Europ. J. Combinatorics (2002) **23**, 257–274
doi:10.1006/eujc.2002.0574
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Generalized Pigeonhole Properties of Graphs and Oriented Graphs*

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A relational structure A satisfies the $\mathcal{P}(n, k)$ property if whenever the vertex set of A is partitioned into n nonempty parts, the substructure induced by the union of some k of the parts is isomorphic to A. The $\mathcal{P}(2, 1)$ property is just the pigeonhole property, $(\mathcal{P})$, introduced by Cameron, and studied by Bonato, Delić and Cameron. We classify the countable graphs, tournaments, and oriented graphs with the $\mathcal{P}(3, 2)$ property.

1. INTRODUCTION

Vertex partition properties of relational structures have been studied by numerous authors; see, for example, [2–5, 7–9, 11] and [12]. One such property that has received some attention recently is the *pigeonhole property*, $(\mathcal{P})$: a relational structure A has $(\mathcal{P})$ if for every partition of the vertex set of A into two nonempty parts, the substructure induced by some one of the parts is isomorphic to A. This property was introduced by Cameron in [5], who in Proposition 3.4 of [6] classified the countable graphs with the $(\mathcal{P})$ property; remarkably, there are only four: the graph with one vertex, the countably infinite clique and its complement, and R, the countably infinite random graph. The countable tournaments with the $(\mathcal{P})$ property were classified in [3]; in this case, there are $\sim_1$ many such tournaments: the countable ordinal powers of the first infinite ordinal ω and their reversals, and T^{∞}, the countably infinite random tournament. (As noted in [3], the classification of the countable oriented graphs with the $(\mathcal{P})$ property is open. The problem reduces to classifying orientations of R satisfying property $(\mathcal{P})$.)

A natural generalization of the pigeonhole property is to allow for partitions of the vertex set into n nonempty parts, and insist that for some $1 \leq k < n$, the substructure induced by the union of some k of the parts is isomorphic to the original structure. We call this the $\mathcal{P}(n, k)$ *property*, and we say that A *has* $\mathcal{P}(n, k)$ or that A *is a* $\mathcal{P}(n, k)$ *structure*. (Then $(\mathcal{P})$ becomes the $\mathcal{P}(2, 1)$ property.) For example, the countably infinite clique and its complement have $\mathcal{P}(n, k)$ for all values of the parameters n and k.

This property was discovered in the summer of 2000 by Cameron, and is similar to the p-indivisibility property (see [12]). The $\mathcal{P}(n, k)$ properties are examples of what we refer to as *generalized pigeonhole* or *fractal* properties of relational structures.

At a conference in the summer of 2000 in honour of Fraïssé's 80th birthday, Cameron asked which countable graphs have $\mathcal{P}(3, 2)$. (See also Problem 26 of Cameron's problem web page: `http://www.maths.qmw.ac.uk/~pjc/oldprob.html`.) In this paper, we give a complete answer to this problem (see Section 2), and furthermore, we give a complete classification of *all* oriented graphs with the $\mathcal{P}(3, 2)$ property.

In Section 2 we give the classification of the countable graphs with the $\mathcal{P}(3, 2)$ property. In contrast to the case for the $\mathcal{P}(2, 1)$ property, Theorem 1 implies that R does not satisfy the $\mathcal{P}(n, n-1)$ property if $n > 2$. In Section 3 we give the classification of the countable linear orders (that is, transitive tournaments) with the $\mathcal{P}(3, 2)$ property. The classification breaks down into two cases: when there is a first or last element (see Theorems 3 and 4) or when there is neither a first nor last element (see Theorem 5). In Section 4 we prove in Theorems

*1991 Mathematics Subject Classification: 05C75, 05C20.

0195–6698/02/040257 + 18 $35.00/0

6 and 7 that a countable $\mathcal{P}(3,2)$ tournament must be a *scattered* linear order (that is, it does not contain a dense suborder). This result, along with the results of Section 3, give a complete classification of the countable tournaments with the $\mathcal{P}(3,2)$ property. The case of countable oriented graphs with the $\mathcal{P}(3,2)$ property is covered by Theorem 8 in Section 5, which makes use of the results from all of the previous sections. We close with a brief section containing some open problems.

Unless otherwise stated, all structures (that is, graphs or oriented graphs) are countable, nonempty, and do not have loops or multiple edges. If A is a structure, $V(A)$ is the set of vertices of A, $E(A)$ is the set of edges of A if A is a graph, and the arcs (or directed edges) of A if A is an oriented graph. If $B \subseteq V(A)$, then we write $A \restriction B$ for the substructure induced on B; if C is an induced substructure of A we write $C \leq A$. We write $A \cong B$ if A and B are isomorphic. If A is a structure and $X \subseteq V(A)$, then the structure $A - X$ results by deleting X and all edges or arcs incident with a vertex in X. If $X = \{x\}$, then we simply write $A - X = A - x$. If G is a graph and $x \in V(G)$, then the *neighbour set* of x, denoted $N(x)$, is the set of vertices joined to x; the elements of $N(x)$ are the *neighbours* of x. The *co-neighbour set* of x, denoted $N^c(x)$, is the set of vertices that are neither joined nor equal to x; the elements of $N^c(x)$ are the *nonneighbours* of x.

ω is the set of natural numbers (considered as an ordinal), and $\aleph_0$ is the cardinality of ω. The proper class of ordinals is denoted ON. The order-type of the rationals is η. We assume familiarity with basic results on linear orders. We refer the reader to Rosenstein [10] throughout the article for specific results on linear orders.

The clique (or complete graph) of cardinality α is denoted K_α. The complement of a graph G is denoted $\overline{G}$; the *converse* of an oriented graph O is denoted O^*. If O is an order, then we say that O^* is the *reversal* of O. Given two graphs G, H, the *join* of G and H, written $G \vee H$, is then the graph formed by adding all edges between vertices of G and H; the *disjoint union* of G and H is written $G \uplus H$. If α is a cardinal, the graph αG consists of α disjoint copies of G. The (linear) sum of (linear) orders $(L_i : i \in I)$ is denoted $\sum_{i \in I} L_i$; the sum of two orders L and M is denoted $L + M$.

2. The Graphs With the $\mathcal{P}(3,2)$ Property

In this section, the $\mathcal{P}(3,2)$ graphs are classified. In order to accomplish this, we first introduce some terminology. Recall from [1] that a graph is *n-existentially closed* or *n-e.c.* if for each n-subset S of vertices, and each subset T of S (possibly empty), there is a vertex not in S joined to each vertex of T and no vertex of $S \backslash T$. The infinite random graph R is the unique graph that is n-e.c. for all $n \geq 1$. An *extension* of a subset $X \subseteq V(G)$ is a vertex z not in X joined to the vertices of X in some fixed way; we say that z *extends* X. If r is a positive integer, then the set X is *r-extendable* if one can extend X in G in r different ways. If X is $2^{|X|}$-extendable, then we say that X is *extendable*. Each n-subset of $V(G)$ is extendable if and only if G is n-e.c. Our first step in the classification of the $\mathcal{P}(3,2)$ graphs is the following theorem.

THEOREM 1. *For each $n > 2$, there is no $(n-1)$-e.c. $\mathcal{P}(n, n-1)$ graph.*

PROOF. Suppose that G is an $(n-1)$-e.c. $\mathcal{P}(n, n-1)$ graph. Fix a set of n vertices of G, $X = \{a_1, \ldots, a_n\}$. Partition $V(G)$ into parts $A_1, \ldots, A_n$ so that

$$A_i = \{a_i\} \cup S_i,$$

where S_i is the set of vertices y joined to every a_j, where $j \in \{1, \ldots, n\} \setminus \{i, i-1\}$, and y is not joined nor equal to a_{i-1} (where the indices are ordered cyclically mod n). Each set S_i is nonempty by hypothesis. The remaining vertices of G belong to A_1.

Fix $i \in \{1, \dots, n\}$. If we consider the graph $H_i = G \restriction (V(G) \setminus A_i)$, then there is no vertex in H_i that is joined to the vertices in $X \setminus \{a_i, a_{i-1}\}$, and not joined nor equal to a_{i-1}. For the $\mathcal{P}(n, n-1)$ property, one of the graphs H_i is isomorphic to G. This contradicts that G is $(n-1)$-e.c. □

Observe that Theorem 1 implies, perhaps surprisingly, that the random graph R does *not* have $\mathcal{P}(n, n-1)$, when $n \geq 3$.

A vertex $x \in V(G)$ is *isolated* if it has no neighbours, and *universal* if it is isolated in $\overline{G}$. A pair of vertices $\{x, y\}$ of G is an *interval* if for every $z \in V(G) \setminus \{x, y\}$, x is joined to z if and only if y is joined to z; it is an *anti-interval* if for every $z \in V(G) \setminus \{x, y\}$, x is joined to z if and only if y is not joined to z. If xy is an edge of G, then we say it is either a *full interval* or *full anti-interval*.

THEOREM 2. *The countable* $\mathcal{P}(3, 2)$ *graphs are the one-vertex graph, the two-vertex and* $\aleph_0$*-vertex cliques and their complements, and the graphs*

$$K_1 \uplus K_{\aleph_0}, K_1 \vee \overline{K_{\aleph_0}}, \overline{K_{\aleph_0}} \vee \overline{K_{\aleph_0}}, K_{\aleph_0} \uplus K_{\aleph_0}, \overline{K_{\aleph_0}} \uplus K_{\aleph_0}, \overline{K_{\aleph_0}} \vee K_{\aleph_0}.$$

PROOF. We leave the proof of sufficiency as an exercise for the reader. For necessity, let G be an infinite $\mathcal{P}(3, 2)$ graph. We may assume, by Theorem 1, that G is not 2-e.c. We note first that if G has exactly one isolated vertex x, then $G - x$ is a $\mathcal{P}(2, 1)$ graph. The graph $R \uplus K_1$ does not have $\mathcal{P}(3, 2)$. To see this, fix $y \in V(R)$, consider the partition $\{x, y\}$, $N(y)$, $N^c(y) \cap V(R)$, and use the facts that $R - y \cong R$, and that R has no universal or isolated vertex. Hence, $G - x$ must be $K_{\aleph_0}$, and the characterization holds. The case if G has some unique universal vertex is similar.

Let us now prove that G has an interval. Let $V = V(G)$. If G has more than one isolated (or universal) vertex, then it certainly has an interval (any two isolated vertices or any two universal vertices). So we can assume, without loss of generality, that G has no isolated nor universal vertices.

By Theorem 1, G has a nonextendable pair x, y of vertices. Partition $V \setminus \{x, y\}$ into four subsets

$$S_{00}, S_{01}, S_{10}, S_{11},$$

where S_{00} contains the vertices not joined to x and y, S_{01} contains the vertices not joined to x and joined to y, S_{10} contains the vertices joined to x but not y, and S_{11} contains the vertices joined to both x and y.

Suppose first that $\{x, y\}$ is 3-extendable.

Case 1. $S_{11} = \emptyset$. We partition V into $\{x\} \cup S_{01}$, $\{y\} \cup S_{10}$ and S_{00}. Since G is a $\mathcal{P}(3, 2)$ graph, the subgraph induced by the union of two of these subsets is isomorphic to G. Two cases give isolated vertices, and we must have that $G \cong G \restriction (\{x, y\} \cup S_{01} \cup S_{10})$, in which case $\{x, y\}$ is 2-extendable; therefore, there is a 2-extendable pair of distinct vertices in G.

Case 2. $S_{10} = \emptyset$. We partition V into $\{x\} \cup S_{00}$, $\{y\} \cup S_{11}$ and S_{01}. Since G is a $\mathcal{P}(3, 2)$ graph, the subgraph induced by the union of two of these subsets is isomorphic to G. Two cases give an isolated or a universal vertex, and we must have that $G \cong G \restriction (\{x, y\} \cup S_{00} \cup S_{11})$, in which case $\{x, y\}$ is 2-extendable.

The other cases are equivalent. If $\{x, y\}$ is 1-extendable, then we conclude that G has a universal or an isolated vertex, or that $\{x, y\}$ is an interval.

Finally consider the case when there exists a pair $\{x, y\}$ which is 2-extendable and, to obtain a contradiction, assume that there is no interval. The pair $\{x, y\}$ must then be an anti-interval. By taking complements if necessary, we can assume that $\{x, y\}$ is a full anti-interval. Enumerate the full anti-intervals of G as

$$\{x_1, y_1\}, \{x_2, y_2\}, \ldots .$$

If two full anti-intervals intersect, then an interval is created, so we assume that all these pairs are disjoint.

Denote by X the union of the x_i's, by Y the union of the y_i's and by S the set $V \setminus (X \cup Y)$. We show first that S is empty. Otherwise, by considering the partition X, Y, S of G, we deduce that G is isomorphic to its restriction on, say, $X \cup S$ (and not on $X \cup Y$, since in this case, every vertex of G would be contained in a full anti-interval). The crucial fact is now that every full anti-interval of G restricted on $X \cup S$ is also a full anti-interval of G, and this is impossible. Therefore, $S = \emptyset$; in particular, the full anti-intervals of G form a perfect matching (that is, a set of pairwise nonincident edges). Now the partition

$$\{x_1\}, \{y_1\}, V \setminus \{x_1, y_1\}$$

gives a contradiction.

Thus, G has an interval, and by taking complements if necessary, we can assume that there exists a full interval $\{x, y\}$. The relation

$$x \sim y \text{ if and only if } \{x, y\} \text{ is a full interval,}$$

is an equivalence relation. Name the partition of G into its $\sim$-equivalence classes a *full partition*, with its classes named *full classes*. The full classes are cliques. It is routine to check that if an induced subgraph H of G has at least one vertex in each full class of G, then the full partition of H is the restriction of the full partition of G. Suppose, to obtain a contradiction, that a full class $\{x, y\}$ of G contains exactly two vertices. Then the partition $\{x\}, \{y\}, V \setminus \{x, y\}$ implies that some full classes of G are singletons. Now enumerate the full classes of G which have exactly two elements

$$\{x_1, y_1\}, \{x_2, y_2\}, \ldots .$$

The partition X, Y, S, where X is the union of the x_i's, Y is the union of the y_i's, and S is the set $V \setminus (X \cup Y)$, gives a contradiction.

If one full class of G is finite and has exactly three vertices x, y, z, then the partition

$$\{x\}, \{y\}, V \setminus \{x, y\}$$

gives a full class with two elements. More generally, we can prove that there are no full classes with exactly n elements, where $n \geq 3$. We may therefore suppose that every full class has 1 or $\aleph_0$ many vertices. If there exist at least two infinite full classes X, Y, then $G \cong G \restriction (X \cup Y)$. To see this, fix $\{x, x'\} \subseteq X$, $\{y, y'\} \subseteq Y$, and consider the partition

$$V \setminus (X \cup Y), \{x, x'\} \cup Y \setminus \{y, y'\}, \{y, y'\} \cup X \setminus \{x, x'\}$$

of $V(G)$. In this case, G or $\overline{G}$ is $K_{\aleph_0} \uplus K_{\aleph_0}$: since X and Y are full classes, if one vertex of X is joined to a vertex of Y, then every vertex of X is joined to every vertex of Y. For that reason, G is one of $\overline{K_{\aleph_0}} \vee \overline{K_{\aleph_0}}$ or $K_{\aleph_0} \uplus K_{\aleph_0}$.

Assume that G has exactly one infinite full class C. By a partition argument, we can assume that C is joined or not joined to all the vertices of $V \setminus C$. To see this, let W be the set of vertices not in C. Each vertex in W is either joined to each vertex of C or to no vertex of C. Let A be the set of vertices in W joined to each vertex of C, and let B be the set of vertices in W joined to no vertex of C. Assume that both A and B are nonempty. Consider the partition A, B, C of V. If $G \restriction (C \cup X) \cong G$, where $X \in \{A, B\}$ then we obtain the desired conclusion. Suppose that $G \cong G \restriction (A \cup B)$ via an isomorphism f. Then $f(C) = C'$ is an infinite full class in $H = G \restriction W$. If C' is contained entirely in A or B, then C' is also a full class in G, which gives a contradiction. Hence, $C' \cap X \neq \varnothing$, where $X \in \{A, B\}$. Then one of $C' \cap A$ or $C' \cap B$ is infinite; suppose that $C' \cap A$ is infinite (the other case is similar). Then it is straightforward to check that any pair $\{x, y\}$ of distinct vertices in $C' \cap A$ is a full interval in G, which gives a contradiction.

Suppose that $G = C \uplus W$. Fix a partition A, B of W. As we have discussed earlier, $G \ncong G \restriction (A \cup B)$. Hence, by the $\mathcal{P}(3, 2)$ property, we must have that $G \cong G \restriction (V(C) \cup X)$, where $X \in \{A, B\}$, via an isomorphism f. It is not hard to see that $f(C) = C$. From this it follows that $H = G \restriction V(W)$ must have $\mathcal{P}(2, 1)$. The only case that does not give a contradiction is for H to be either K_1 or $\overline{K_{\aleph_0}}$.

The final case is when $G \equiv C \vee W$. By taking complements, we may assume that G has infinitely many isolated vertices, and $G = I \uplus W$ where I is the set of isolated vertices of G. (In fact, $G = I \uplus W^c$. For ease of notation, we write W rather than W^c.)

If one vertex of W is universal in W, then the conclusion follows: partition $V(G)$ into the set U of universal vertices in W, the set $V(W) \setminus U$, and $V(I)$. Then $G \cong G \restriction (U \cup V(I))$ and so $G \cong K_{\aleph_0} \uplus \overline{K_{\aleph_0}}$.

Suppose, in order to reach a contradiction, that no vertex of W is universal. We prove first that G has some vertices with degree 1. Suppose that there exists $x \in V(W)$ such that $W - x$ is isomorphic to G via an isomorphism f. Then $f(I)$ is a set of isolated vertices in $W - x$. Since no vertex is isolated in W (by choice of I), it follows that each vertex of $f(I)$ is of degree 1 in G.

Now suppose that there is no $x \in V(G)$ so that $W - x$ is isomorphic to G. Fix $x \in V(W)$. Then, by hypothesis, $A = N(x) \subsetneq V(W)$ and $B = N^c(x) \cap V(W)$ are nonempty, with $|A| \geq 2$.

Fix $a \in A$. Consider the partition

$$V(I) \cup \{x\},\ A \setminus \{a\},\ B \cup \{a\}$$

of $V(G)$. If $V(I) \cup \{x\}$ is deleted, then we are left with $W - x$, which, by hypothesis, is not isomorphic to G. Now suppose that $G \cong G \restriction (V(I) \cup \{x\} \cup A \setminus \{a\})$ via an isomorphism f. Then $f(I) = I$ and $f(W) = G \restriction (\{x\} \cup A \setminus \{a\})$. But x is universal in $G \restriction (\{x\} \cup A \setminus \{a\})$ which would imply the contradiction that W also has a universal vertex. Hence,

$$G \cong G \restriction (V(I) \cup \{x\} \cup B \cup \{a\}) = H;$$

but x has degree 1 in H, and so some vertex of G has degree 1.

Therefore, G has some vertices of degree 1, and some vertices with degree 0. Define the *reduction* of a graph G to be the graph G' obtained from G by deleting the vertices of G with degree 0 and 1 (note that G' may be empty).

We may iterate the number of reductions (possibly taking transfinitely many reductions) until either the empty graph is obtained, or we obtain a graph with no vertex of degree 0 or 1. In the latter case, the induced subgraph obtained is unique. We call this unique induced subgraph the *nucleus* of G, and is denoted $Nu(G)$. We leave it as an exercise to check that the vertices not in $Nu(G)$ induce a forest (that is, a graph with no finite circuits).

Case 1. $Nu(G)$ is empty.

In this case, G is a forest with some isolated vertices. If all vertices are isolated, we are done. If not all vertices are isolated, let X be the set of nonisolated vertices. Since $H = G \restriction X$ is 2-colourable with no isolated vertex, we may partition H into two nonempty independent sets A, B which correspond to the two colours. The partition, A, B, I of $V(G)$ gives a contradiction: deleting either A or B leaves only isolated vertices, and deleting $V(G) \setminus (A \cup B)$ leaves no isolated vertices.

Case 2. $Nu(G)$ is not empty.

In this case, either there is an edge between $Nu(G)$ and $G \setminus V(Nu(G))$ or not. Suppose that there is no such edge. Then G is the disjoint union of $Nu(G)$ and a forest F. Fix some 2-colouring of F into nonempty independent sets A and B. Consider the partition

$$V(Nu(G)), A, B.$$

Deleting $V(Nu(G))$ leaves a graph with an empty nucleus; deleting A or B results in a graph with no vertex of degree 1.

The only remaining case is that $Nu(G)$ is not empty and there is some edge between a vertex of $Nu(G)$ and some vertex of $V(G) \setminus V(Nu(G))$. In this case, we denote by O the set of vertices of $V(G) \setminus V(Nu(G))$ joined to some vertex of $Nu(G)$. The partition

$$V(I), O, V(G) \setminus (O \cup V(I))$$

gives a contradiction. To see this, note that deleting $V(I)$ leaves a graph with no isolated vertex. Deleting O leaves a graph with the same nucleus as G, but with no vertex outside $Nu(G)$ joined to a vertex of $Nu(G)$. Deleting $V(G) \setminus (O \cup V(I))$ leaves a forest which as we have determined earlier, must be the complement of a clique. This contradiction completes the proof. □

3. Linear Orders With the $\mathcal{P}(3, 2)$ Property

We divide the classification of the $\mathcal{P}(3, 2)$ linear orders into cases depending on whether or not there are endpoints. We will make use of the following property of oriented graphs.

Principle of Directional Duality. For each property of oriented graphs, there is a corresponding property obtained by replacing every concept by its converse.

Since the only finite oriented graphs with the $\mathcal{P}(3, 2)$ property are the one and two element linear orders, we will consider only infinite linear orders.

3.1. The case when there is a source or sink.

We first consider the case of the well-orders with $\mathcal{P}(3, 2)$.

THEOREM 3. *The countable ordinals with the* $\mathcal{P}(3, 2)$ *property are*

$$L = \omega^\alpha m + \omega^\beta n,$$

where α, β, m, n *are countable ordinals and* $0 < m + n \leq 2$, $\alpha + \beta > 0$.

PROOF. Suppose that L is an ordinal that satisfies $\mathcal{P}(3, 2)$. By Cantor's normal form theorem (see Theorem 3.46 of [10]), there are ordinals $\alpha_1 > \cdots > \alpha_k$ for $k \in \omega - \{0\}$, and $n_1, \ldots, n_k \in \omega - \{0\}$ such that

$$L = \omega^{\alpha_1} n_1 + \cdots + \omega^{\alpha_k} n_k.$$

By the $\mathcal{P}(3, 2)$ property, we have that $k \leq 2$. Otherwise, consider the partition

$$\omega^{\alpha_1} n_1, \omega^{\alpha_2} n_2, \omega^{\alpha 3} n_3 + \cdots + \omega^{\alpha_k} n_k$$

to obtain a contradiction. In a similar fashion, we have that $n_1 + n_2 \leq 2$.

For sufficiency, consider the case when $m = n = 1$ (the other cases are similar). Suppose that the vertices of $L = \omega^\alpha + \omega^\beta$ are partitioned into A, B, C. Define $X_i = X \cap \omega^i$ where $X \in \{A, B, C\}$ and $i \in \{\alpha, \beta\}$. By the $\mathcal{P}(2, 1)$ property, there are $Y, Z \in \{A, B, C\}$ so that the suborders on Y_α and Z_β are isomorphic to ω^α and ω^β, respectively. If $Y = Z$, choose some $W \in \{A, B, C\} \backslash \{Y\}$. Now $\omega^i \leq \omega^i \restriction (Y_i \cup W_i) \leq \omega^i$ so that $\omega^i \cong \omega^i \restriction (Y_i \cup W_i)$. (We use here the property that if two ordinals are mutually embeddable they are isomorphic; see Theorem 3.14 of [10].) Hence, $L \restriction (Y \cup W) \cong L$. If $Y \neq Z$, by a similar argument, $L \restriction (Y \cup Z) \cong L$. □

Since $\mathcal{P}(3, 2)$ is preserved by taking reversals, Theorem 3 classifies the reversals of ordinals with the $\mathcal{P}(3, 2)$ property. To complete the classification of the $\mathcal{P}(3, 2)$ linear orders with an endpoint we prove the following theorem.

THEOREM 4. *The countable linear orders L with the $\mathcal{P}(3, 2)$ property with an endpoint and with the property that $L, L^* \notin ON$, are*

$$\omega^\alpha + (\omega^\beta)^*,$$

where α, β are nonzero countable ordinals satisfying $\alpha + \beta > 0$.

PROOF. The argument for sufficiency uses the facts that ω^α and $(\omega^\beta)^*$ satisfy $\mathcal{P}(2, 1)$. Since the details are similar to the proof of sufficiency of Theorem 3, they are omitted.

For necessity, suppose that L satisfies the hypotheses of the theorem. By the principle of directional duality, we can assume, without loss of generality, that L has a first element 0. By hypothesis, we may assume that L is not a well-order.

We write $L = (A, C)$, where $L = A + C$ and A is the maximal initial section of L which is well-ordered. Since $0 \in A$, A is nonempty. It is not hard to see that if L is isomorphic to an order $L' = (A', C')$, then A is isomorphic to A' and C is isomorphic to C'.

We claim that both A and C satisfy $\mathcal{P}(2, 1)$. Once the claim is proven, the proof of the theorem will follow. Partition A into nonempty parts A_1 and A_2, and partition C into nonempty parts C_1 and C_2. Assume, for instance, for $\mathcal{P}(3, 2)$, that $L \cong L \restriction (A_1 \cup C)$ and $L \cong L \restriction (A \cup C_1)$. Since $L \restriction (A_1 \cup C) = (A_1, C)$, we have that $A_1 \cong A$ and so A satisfies $\mathcal{P}(2, 1)$. Suppose for the $\mathcal{P}(3, 2)$ property that $L \restriction (A \cup C_1) \cong L$. Set $L \restriction (A \cup C_1) = (A', C')$, noting that $A \subseteq A'$. Since $(A, C) \cong (A', C')$, we have that $A = A'$, and thus $C \cong C' = C_1$. Thus, C satisfies $\mathcal{P}(2, 1)$. □

3.2. The linear orders without endpoints with the $\mathcal{P}(3, 2)$ property. In the case when there are no endpoints we have the following classification of the countable $\mathcal{P}(3, 2)$ linear orders.

THEOREM 5. *The countable $\mathcal{P}(3, 2)$ linear orders without endpoints are the following linear orders and their converses: $(\omega^\alpha)^* + \omega^\beta$, where α, β are nonzero ordinals, and $\omega^\gamma \cdot \omega^* + \omega^\delta$ for some ordinals satisfying $0 \leq \gamma$ and $0 < \delta$.*

PROOF. Let L be a $\mathcal{P}(3, 2)$ linear order. We define the equivalence relation $\equiv$ on L: $x \equiv y$ if the interval $[x, y]$ of L is finite. (For more on this equivalence relation, see Section 4.2 of [10].) We first prove that every $\equiv$-class of L is infinite. To see this, note that $\mathcal{P}(3, 2)$ implies that every finite $\equiv$-class is a singleton. Indeed, if there exists a finite $\equiv$-class with exactly n elements, for some $n > 1$, then partition $V = V(L)$ into A, B, C, where A contains exactly one element in all the $\equiv$-classes with exactly n elements, B contains the other elements in the $\equiv$-classes with exactly n elements, and C contains the elements not in $A \cup B$. This partition either yields singleton $\equiv$-classes, or forces each $\equiv$-class to have exactly n elements; a suitable partition proves the latter case to be impossible.

Denote by S the set of singleton $\equiv$-classes. Suppose for a contradiction that there exists two elements x and y in S. Without loss of generality, we may assume that $x < y$. We write $L = A + x + B + y + C$. Choose $a \in V(A)$ and $c \in V(C)$. We claim that the partition

$$(V(A) \setminus \{a\}) \cup \{x\}, (V(C) \setminus \{c\}) \cup \{y\}, V(B) \cup \{a\} \cup \{c\}$$

violates $\mathcal{P}(3, 2)$. To see this, observe that the only case that does not have endpoints is $L \cong L \restriction V(L) \setminus (V(B) \cup \{a\} \cup \{c\})$. But this case is also impossible since $\{x, y\}$ is now an $\equiv$-class. If S has exactly one element x, then we may write $L = A + x + B$, with A and B nonempty (otherwise, L would have an endpoint). But then the partition $A, \{x\}, B$ violates $\mathcal{P}(3, 2)$.

Therefore, every $\equiv$-class is infinite. We next prove that for every partition into two summands $L = A + B$, either A is the reverse of an ordinal or B is an ordinal. Assume that this is not the case, and some fixed partition $A + B$ does not satisfy this. There exists an initial section S_A in A with no maximum and a final section S_B in B without minimum. We first prove that we can suppose that $L = S_A + C + S_B$ with C nonempty. Otherwise, assume that $L = S_A + S_B$ and fix a vertex $a \in S_A$ and $b \in S_B$. We partition L into

$$V(X), \{a\} \cup V(Y) \cup \{b\}, V(Z),$$

where $L = X + a + Y + b + Z$. The sets $V(X), V(Z)$ are nonempty to avoid endpoints. To avoid endpoints and to satisfy $\mathcal{P}(3, 2)$, we must have that $L \cong L \restriction (V(X) \cup V(Z)) = L'$. Since $L = S_A + S_B$, we can find in L' an initial section S'_A without a maximum and a final section S'_B without a minimum so that $L' = S'_A + S'_B$. If $S'_A = X$ and $S'_B = Z$, then we may choose $C = a + Y + b$. Suppose now that $S'_A \subsetneqq X$. (The case when $X \subsetneqq S'_A$ is similar and so omitted.) Let $S' = S'_A$ and $S'' = S_B$. Then S' is an initial section with no maximum and S'' is a final section with no minimum, and we may choose (the nonempty set) C to be the vertices greater than S' but less than S''.

Thus, there exists a partition $S_A + C + S_B$ with C nonempty. Fix $a \in S_A$, $b \in S_B$ and $c \in C$. By considering the following partition for $\mathcal{P}(3, 2)$

$$(S_A \setminus \{a\}) \cup \{c\}, \{a\} \cup (V(C) \setminus \{c\}) \cup \{b\}, S_B \setminus \{b\},$$

we obtain either an endpoint, or $\{c\}$ as an $\equiv$-class. Each case gives a contradiction.

We may therefore assume that $L = A + O$ where A is a linear order and O is an ordinal (which is a limit ordinal since L has no greatest element).

Case 1. Suppose that $L = (O')^* + A'$, for some ordinal O' and some linear order A'.

Then O' is a limit ordinal (since L has no least element), and $L = (O')^* + A'' + O$, where A'' is a linear order. If A'' is nonempty, then we may consider the partition $(O')^* \setminus \{x\}, A'' \cup \{x, y\}, O \setminus \{y\}$, where $x \in (O')^*$ and $y \in O$, to reduce to the case when $A'' = \emptyset$. The choice of $(O')^*$ and O are unique in this notation, and thus, $(O')^*$ and O have $\mathcal{P}(2, 1)$. So $L = (\omega^\alpha)^* + \omega^\beta$, for some ordinals $\alpha, \beta > 0$.

Case 2. No initial section of L is the reverse of an ordinal, and so every proper final section of L must be an ordinal.

Write $L = A + B$, where B is the least nonzero ordinal with this property. It is straightforward to check that B has $\mathcal{P}(2, 1)$, and is therefore infinite (since L has no endpoints). The linear order B, which is a countable ordinal power of ω, has the property that $O + B = B$ when O is an ordinal satisfying $O < B$.

Case 2.1. Suppose that there is a decomposition $L = A + C + B$, where C is an ordinal satisfying $C > B$.

Hence, there is an ordinal C' so that $C = B + C'$ so that $L = X + B_1 + C' + B_2$, where X is some linear order, and $B_1, B_2 \cong B$. Partition L into

$$V(X) \cup V(B_1), V(C') \cup \{x\}, V(B_2) \setminus \{x\},$$

where $x \in V(B_2)$. Deleting $V(X) \cup V(B_1)$ leaves an ordinal. Deleting $V(B_2) \setminus \{x\}$ leaves a last element. Therefore, $L \cong L \restriction (V(X) \cup V(B_1) \cup V(B_2 - x))$. Since B has $\mathcal{P}(2, 1)$, $B_2 - x \cong B_2$, so $L \cong X + B + B$.

Applying this argument inductively gives either that $L \cong Z + B \cdot \omega^*$ or $L \cong Z + B \cdot n$, where Z has no proper final section equal to B and $n \geq 1$. This last case gives directly by $\mathcal{P}(3, 2)$ that $n = 1$, so $L \cong Z + B$. Recall that $L \cong X + B_1 + B_2$, where $B_1, B_2 \cong B$. Suppose that $Z + B$ is isomorphic to $X + B_1 + B_2$ via an isomorphism f. By the choice of Z, $f(B)$ properly contains B_2. Since $B + B > B$, $f(B)$ cannot properly contain $B_1 + B_2$. Therefore, there are nonzero ordinals α, β so that $B_1 = \alpha + \beta$ and $f(B) = \beta + B_2$. It follows that $\beta \not\cong B$, and so by $\mathcal{P}(2, 1)$, $\alpha \cong B$. But then $f(Z) \cong X + \alpha \cong X + B$, which contradicts that Z has no final section equal to B.

Thus, $L = Z + B \cdot \omega^*$, where Z has no final section equal to B. Fix $z \in V(Z)$. Then the final section $\{x : x \geq z\}$ contains ω^*, which is a contradiction since we are in Case 2. Hence, $L = B \cdot \omega^*$, and so L has the desired structure.

Case 2.2. Every partition $L = A + C + B$ satisfies $C < B$, and thus, $C + B = B$.

Thus, every proper final section of L is isomorphic to B, where $B = \omega^\delta$ for some nonzero ordinal δ. An element x of L is a *bad cut* if, writing $L = L_1 + x + R$, the order L_1 has the property that all its proper final sections are isomorphic. If x is a bad cut, then we claim that every proper final section of L_1 is isomorphic to ω^γ, for some countable $\gamma \in ON$. To see this, fix a proper final section S of L_1. Since $S + x + R$ is a final section of L, S is an ordinal. If $S = \alpha + \beta$, where $\beta \neq 0$, then β is a proper final section of L_1 and so equals S. The ordinal S is therefore additively indecomposable and the claim follows (see Exercise 10.4 (6) of [10]). We say that the *type* of the bad cut x is γ.

If x, y are bad cuts, and $x < y$ in L, then the type of x is certainly strictly smaller than the type of y; and from this, if there is a bad cut, then there exists a minimum bad cut b. In other words, for every $y < b$, writing $L = L_y + y + R$, the order L_y can be partitioned in a unique way into $L_y = X + Y$, where Y is an ordinal and every proper final section of X is greater than or equal to Y. (Every proper final section of X is a suborder of a proper final section of L, and so is an ordinal.)

If there are no bad cuts, then choose y to be any element of L. Otherwise, choose $y < b$. We decompose L as follows. Let $L = L_1 + y + R$ and $L_1 = L_2 + A_1$, where $A_1 \cong \omega^{\alpha_1}$, is the unique partition of L_1 such that every proper final section of L_2 has ordinal type greater or equal to ω^{α_1}. More generally, we define $L_i = L_{i+1} + A_i$, where $A_i \cong \omega^{\alpha_i}$, as the unique partition of L_i such that every proper final section of L_{i+1} has ordinal type at least ω^{α_i}. By this decomposition, we may write

$$L = X + \sum_{i \in \omega^*} \omega^{\alpha_i}$$

with $\alpha_0 = \delta$, the order-type of B. Since every proper final section of L is an ordinal, X is empty.

The increasing ordinal sequence

$$\alpha_1, \alpha_2, \alpha_3, \alpha_4, \ldots$$

is denoted by $s(y)$. If there is a bad cut and $y, y' < b$, or there is no bad cut and y, y' are arbitrary, then the sequences $s(y)$ and $s(y')$ are equal after some finite number of terms. To see this, suppose that $y' < y$ and y' belong to A_i in the decomposition of L which starts at y. Then the sequence $s(y')$, after its first i terms is equal to $\alpha_{i+1}, \alpha_{i+2}, \ldots$. So two decompositions of $L = \sum_{i \in \omega^*} \omega^{\alpha_i}$, where $\alpha_i \leq \alpha_j$, for every $0 < i < j$, must be the same up to a finite number of terms.

If in addition L has $\mathcal{P}(3, 2)$, then we claim that for every decomposition, the sequence (α_i) is constant after a finite number of terms. Otherwise, the partition O, E, ω^{α_0}, where O is the union of the ω^{α_i} with i odd, and E is the union of the ω^{α_i}, with $i > 0$ even, violates $\mathcal{P}(3, 2)$. To see this, note first that since we are in Case 2.2, then we cannot have that $L \cong L \restriction (O \cup E)$. Suppose that $L \cong L \restriction (O \cup \omega^{\alpha_0}) = L'$ (the other case is similar). L' gives rise to the sequence

$$\beta = (\beta_i) = (\alpha_0, \alpha_1, \alpha_3, \ldots).$$

Let α be the sequence $(\alpha_i : i \in \omega)$. By the last sentence of the previous paragraph, we must have that there is a $k_0 \in \omega$ so that for $k > k_0$, $\beta_k = \alpha_k$. But then we obtain the equalities

$$\alpha_{(2k_0-1)+2j} = \alpha_{k_0+j},$$

where $j > 0$. But since α is increasing, these equalities imply that α is constant after α_{k_0}.

Hence,

$$L = \omega^\gamma \cdot \omega^* + \omega^{\alpha_{k_0}} + \cdots + \omega^{\alpha_1} + \omega^{\alpha_0},$$

for some γ such that $0 < \alpha_0 \leq \cdots \leq \alpha_{k_0} \leq \gamma$. The $\mathcal{P}(3, 2)$ property implies that

$$L = \omega^\gamma \cdot \omega^* + \omega^\delta,$$

where $\delta = \alpha_0$. □

4. The Tournaments With the $\mathcal{P}(3, 2)$ Property

The notions of an *r-extendable* set of vertices in a tournament and an *n-e.c. tournament* are similar to the corresponding notions for graphs, and so we omit the definitions. The *infinite random tournament*, written T^∞, is the unique tournament that is n-e.c. for all $n \geq 1$.

The following definitions apply in any oriented graph. The *in-neighbours* of vertex x are the vertices y so that (y, x) is an arc; the *out-neighbours* of x are the vertices y so that (x, y) is an arc. A vertex x is a *source* if it has no in-neighbours, and a *sink* if it has no out-neighbours. If (x, y) is an arc, then we say that x *dominates* y and y is *dominated by* x.

Following the proof of Theorem 1, no 2-e.c. $\mathcal{P}(3, 2)$ tournament exists. The proof of this is nearly identical to the proof of Theorem 1 and is therefore omitted. What remains is to classify the $\mathcal{P}(3, 2)$ tournaments which fail to be 2-e.c. We prove in the following theorem that every $\mathcal{P}(3, 2)$ tournament is a linear order. Theorems 3–6 finish the classification of the $\mathcal{P}(3, 2)$ tournaments. For nonempty sets of vertices A and B, the notation $A \to B$ means that each vertex of A dominates each vertex of B.

Theorem 6. *The tournaments with the $\mathcal{P}(3, 2)$ property are linear orders.*

PROOF. Let T be a $\mathcal{P}(3,2)$ tournament. We may assume that T is infinite. As in the proof of Theorem 2, we first prove that T has an *interval*: two distinct vertices x, y with the same out-neighbourhood in $V(T) \setminus \{x, y\}$.

If T has a source x, then we prove there is an interval. (The case when T has a sink follows by directional duality.) Choose y in $V(T) \setminus \{x\}$ and partition $V(T) \setminus \{x, y\}$ into the out-neighbours A of y, and the in-neighbours B of y different from x. Consider the partition $\{x, y\}, A, B$.

If $T \cong T \restriction (A \cup \{x, y\})$, then $\{x, y\}$ is an interval, so we may assume that $B \neq \emptyset$. Suppose that $T \cong T \restriction (B \cup \{x, y\}) = X$. Then y is a sink in X. We claim that an interval exists. To see this, consider the partition $\{x\}, \{y\}, B$. It follows that $X \restriction (\{z\} \cup B) \cong X$, where z is either x or y. If $X \restriction (\{x\} \cup B) \cong X$, then since $X \cong T$, it follows that $X \restriction (\{x\} \cup B)$ has a sink, s, which must be in B. But then $\{y, s\}$ is an interval in X, and therefore, there is an interval in T. If $X \restriction (\{y\} \cup B) \cong X$, then since $X \cong T$, $X \restriction (\{y\} \cup B)$ has a source, t, which must be in B. But then $\{x, t\}$ is an interval in X, and we conclude that there is an interval in T.

The final case is if $T \cong T \restriction (A \cup B)$; then there exists a source x' in $A \cup B$. If x' belongs to B, then x' dominates every vertex in T except x, thus, $\{x, x'\}$ is an interval in T. If $x' \in A$, then partition T into

$$\{x\}, (A \setminus \{x'\}) \cup \{y\}, B \cup \{x'\}.$$

Deleting $B \cup \{x'\}$ gives the interval $\{x, y\}$. Deleting $\{x\}$ gives a source s in $T - x$. Since (y, x') is an arc, $s \neq x'$. Since x' dominates each vertex of $(A \cup B) \setminus \{x'\}$, we must have that $s = y$. Hence, $B = \emptyset$. Thus, $\{x, y\}$ is an interval. Finally, deleting $(A \setminus \{x'\}) \cup \{y\}$ gives that $\{x, x'\}$ is an interval.

Next, we assume that T has neither a source nor a sink. From the tournament analogue of Theorem 1, it follows that T has a nonextendable pair of vertices. If x, y is one such pair of nonextendable vertices in $V(T) = V$, then partition $V \setminus \{x, y\}$ into four subsets

$$S_{00}, S_{01}, S_{10}, S_{11},$$

where S_{00} is the set of vertices dominating x and y, S_{01} is the set of vertices dominating x and not y, S_{10} is the set of vertices dominating y but not x, and S_{11} is the set of vertices dominated by x and y.

Suppose first that x, y is 3-extendable.

Case 1. $S_{11} = \emptyset$. We partition V into $\{x\} \cup S_{01}$, $\{y\} \cup S_{10}$ and S_{00}. Since T is a $\mathcal{P}(3,2)$ tournament, the induced subtournament on the union of two of these subsets is isomorphic to T. Two cases give sinks, so the sole remaining case is $T \restriction (\{x, y\} \cup S_{01} \cup S_{10}) \cong T$, in which case x, y is 2-extendable in the induced subtournament, and so there is a 2-extendable pair of vertices in T.

Case 2. $S_{10} = \emptyset$. We partition V into $\{x\} \cup S_{00}$, $\{y\} \cup S_{11}$ and S_{01}. Two cases for $\mathcal{P}(3,2)$ give a source or a sink, so the sole remaining case is $T \restriction (\{x, y\} \cup S_{00} \cup S_{11}) \cong T$, in which case $\{x, y\}$ is an interval.

The other cases are similar. If x, y is 1-extendable, then $\{x, y\}$ is an interval or an *anti-interval*: a pair of vertices $\{a, b\}$ such that whenever (a, z) is an arc, then (z, b) is an arc, where $z \neq a, b$.

Consider the final case when there exists a pair x, y which is 2-extendable and assume that $\{x, y\}$ is neither an interval nor an anti-interval. The sole case then (by directional duality) is when S_{01} and S_{11} are nonempty. The partition

$$\{x\}, S_{01} \cup \{y\}, S_{11}$$

then gives either a source or an anti-interval. If we have a source, then we obtain an interval by previous arguments.

To prove that we have an interval, it is enough to show that the existence of an anti-interval $\{a, b\}$ in T gives a contradiction or an interval. By directional duality, we may suppose that (a, b) is an arc. Throughout, when referring to an interval or an anti-interval $\{a, b\}$, it will be implicitly assumed that (a, b) is an arc.

If T has two distinct anti-intervals $\{a, b\}$, $\{c, d\}$ which intersect, then if $b = d$, $\{a, c\}$ will be an interval. A similar conclusion holds when $a = c$. We can therefore assume that $b = c$ or $a = d$. Without loss of generality, suppose $b = c$, and so (d, a) is an arc. The set of vertices

$$V(T) \setminus \{a, b, d\}$$

admits a partition into A the out-neighbours of a not equal to b, and B the in-neighbours of a not equal to d. Observe that $A \to b \to B$ and $B \to d \to A$. The partition $A \cup \{a\}, \{b\}, B \cup \{d\}$ gives either the interval $\{a, d\}$, or b as a source or sink. In any case, we have an interval.

Thus, we can assume that the anti-intervals are disjoint. Enumerate the anti-intervals of T as

$$\{x_1, y_1\}, \{x_2, y_2\}, \ldots .$$

Denote by X the union of the x_i's, by Y the union of the y_i's, and by S the set $V \setminus (X \cup Y)$. We first reduce to the case when S is empty. Otherwise, by considering the partition X, Y, S of T, we deduce that T is isomorphic to $T \restriction (X \cup S)$ or $T \restriction (Y \cup S)$ (and not to $T \restriction (X \cup Y)$, since in that case, every vertex of T would be contained in an anti-interval and so S would be empty). Suppose that $T \cong T \restriction (X \cup S)$ (the other case is similar). Every anti-interval of $T \restriction (X \cup S)$ is an anti-interval of T, which gives a contradiction.

We may therefore assume that S is empty; in particular, the anti-intervals of T form a perfect matching (that is, a set of pairwise nonincident directed edges). Now the partition

$$\{x_1\}, \{y_1\}, T \setminus \{x_1, y_1\}$$

gives a contradiction.

We conclude that T has an interval. We now introduce an extension of the notion of interval. A *chain-interval* is a subset S of V such that $T \restriction S$ is a linear order, and every element outside of S either dominates S or is dominated by S. An important property of chain-intervals is that a (not necessarily finite) union of pairwise intersecting chain-intervals is a chain-interval. Thus, using Zorn's lemma, we may consider maximal chain-intervals of T; moreover, the set of vertices of T is partitioned into chain-intervals. By the fact that T has an interval, there exists one nontrivial chain-interval. By the $\mathcal{P}(3, 2)$ property and an argument similar to one in the proof of Theorem 2, T has either two infinite chain-intervals, which results in a linear order, or a unique infinite chain-interval and possibly some singleton chain-intervals.

We consider the case when there is a unique infinite chain-interval C. If $C = T$, then the theorem follows, so we may assume C is a proper subtournament of T. The tournament C satisfies $\mathcal{P}(2, 1)$ by uniqueness. We assume that $C = \omega^\alpha$, where α is a nonzero ordinal. The case when C is the reversal of an ordinal follows by directional duality. Let us denote by A and B the partition of $V(T) \setminus C$ such that $A \to C$ and $C \to B$. Now consider for $\mathcal{P}(3, 2)$ the partition A, B, C.

If $T \cong T \restriction (A \cup B)$, then there exists a unique infinite chain-interval C' in $A \cup B$. Let $T' = T \restriction (A \cup B)$. Denote by A' and B' the intersection of A and B with C', respectively. Since C is the unique infinite chain-interval of T and has order-type ω^α, in order to avoid in C' an interval of T (which would be disjoint from C, and thus violate our hypothesis that there is a unique nontrivial chain-interval in T), it is necessary that the successor and the predecessor in C', if any, of an element of A' are elements of B', and that the successor and the predecessor of an element of B' are elements of A'. In particular, the order-types of A' and B' are exactly the order-type of C', which is the order-type of C. (We are using the crucial fact here that C has order-type ω^α.) Consider now the partition

$$A', B', A \cup B \setminus (A' \cup B')$$

of $V(T')$. If $T' \cong T' \restriction (A' \cup B')$ we are done, since T' and hence, T, are linear orders. If $T' \cong T' \restriction (V(T') \setminus B')$, then A' and C are infinite chain-intervals of T. Since there exists at most one infinite chain-interval in T, A' and C must be contained in a larger infinite chain-interval of T, which must be isomorphic to C by uniqueness. Since the order-type of $A' + C$ is $C + C$, and the order-type of A' is C, we violate the left-cancellation law of ordinals (see Theorem 3.10 of [10]). The same contradiction occurs if $T \cong T \restriction (V \setminus A')$: we would obtain the conclusion that the order-type of $C + B'$ is the order-type of C.

We must therefore have that $T \cong T \restriction (A \cup C)$ or $T \cong T \restriction (B \cup C)$. By directional duality, we now have the following situation: T is isomorphic to $C \to B$, where C is an ordinal power of ω or the reversal of such an ordinal, and B has no nontrivial chain-intervals.

We now prove that there is some vertex in B of in-degree 1. Fix a vertex x in B. We denote by X and Y the in-neighbours and out-neighbours of x in B, respectively. Observe that since C is a maximal chain-interval, X is nonempty. Let $y \in X$. If $X \setminus \{y\} = \emptyset$, then x has in-degree 1 in T. We may therefore assume that $X \setminus \{y\} \neq \emptyset$. We partition $V(T)$ into

$$C \cup \{x\}, Y \cup \{y\}, X \setminus \{y\}.$$

If $T \cong T \restriction ((C \cup \{x\} \cup (X \setminus \{y\})) = T'$, then x is a sink in T'. Consider the partition of $V(T')$ into

$$C, X \setminus \{y\}, \{x\}.$$

Deleting $X \setminus \{y\}$ leaves $C \to x$, which is a linear order. If $T' \cong T'(X \setminus \{y\} \cup \{x\}) = T''$, then T'' has a chain-interval C'' isomorphic to C. It is not hard to see that C'' is a chain-interval of T', and by the maximality and uniqueness of C, we must have that C and C'' are contained in a chain-interval of T' isomorphic to C. If C is an ordinal power of ω, then this violates the left-cancellation law for ordinals. If the order-type of C is $(\omega^\alpha)^*$ for some nonzero ordinal α, then we may use the fact that $(\omega^\alpha)^* \to (\omega^\alpha)^* \ncong (\omega^\alpha)^*$ to obtain a contradiction.

This forces $T' \cong T' - x$, which is impossible: $T' - x$ would contain a sink x', which in turn, with x, would be a nontrivial chain-interval in T' disjoint from C, which as before would give a contradiction.

If $T \cong T \restriction ((X \cup Y) \setminus \{x\})$ via an isomorphism f, then the image under f, say C', of C in $X \cup Y$ would alternate from X to Y. Suppose that C'_X is the part of C' intersecting X; C'_Y is defined similarly. We consider the partition

$$C'_X, C'_Y, V \setminus (C'_X \cup C'_Y).$$

As in an argument earlier, this case gives either a contradiction or gives that T is a linear order.

Thus, $T \cong T \restriction (C \cup \{x\} \cup Y \cup \{y\})$ via an isomorphism g. In other words, (with the notation that $T = C \to B$) B has a vertex of in-degree 1 relative to B (the pre-image of x under g); we denote it by x_0.

Given a tournament T', the *chain-reduction* of T' is the operation in which we delete all the vertices of a maximal linear order L satisfying $T' = L \to A$. A *point-reduction* of T' is the tournament obtained from T by deleting one vertex of in-degree 1. A *reduction* of T' is obtained by applying a chain-reduction followed by one point-reduction to T'. A tournament which is unchanged by a reduction is *reduced*. Applying some number of reductions to T' (beginning with the chain-reduction of deleting C followed by the point-reduction of deleting x_0; possibly transfinitely many reductions may result after this initial reduction), the process eventually terminates in the empty tournament or a reduced tournament. In the latter case, we call the resulting reduced tournament a *nucleus* of T.

The nucleus is unique, and is written $Nu(T)$. To see this, note first that chain-reductions are unique. After a chain-reduction, the number of vertices of in-degree 1 is between one and three. If there is a unique in-degree 1 vertex x after a chain-reduction, the next point-reduction must delete x.

If there are two vertices x, y of in-degree 1 after a chain-reduction, then the deletion of x will make either y a source, or y a vertex with one in-neighbour z. Since z has in-degree at least 2, the tournament resulting from the deletion of x has no linear order which dominates every other vertex. (Suppose L were such a linear order. Note that if z belongs to L, then each of the in-neighbours of z belongs to L. If y were in L, then since y has in-degree 1, L can have only the vertices y and z, which is a contradiction. If y were not in L, then since y has in-degree 1, L can have only the vertex z, which is a contradiction.) The next reduction consists of one point-reduction: the deletion of y. Therefore, deleting first x and then y, or the contrary, will result in the same tournament after two reductions.

If there are three vertices of in-degree 1 after a chain-reduction, say x, y, and z, they form a directed cycle which dominates all the remaining vertices. Thus, deleting any of these vertices will make the other two into a linear order which dominates all the remaining vertices. This linear order must be deleted in the next chain-reduction, and so the deletion of any one of x, y, and z results in the same tournament after two reductions.

The reduction process defines a linear order L on the vertices not in the nucleus: $x <_L y$ if x has been deleted before y in the reduction, or x and y were deleted in the same chain-reduction and x is less than y in this chain. Thus, for any element x of L, the in-degree of x in the induced subtournament of T' on the set

$$\{y \in L : x < y \text{ in } L\} \cup \{x\}$$

is at most 1.

Case 1. $Nu(T)$ is empty.

In this case, we make use of the linear order L on V defined earlier. Consider the graph G of the oriented graph on T whose arcs are the arcs which are not in L. The vertices outside $Nu(T)$ form a forest in G; hence, in this case, the graph G itself is a forest. The vertices in C are isolated in G, and B gives rise to a forest F. Recall that $T = C \to B$, with C the unique infinite chain-interval of T. Consider a fixed 2-colouring of B with nonempty independent sets B_1, B_2. Consider the partition $V(C), B_1, B_2$. Deleting $V(C)$ leaves a tournament with no nontrivial chain-interval which is a contradiction. Finally, the induced subtournaments on $V(C) \cup B_1$ and $V(C) \cup B_2$ are linear orders: the linear order L restricted to these sets coincides with T.

Case 2. $Nu(T) = N$ is nonempty.

In this case, it is straightforward to see that N and C are disjoint. Partition $V(T)$ into

$$V(C), V(N), V(T) \setminus (V(C) \cup V(N)).$$

The set $V(T) \setminus (V(C) \cup V(N))$ is not empty since it contains x_0 (our vertex of in-degree 1 in B). If $T \cong T \restriction (V(C) \cup V(N)) = T'$ via an isomorphism f, then C is the unique nontrivial chain-interval of T' (this follows as earlier by left-cancellation for ordinals and the fact that $(\omega^\alpha)^* \to (\omega^\alpha)^* \ncong (\omega^\alpha)^*$). Hence, $f(B) = N$. But B has a vertex of in-degree 1, while N does not. Deleting $V(C)$ would result in $T \cong T \restriction B$ via an isomorphism h. But, as described earlier, considering the image of C under h gives a contradiction. We must therefore have that $T \cong T \restriction (V(T) \setminus V(N)) = S$. In this case, we consider the graph G of the oriented graph on arcs of S which are not in L. Deleting N from T leaves C (since C is deleted in the first chain-reduction), and a set F which is a forest in G. If F is empty, then we are finished, since L is isomorphic to C, and so C is a linear order. Assume that F is nonempty. To finish, apply the same argument to S as the one applied to T in Case 1. □

The order type of the rationals is denoted η, and a linear order is *scattered* if it does not contain η as a suborder. Although we do not yet know a classification of the $\mathcal{P}(n, k)$ linear orders for all possible values of the parameters n and k, the following theorem does give some insight into their structure.

THEOREM 7. *If L is a countable $\mathcal{P}(n, k)$ linear order, where $1 \leq k < n$, then L is scattered.*

PROOF. To every countable linear order L, we associate the (countable) set $I(L)$ of intervals of L which are of ordinal order-type. Thus, there is a minimum countable ordinal, $\alpha(L)$, so that no element of $I(L)$ is greater or equal to $\alpha(L)$.

Suppose, to obtain a contradiction, that L is not scattered and satisfies $\mathcal{P}(n, k)$. Then we can find n disjoint intervals $I_1, \ldots, I_n$ of L, each of them containing a suborder of order-type η. Partition each I_j into A_j and B_j in such a way that both $\alpha(A_j)$ and $\alpha(B_j)$ are greater than $\alpha(L)$. To see that this is possible, we apply the following claim with $\beta = \alpha(L)$, $\gamma = \alpha(A_j)$ and $\delta = \alpha(B_j)$.

CLAIM. *For fixed countable $\beta, \gamma, \delta \in ON$ so that $\gamma, \delta > \beta$, there is a partition of I_j into A_j and B_j so that γ is an interval of A_j and δ is an interval of B_j.*

To prove the claim, note that since η is a suborder of I_j, we may embed γ and δ in I_j in such a way so that there are x, y, z so that $x < \gamma < y < \delta < z$ in the embedding. Define A_j to be the vertices of $\gamma \cup (\{r : y < r < z\} \setminus \delta)$, and define B_j to be the vertices of $(\{s : s \leq y\} \setminus \gamma) \cup \delta \cup \{t : t \geq z\}$. It is routine to check that γ is an interval in the suborder on A_j and δ is an interval in the suborder on B_j.

Let $S = L \setminus \left(\bigcup I_i\right)$. The partition

$$S \cup A_1 \cup B_n,\ A_2 \cup B_1,\ A_3 \cup B_2, \ldots, A_n \cup B_{n-1}$$

of L violates $\mathcal{P}(n, k)$ since the induced suborder on the union of any k of these subsets is a linear order L' with $\alpha(L') > \alpha(L)$. □

5. THE ORIENTED GRAPHS WITH THE $\mathcal{P}(3, 2)$ PROPERTY

An oriented graph O with the $\mathcal{P}(3, 2)$ property must have an underlying graph which has $\mathcal{P}(3, 2)$. In order to characterize the $\mathcal{P}(3, 2)$ oriented graphs, we may therefore exploit Theorem 2. Theorem 8 also classifies the countable $\mathcal{P}(3, 2)$ orders. An oriented graph is *independent* if it has no directed edges.

THEOREM 8. *The infinite oriented graphs with the $\mathcal{P}(3,2)$ property that are neither independent nor tournaments are (up to converses) the following:*

$$K_1 \uplus \omega^\alpha, \omega^\alpha \uplus \omega^\beta, \omega^\alpha \uplus \overline{K_{\aleph_0}}, K_1 \to \overline{K_{\aleph_0}}, \overline{K_{\aleph_0}} \to \overline{K_{\aleph_0}}, \overline{K_{\aleph_0}} \to \omega^\alpha, \omega^\alpha \to \overline{K_{\aleph_0}},$$

where α and β are countable ordinals.

PROOF. Consider orientations of the infinite $\mathcal{P}(3,2)$ graphs G that are neither cliques nor complements of cliques. These will give all the infinite $\mathcal{P}(3,2)$ oriented graphs that are neither tournaments nor independent.

Case 1. $G = K_1 \uplus K_{\aleph_0}$.

In this case, the infinite clique must be an orientation of a $\mathcal{P}(2,1)$ tournament, which must be a linear order: the infinite random tournament, T^∞, along with an isolated vertex does not have $\mathcal{P}(3,2)$. To see this, let x be the isolated vertex, and fix y a vertex of T^∞. Let O be the set of out-neighbours of y and I the set of in-neighbours of y. The conclusion follows by considering the partition $\{x, y\}$, O, I.

Case 2. $G = K_1 \vee \overline{K_{\aleph_0}}$.

Partition $\overline{K_{\aleph_0}}$ into O, the out-neighbours of K_1, and I, the in-neighbours of K_1. Since the oriented subgraph induced by $O \cup I$ has no edges, by the $\mathcal{P}(3,2)$ property we have that K_1 is a source or sink.

Case 3. $G = \overline{K_{\aleph_0}} \vee \overline{K_{\aleph_0}}$.

Denote the join-components as X and Y. Fix $x \in X$. Let O be the out-neighbours of x in Y, and let I be the in-neighbours of x in Y. By $\mathcal{P}(3,2)$, we conclude that there is a source or a sink. Since x was arbitrary, we can conclude there exist at least two sources or two sinks; without loss of generality, suppose that there are two sources and they belong both to X. In particular, Y is determined by having no sources. We partition X into its set of sources S minus one called s, the set $X \setminus S$, and Y. If we delete S, then we are left with an oriented graph with exactly one source s, giving a contradiction. To see this, note that there are no sources in Y since $s \in X \setminus S$ is a source. Any source in $X \setminus S$ would be a source in $X \cup Y$. If Y is deleted, then we are left with an independent set. Therefore, the oriented graph induced by $S \cup Y$ is isomorphic to the original oriented graph, which must be $\overline{K_{\aleph_0}} \to \overline{K_{\aleph_0}}$.

Case 4. (a) $G = K_{\aleph_0} \uplus K_{\aleph_0}$, *(b)* $G = \overline{K_{\aleph_0}} \uplus K_{\aleph_0}$.

In either case, write $G = X \uplus Y$, where $X, Y \in \{\overline{K_{\aleph_0}}, K_{\aleph_0}\}$. It is straightforward to see that if X and Y are complete, then they have $\mathcal{P}(2,1)$. In case (a), we obtain the disjoint union of two $\mathcal{P}(2,1)$ tournaments, which must be linear orders. In case (b), we obtain the disjoint union of a $\mathcal{P}(2,1)$ linear order and an infinite independent set.

Case 5. $G = \overline{K_{\aleph_0}} \vee K_{\aleph_0}$.

Name the join-components X, Y, respectively. A similar argument as in Case 4 establishes that Y has $\mathcal{P}(2,1)$, and so is a linear order. A similar argument as in Case 3 establishes that we must have that $X \to Y$ or $Y \to X$. Therefore, in this case, we obtain (up to converses) $L \to I$, where L is a $\mathcal{P}(2,1)$ linear order, and I is an infinite independent set. □

6. COMMENTS AND PROBLEMS

For a given integer $n \geq 2$, we may construct several examples of $\mathcal{P}(n, n-1)$ graphs as follows. If G and H are graphs and $x \in V(G)$, then by *substituting* x in G by H we mean expanding x to a copy of H and then joining every vertex of H to the neighbours of x in G. Fix a graph G with $n-1$ vertices. Substitute either $K_{\aleph_0}$ or $\overline{K_{\aleph_0}}$ for some of the vertices of G. It is not hard to see that the resulting graphs have $\mathcal{P}(n, n-1)$; in fact, the $\mathcal{P}(2, 1)$ graphs, except R, are of this form, and *all* the $\mathcal{P}(3, 2)$ graphs are of this form. Unfortunately, there are examples of $\mathcal{P}(n, n-1)$ graphs, for each $n \geq 4$, which are not of this form. For example, the graph

$$(n-4)K_{\aleph_0} \uplus \aleph_0 K_2 \uplus \overline{K_{\aleph_0}}$$

has $\mathcal{P}(n, n-1)$.

The outstanding open problem we present is the one of classifying the $\mathcal{P}(n, k)$ graphs, tournaments, and oriented graphs, when $n > 3$ and $1 \leq k < n$. Theorems 1 and 7 put some restrictions on such structures. A related problem is whether there are only finitely many $\mathcal{P}(n, n-1)$ graphs when $n > 3$. The evidence so far suggests this question will be answered affirmatively; if so, is there a nonconstructive proof? Another problem is whether a $\mathcal{P}(n, n-1)$ structure also satisfies $\mathcal{P}(n+1, n)$ when $n \geq 3$.

The *age* of a graph G is the set of isomorphism types of induced subgraphs of G. For example, the age of the infinite random graph R is the set of isomorphism types of all countable graphs, while the age of $K_{\aleph_0}$ is the set of isomorphism types of all countable cliques. An age $\mathcal{A}$ has *polynomial profile* if there is a polynomial function $f : \omega \rightarrow \omega$ so that the number of isomorphism types of n-vertex graphs in $\mathcal{A}$ is bounded above by $f(n)$. Hence, $K_{\aleph_0}$ has polynomial profile while R does not. We conjecture that an age $\mathcal{A}$ of a countable graph has polynomial profile if and only if $\mathcal{A}$ is the age of a countable graph satisfying the $\mathcal{P}(n, n-1)$ property for some $n > 2$.

Acknowledgements

AB gratefully acknowledges support from a Natural Science and Engineering Research Council of Canada (NSERC) research grant. DD gratefully acknowledges support from a NSERC Postdoctoral Fellowship.

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Received 15 May 2001 and accepted 10 January 2002

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