+$ is axiomatized by $Th(\mathcal{K}^+) \cup T' \cup T''$.

2) Let $\mathcal{K}$ be the class of graphs. Let $G \in (\mathcal{K})_{fin}$ be a core graph (see Definition 3.6 above). Let $|G| = n \geq 2$, with $G = \{1, \ldots, n\}$. Let $L = \{E, P_1, \ldots, P_n\}$, with E 2-ary, and each P_i 1-ary. Define the L' class $C(G)'$ to be the $\forall_1$ class defined by:

$$\forall x(\bigvee_{i=1}^{n} P_i(x));$$
$$\forall x(\bigwedge_{1 \leq i < j \leq n} \neg(P_i(x) \wedge P_j(x)));$$

if $\neg E^G ij$ then there is a sentence:

$$\forall xy((P_i(x) \wedge P_j(y)) \rightarrow \neg Exy).$$

$C(G)$ is the class of $\{E\}$-reducts of $C(G)'$; $C(G)'$ is the class of G-*coloured graphs.*

Using Proposition 2.18 it can be shown that $C(G)'$ is closed under unions. Hence, Corollary 4.18 and Theorem 4.33 apply to $C(G)'$, in turn yielding an isometric universal countable $C(G)$-structure.

$((C(G)')^+)^{mc}$ may not have any connected e.c.'s.

For example, let $A = K_3$ and B be the graph depicted in Figure 4.3.

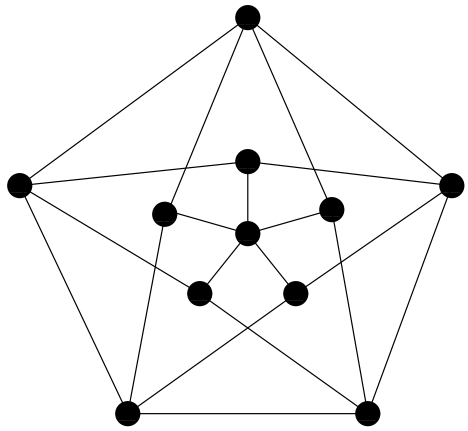

Figure 4.3: The graph B.

B has the property that $\chi(B) = 4$ (that is, B is 4-colourable but not 3-colourable) and B contains no triangles. From these two facts it follows that A and B are a $\rightarrow$-antichain. A and B are cores as they are both (point) critical graphs.

Let $G = A \uplus B$. Then G is core: let $f : G \rightarrow G$ be a homomorphism. Then $f(A)$ must be connected, and so $f(A) \subseteq A$. Similarly, $f(B) \subseteq B$. But since A and B are core's, $f \restriction A$ and $f \restriction B$ are both onto A and B, respectively; hence, f is onto.

Each H that maps homomorphically onto G must be disconnected. It follows that elements of $((C(G)')^+)^{mc}$ are disconnected. In particular, by Theorem 4.28, $((C(G)')^+)^{mc}$ is axiomatized by $Th(\mathcal{K}^+) \cup T^* \cup T' \cup T''$.

Another (and more trivial) example is to let L consist of a fixed finite number of unary predicates and let $\mathcal{K} = \mathcal{K}(L)$. In $\mathcal{K}$, every model has only singleton components.

4.5 **Nqfa**(3) of the model companion of $\mathcal{K}^+$

The present section is devoted to proving the following theorem.

Theorem 4.34 *Let $\mathcal{K}$ be $\forall_1$ free amalgamation class with edges. Then $(\mathcal{K}^+)^{mc}$ is non-finitely axiomatizable modulo*

$$\Gamma = Th(\mathcal{K}^+) \cup T^* \cup T'' \cup \Theta,$$

where Θ is as in Definition 4.19 (1) (in which case we say that $\mathcal{K}^{mc}$ is ***nqfa***(3).*)*

If $B \in \mathcal{K}$ is connected, $diam(B)$ is the diameter of $G(B)$.

Recall that we are assuming that $\mathcal{K}$ is closed under unions.

Lemma 4.35 *Fix $A \in \mathcal{K}$ an edge, with $|A| = n \geq 2$.*

1. *For $m \geq 1$, there is $S_m(A) \in \mathcal{K}$ with graph as in Figure 4.4.*
2. *For all $m \geq 2$, there is $S_{m,m}(A) \in \mathcal{K}$ with graph as in Figure 4.5.*
3. *For all $m \geq 2$, $S_m(A)$ and $S_{m,m}(A)$ in items (1) and (2) may be constructed so that*

$$S_m(A) \leq^i S_{m,m}(A).$$

Proof. (1) Let $S_1 = A$.

Fix $x \in A$.

Assume S_m is defined.

Let A_{m+1} be a copy of A, with $x \in A_{m+1}$ represented by x_{m+1} disjoint from S_m.

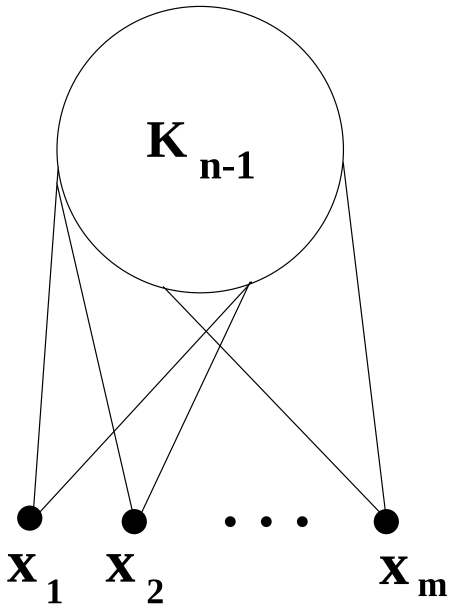

Figure 4.4: S_m. Each x_i is adjacent to all of K_{n-1}.

Let S_{m+1} be the free amalgam of S_m and A_{m+1} over $A \restriction A - \{x\}$.

(2) Let $S_{m,m}$ be the free amalgam of S_m and S_m over $A \restriction \{x_{1,}, \ldots, x_m\}$.

(3) Immediate, as $diam(S_m) = 2$. $\square$

Remark 4.36 When A is clear from context, we write S_m and $S_{m,m}$ for $S_m(A)$ and $S_{m,m}(A)$, respectively.

4.5.1 Proof of Theorem 4.34

We define a set of countable structures $\{M_r : r \geq 1\}$ so that for each $r \geq 1$, $M_r \models \{\varphi'_r \wedge \neg\varphi'_{r+2n-1}\} \cup \Gamma$.

If $(\mathcal{K}^+)^{mc}$ were qfa(3) then by Compactness and Theorem 4.33, $(\mathcal{K}^+)^{mc}$ would

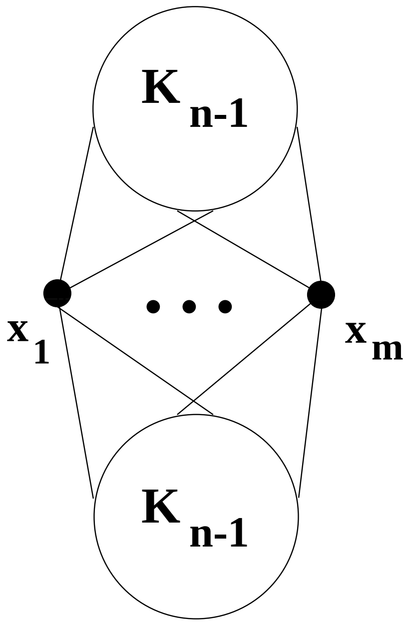

Figure 4.5: $S_{m,m}$.

be axiomatized by $\Gamma \cup \{\varphi'_r\}$, for some $r \geq 1$.

But then $M_r \in (\mathcal{K}^+)^{mc}$, contradicting the fact that $Th((\mathcal{K}^+)^{mc}) = Th(M^+)$ and $M^+ \models \varphi'_{r+2n+1}$.

For $r \geq 1$ define $M_r \in \mathcal{K}^+$ as follows.

1. Let $M_r^0 = S_{r+2}^+ \in \mathcal{K}^+$.

2. For $s \geq 0$ assume $M_r^s \in \mathcal{K}_{fin}^+$ and $M_r^0 \leq M_r^s$.

3. Form $M_r^{s+1\prime\prime} \in \mathcal{K}_{fin}^+$ by forming all extensions of at most r-element substructures of M_r^s to at most $r+1$ element $\mathcal{K}^+$-structures (accomplished by iterated amalgamation); this makes sense as the number of isomorphism types of n-element $\mathcal{K}^+$-structures is finite for all $n \geq 1$.

Let $M_r^{s+1\prime}$ be the disjoint union of $M_r^{s+1\prime\prime}$ along with one isomorphic copy of every $\mathcal{K}^+$-structure of size $\leq s+1$. Then $M_r^{s+1\prime} \in \mathcal{K}^+_{fin}$.

Let M_r^{s+1} be $can_{(M_r^{s+1\prime})^-}(M_r^{s+1\prime})^+ \in \mathcal{K}^+_{fin}$ (relative to some set of paths). Note that

$$M_r^s \leq M_r^{s+1\prime\prime} \leq M_r^{s+1\prime} \leq M_r^{s+1},$$

the last by Lemma 4.16 (3).

Let

$$M_r = \bigcup_{s\in\omega} M_r^s.$$

Then $M_r \in \mathcal{K}^+$ by Remark 4.4 and Theorem 2.4.4 of [23].

By construction,

$$M_r \models Th(\mathcal{K}^+) \cup \{\varphi_r'\} \cup \Theta,$$

as the reader can check. To see that $M_r \models T''$, let $A, \bar{a}$ be as in the definition of φ_n'', and let $\bar{c}$ be a finite subset of M_r so that

$$M_r \models qft'_{A,\bar{a}}(\bar{c}).$$

Let $s \geq 1$ be chosen so that $\bar{c} \subseteq M_r^s$. The restriction of M_r^{s+1} to a set of minimal paths connecting elements of $\bar{c}$ is then a connected substructure of M_r containing $\bar{c}$ with order $\leq \beta(\bar{a})$ (by Lemma 4.22).

To see that

$$M_r \models T^*,$$

let A be a finite subset of M_r and let $B \in \mathcal{K}^+_{fin}$ be connected and disjoint from A, with $|B| = s$. Then $A \subseteq M_r^t$ for some $t \geq 0$. Let $u = \max(s, t)$. Then B is realized in $M_r^{u+1\prime}$ in a different component of M_r than A.

Claim: For $r \geq 2$, $M_r \not\models \varphi'_{r+2n-1}$.

Consider the $\leq$-extension of M_r^0 by $C = S^+_{r+2,r+2}$. If

$$M_r \models \varphi'_{r+2n-1}$$

then as $M_r^0 \leq M_r$ and $M_r^0 \leq C$ (as $diam(M_r^0) = 2$), M_r would realize the extension of M_r^0 by C.

We find a contradiction.

The idea is to show inductively that C is not realized in the chain of structures defining M_r.

1. C is not realized in M_r^0 as $|C| > |M_r^0|$.
2. Assume C is not realized in M_r^s.
   - (a) C is not realized in $M_r^{s+1\prime\prime}$: no element in $M_r^{s+1\prime\prime} - M_r^s$ is adjacent in $G((M_r^{s+1\prime\prime})^-)$ to more than r elements in M_r^s (amalgamation in $\mathcal{K}^+$, while not free, never forces nonadjacent points in the amalgamated structures to become adjacent in the amalgam).
   - (b) C is realized in $M_r^{s+1\prime} - M_r^{s+1\prime\prime}$: impossible as C is connected.
   - (c) C is realized in $M_r^{s+1} - M_r^{s+1\prime}$: elements in $M_r^{s+1} - M_r^{s+1\prime}$ are adjacent in $G((M_r^{s+1})^-)$ to at most two elements of M_r^0. But $r \geq 2$ so that there are elements in $C - M_r^0$ adjacent in $G(C)^-$ to at least 3 elements of M_r^0.

This completes the proof of Theorem 4.34.

Chapter 5

Open Problems

For the benefit of the reader, we collect the open problems stated in this thesis.

1. (page 25) Given L a relational language, classify those $\forall_1$ L-classes that are in $MC(L)$.

2. (page 27) Do the classes of width n-orders, for $n > 2$, have model companions? The same question for graphs omitting a single cycle of length > 3.

3. (page 80) Given $\mathcal{K} \subseteq \mathcal{K}(L)$ an $\forall_1$ class, classify those cores $A \in \mathcal{K}_{fin}$ so that $C!_{\mathcal{K}}(A) = C *_{\mathcal{K}} (A)$.

4. (page 87) Find a necessary and sufficient condition characterizing $\forall_1$ classes with fixations.

5. (page 105) If $\mathcal{K}$ is an $\forall_1$ free amalgamation class with edges without a type 3 axiomatization, and $A \in \mathcal{K}_{fin}$ is a non-unit core, is $C_{\mathcal{K}}(A)^{mc}$ is nqfa(2)? Is $C!_{\mathcal{K}}(A)_{fin}$ cofinal in $C_{\mathcal{K}}(A)_{fin}$?

6. (page 135) Find a necessary and sufficient condition for a free amalgamation class to have connected distanced e.c.'s.

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