

# The Structure and Automorphisms of Semi-directed Graphs\*

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Complex real-world networks such as the web graph are often modelled as directed graphs evolving over time, where new vertices are joined to a constant  $m$  number of existing vertices of prescribed type. We consider a certain on-line random construction of a countably infinite graph with out-degree  $m$ , and show that with probability 1 the construction gives rise to a unique isomorphism type. We show that random semi-directed graphs are prime models in a certain first-order theory. We study algebraic properties of random semi-directed graphs; in particular, we prove that their automorphism groups embed all countable groups.

*Keywords:* Universal graphs, semi-directed graphs, web graph, adjacency property, automorphism group, prime model, undecidable universal theory

## 1 INTRODUCTION

Complex networks arise in many real-world contexts, ranging from the web graph to networks arising in the biological and social sciences. Such networks are usually modelled as directed graphs (digraphs) that evolve over time, where new vertices and edges are born over time. A large number of stochastic models for complex networks have been proposed; see [4, 13] for a discussion of such models. For example, in the *preferential attachment model* introduced in [1, 2], new vertices are born over time which have a greater

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probability to join to high degree vertices. The digraphs generated by the preferential attachment models have the property that each vertex has exactly  $m$  out-neighbours. Constant out-degree is in fact, a common assumption in other models of complex networks; see, for example, [1, 2, 13, 20]. Hence, models of complex networks often generate directed graphs satisfying the following properties.

1. *On-line*: digraphs are generated over a countably infinite set of discrete time-steps, with a countable (either finite or countably infinite) set of vertices born at each time-step. At time 0, a fixed *initial digraph*  $H$  is given.
2. *Constant out-degree*: new vertices have edges directed only to existing vertices, and for  $m > 0$  a fixed integer, there are exactly  $m$  such edges.

A digraph  $G$  satisfying these two properties is called *semi-directed with initial graph  $H$  and constant out-degree  $m$* ; we sometimes refer to  $G$  simply as *semi-directed*. The moniker “semi-directed” comes from [3] (see p. 17). It emphasizes that the orientation of edges in a semi-directed graph canonically arises according to time: new vertices may only point to vertices born at earlier time-steps. Note that semi-directed graphs have no infinite directed path emanating from any vertex, are connected, and is acyclic so long as  $H$  is acyclic.

We consider the infinite semi-directed limit graphs that result when time tends to infinity. Analyzing stochastic models by considering the infinite limit is a common technique in the natural sciences. In particular, the existence of a unique limit indicates coherent behaviour of the model, while many distinct limits suggest a sensitivity to initial conditions that is an indicator of chaos. In [8, 19], infinite limits of graphs generated by models of the web graph were investigated. Limits generated by on-line graph processes were, in fact, studied by Fraïssé [16] and others decades prior to birth of the internet.

One of the most studied examples of an infinite limit graph arising from a stochastic model is the infinite random graph. The probability space  $G(\mathbb{N}, p)$  consists of graphs with vertices  $\mathbb{N}$ , so that each distinct pair of integers is joined independently with a fixed probability  $p \in (0, 1)$ . Erdős and Rényi discovered that with probability 1, all  $G \in G(\mathbb{N}, p)$  are isomorphic. The unique isomorphism type of countably infinite graph is named the *infinite random graph*, or the *Rado* graph, and is written  $R$ ; see the survey [10].

Define a deterministic graph  $R^*$  as follows. Let  $R_0$  be the graph with one vertex,  $K_1$ . Assume that for a nonnegative integer  $t \geq 0$ , the graph  $R_t$  is defined and finite. To form  $R_{t+1}$ , for each subset  $S \subseteq V(R_t)$  (possibly empty) add a vertex  $z_S$  joined only to the vertices of  $S$ . The sets  $\{V(R_t) : t \in \mathbb{N}\}$  and

$\{E(R_t) : t \in \mathbb{N}\}$  are well-ordered sets or *chains*. We define

$$V(R^*) = \bigcup_{t \in \mathbb{N}} V(R_t), \quad E(R^*) = \bigcup_{t \in \mathbb{N}} E(R_t).$$

We write  $\lim_{t \rightarrow \infty} R_t = R^*$ , and say that  $R^*$  is the *limit of the chain*  $(R_t : t \in \mathbb{N})$ . The notion of limit extends to any chain  $(G_t : t \in \mathbb{N})$  of graphs.

A graph  $G$  is *existentially closed* or *e.c.* if for all finite disjoint sets of vertices  $A$  and  $B$  (one of which may be empty), there is a vertex  $z \notin A \cup B$  joined to all of  $A$  and to no vertex of  $B$ . By a back-and-forth argument,  $R \cong R^*$  is the unique isomorphism type of countably infinite graphs that is e.c. Further,  $R$  is a *universal* graph: it contains as an induced subgraph an isomorphic copy of each countable graph.

In the present article, we consider structural, logical, and algebraic properties of certain infinite semi-directed graphs that arise naturally as limits of on-line random processes. The current article is the full version of [6]. Analogous to  $R$ , these so-called *random semi-directed* graphs have isomorphism types characterized via a set of adjacency properties (see Theorem 1). As an application of the characterization, random semi-directed graphs are shown to be universal (see Corollary 1). Random semi-directed graphs are generated with probability 1 by a natural stochastic process (see Corollary 2). We explore some of the first-order properties of semi-directed graphs. While semi-directedness is not a first-order property, we show in Theorem 3 that the random semi-directed graphs form a class of prime models in a suitable first-order theory. The automorphism group of  $R$  has been thoroughly investigated (see [10]), while the monoid of  $R$  was studied in [x]. In Section 3 we show that all countable groups embed in the group of a random semi-directed graph. This property of the automorphism group of random graph parallels the universality result of Corollary 1. The universality of the automorphism group of random semi-directed graphs is used to show in Corollary 4 that their universal theories are undecidable.

All graphs we consider are simple (that is, no loops nor multiple edges), directed, and countable. If  $(x, y)$  is a directed edge, then  $y$  is an *out-neighbour* of  $x$ . We say that  $G$  *embeds in*  $H$  and write  $G \leq H$  if  $G$  is isomorphic to an induced subgraph of  $H$ . If  $S \subseteq V(G)$ , then we write  $G[S]$  for the subgraph induced by  $S$  (we omit the subscript  $G$  if it is clear from context). An *independent set* of vertices is a set of vertices with no directed edges between them. The vertices of a semi-directed graph with initial graph  $H$  may be ordered in the following way. A vertex  $x$  not in  $H$  has *height*  $k$  if there is a directed path of length  $k$  from  $x$  ending in a vertex of  $H$ , and there is no such path of length less than  $k$ . Vertices in  $H$  have height 0. The height of a finite set  $S$  of vertices is the maximum height of a vertex in  $S$ . For background in graph theory, the reader is directed to [14, 24].

The *automorphism group* (or *group*) of  $G$  is written  $\text{Aut}(G)$ . We write  $\mathbb{N}$  for the natural numbers,  $\mathbb{N}^+$  for the positive integers, and  $\aleph_0$  for the cardinality of  $\mathbb{N}$ .

## 2 RANDOM SEMI-DIRECTED GRAPHS

We consider the following general framework for limits of semi-directed graphs. A class  $\mathcal{C}$  of digraphs closed under isomorphism is *good* if it both contains infinitely many digraphs, and is *hereditary*: if  $G \in \mathcal{C}$  and  $H \leq G$ , then  $H \in \mathcal{C}$ . For example, the class of all digraphs is good, as the class of linear orders (that is, transitive tournaments).

For the remainder of the article, fix  $m > 0$  an integer,  $\mathcal{C}$  a good class of digraphs, and  $H$  an  $m$ -vertex digraph in  $\mathcal{C}$  (which exists as  $\mathcal{C}$  is good). We define a countably infinite graph  $R_{m,H}(\mathcal{C})$  as follows. Let  $R_0$  be  $H$ . Assume that  $R_t$  is defined and countable so that  $R_0 \leq R_t$ . To form  $R_{t+1}$ , for each induced subdigraph  $S$  of  $R_t$  that has  $m$  vertices and is in  $\mathcal{C}$ , add a vertex  $x_S$  that is joined to each vertex of  $S$  (that is, there are directed edges from  $x_S$  to each vertex of  $S$ ) and no other vertices in  $R_t$ . Define  $R_{m,H}(\mathcal{C}) = \lim_{t \rightarrow \infty} R_t$ . The countably infinite digraph  $R_{m,H}(\mathcal{C})$  is semi-directed by its construction. The idea behind the definition of  $R_{m,H}(\mathcal{C})$  is that all  $m$ -sets of vertices  $S$  that induce a graph in  $\mathcal{C}$  are extended: the vertex  $x_S$  has its out-neighbours equalling  $S$ . We say that the vertex  $x_S$  was *born at time*  $t$ . Observe that vertices born at time  $t$  have height  $t$ .

One of our main results is that the isotype of  $R_{m,H}(\mathcal{C})$  may be captured by a set of simple set of properties. We say that a digraph  $G$  is  $(\mathcal{C}, m)$ -e.c. if for each set  $A$  of  $m$ -vertices which induces a graph in  $\mathcal{C}$  and each finite set  $B$  of vertices disjoint from  $A$ , there is a vertex  $z \notin A \cup B$  so that  $(z, a) \in E(G)$  for all  $a$  in  $A$ , but there are no directed edges between  $z$  and vertices of  $B$ . The  $(\mathcal{C}, m)$ -e.c. property is a directed analogue of the e.c. property, relativized by the parameter  $m$  and by the restriction that  $G[A] \in \mathcal{C}$ . We note that (assuming that the class  $\mathcal{C}$  is axiomatized by finitely many first-order sentences), that the  $(\mathcal{C}, m)$ -e.c. property is first-order expressible in the language of graphs; see Subsection 2.2.

**Theorem 1.** *A countable digraph  $G$  is isomorphic to  $R_{m,H}(\mathcal{C})$  if and only if  $G$  is semi-directed with initial graph  $H$  and constant out-degree  $m$ , each out-neighbour set induces a subdigraph in  $\mathcal{C}$ , and  $G$  is  $(\mathcal{C}, m)$ -e.c.*

*Proof.* As the forward direction is immediate, we prove only the reverse direction. Let  $H'$  be the initial copy of  $H$  in  $G$ . The set  $V(G) \setminus V(H')$  has a *topological sort*: an enumeration  $(x_t : t \in \mathbb{N}^+)$  of  $V(G) \setminus V(H')$  with the

property that if  $(x_i, x_j)$  is a directed edge, then  $i > j$ . To see this, we may choose  $x_1$  to be any vertex with height 1. Assuming that  $\{x_1, \dots, x_t\}$  were chosen, consider a vertex  $u$  of  $V(G) \setminus (V(H') \cup \{x_1, \dots, x_t\})$ . If  $u$  has out-degree 0 in  $G[V(G) \setminus (V(H') \cup \{x_1, \dots, x_t\})]$ , then let  $x_{t+1} = u$ . Otherwise, in  $G[V(G) \setminus (V(H') \cup \{x_1, \dots, x_t\})]$  there is a maximal directed finite path from  $u$ . The end point  $v$  of this path has out-degree 0, and we choose  $x_{t+1} = v$ .

As  $G$  is arbitrary with the given properties, it follows that  $R_{m,H}(\mathcal{C})$  also has a topological sort. Now, let  $(x_t : t \in \mathbb{N}^+)$  and  $(y_t : t \in \mathbb{N}^+)$  be topological sorts of  $V(G) \setminus V(H')$  and  $V(R_{m,H}(\mathcal{C})) \setminus V(R_0)$ , respectively. We proceed by a back-and-forth argument, with  $f_0$  isomorphically mapping  $H'$  in  $G$  to  $H$  at time 0 in  $R_{m,H}(\mathcal{C})$ . For  $t \geq 0$ , suppose that  $f_t$  is a partial isomorphism with domain  $X_t$  containing  $V(H') \cup \{x_1, \dots, x_t\}$  and range  $Y_t$  containing  $V(R_0) \cup \{y_1, \dots, y_t\}$ . We will assume as an additional inductive hypothesis that  $X_t$  and  $Y_t$  are *closed*: all out-neighbours of vertices in the set are in the set itself.

Suppose first that  $t + 1 \geq 1$  is odd. In this case, we go forward. Let  $x$  be the lowest indexed vertex of  $(x_t : t \in \mathbb{N})$  not in  $X_t$ . As the enumeration is a topological sort, the set of out-neighbours  $S_t$  of  $x$  are in  $X_t$ ; as  $X_t$  is closed, no vertex of  $X_t$  points to  $x$ . By hypothesis,  $|S_t| = m$  and  $G[S_t] \in \mathcal{C}$ . As  $R_{m,H}(\mathcal{C})$  satisfies the  $(\mathcal{C}, m)$ -e.c. property, there is a vertex  $y$  whose out-neighbours are exactly  $f_t(S_t)$ . As  $Y_t$  is closed, no vertex of  $Y_t$  points to  $y$ . Note that  $R_{m,H}(\mathcal{C})[f_t(S_t)] \in \mathcal{C}$ . Extend  $f_t$  to  $f_{t+1}$  by mapping  $x$  to  $y$ , and let  $X_{t+1} = X_t \cup \{x\}$  and  $Y_{t+1} = Y_t \cup \{y\}$ . It is straightforward to see that  $f_{t+1}$  is an isomorphism, and that the sets  $X_{t+1}$  and  $Y_{t+1}$  are closed.

The case  $t + 1$  is even is similarly proven by going back, and so is omitted. We therefore have that the union of the chain of partial isomorphisms  $(f_t : t \in \mathbb{N})$

$$F = \bigcup_{t \rightarrow \infty} f_t$$

is an isomorphism of  $G$  with  $R_{m,H}(\mathcal{C})$ .

Analogous to the situation for  $R$  and all countable graphs, the graph  $R_{m,H}(\mathcal{C})$  has the following universal property.

**Corollary 1.** *If  $G$  is a countable semi-directed graph with initial graph  $H$  and constant out-degree  $m$ , so that each out-neighbour set of a vertex of  $G$  induces a subgraph in  $\mathcal{C}$ , then  $G \leq R_{m,H}(\mathcal{C})$ .*

*Proof.* Let  $H'$  be the initial copy of  $H$  in  $G$ , and let  $f_0$  isomorphically map  $H'$  in  $G$  to  $H$  at time 0 in  $R_{m,H}(\mathcal{C})$ . As in the proof of Theorem 1, let  $(x_t : t \in \mathbb{N}^+)$  be a topological sort of  $V(G) \setminus V(H')$ .

For  $t \geq 0$ , suppose that  $f_t$  is an isomorphism with domain  $X_t = V(H') \cup \{x_1, \dots, x_t\}$  whose range equals the set  $Y_t$  of vertices in  $R_{m,H}(\mathcal{C})$ . Consider the vertex  $x_{t+1}$ . The set of  $m$  out-neighbours  $S_t$  of  $x_{t+1}$  are in  $X_t$ , by the properties of the ordering  $(x_t : t \in \mathbb{N}^+)$ . By hypothesis,  $S_t$  induces a subgraph in  $\mathcal{C}$ . It follows that in  $R_{m,H}(\mathcal{C})$ , there is a vertex  $y_{t+1}$  not in  $Y_t$  whose out-neighbours are exactly  $f_t(S_t)$ . We extend  $f_t$  to the isomorphism  $f_{t+1}$  which maps  $x_{t+1}$  to  $y_{t+1}$ .

Define

$$F = \bigcup_{t \rightarrow \infty} f_t.$$

Then  $F$  witnesses that  $G \leq R_{m,H}(\mathcal{C})$ .

### 2.1 A random graph process

We next introduce a random graph process which we name the *Age Dependent Process (ADP)*. The parameters of the process are  $m, \mathcal{C}$ , and  $H \in \mathcal{C}$ . Start with  $G_0 \cong H$  with vertices labelled  $v_1, \dots, v_m$ . For  $t \geq 1$  fixed, assume that a digraph  $G_{t-1}$  has been defined and there are finitely many vertices in  $G_{t-1}$ . At time  $t$ , add a new vertex  $v_{m+t}$ , and choose a set  $S$  of  $m$  distinct vertices from  $V(G_{t-1})$  so that  $S$  induces a subdigraph of  $\mathcal{C}$ , where a vertex  $v_i$  is included in the set independently with probability exponentially proportional to the time it was born. More precisely, denote

$$L_{t-1} = \{(j_1, \dots, j_m) \in \mathbb{N}^m : G_{t-1}[\{v_{j_1}, \dots, v_{j_m}\}] \in \mathcal{C}, \\ v_{j_1}, \dots, v_{j_m} \in V(G_{t-1}) \text{ are distinct}\}.$$

For each  $S = \{v_{i_1}, \dots, v_{i_m}\}$  where  $(i_1, \dots, i_m) \in L_{t-1}$ , define

$$\mu(S) = 2^{-(i_1 + \dots + i_m)}$$

and

$$N_t = \sum_{(j_1, \dots, j_m) \in L_{t-1}} 2^{-(j_1 + j_2 + \dots + j_m)}.$$

In particular,  $N_t$  is the sum of all the  $\mu(S)$ , where  $S$  is a subset of cardinality  $m$  from  $V(G_{t-1})$  such that  $G_{t-1}[S] \in \mathcal{C}$ . The probability that  $S$  is chosen from  $V(G_{t-1})$  equals  $\mu(S)/N_t$ ; this clearly defines a probability measure on  $m$ -subsets  $S$  with  $G_t[S] \in \mathcal{C}$  in  $G_t$ . If  $S$  is so chosen, then add directed edges from  $v_{m+t}$  to each vertex of  $S$ .

**Theorem 2.** Let  $G = \lim_{t \rightarrow \infty} G_t$ , where  $G_t$  is generated by ADP with parameters  $m$ ,  $H$ , and  $\mathcal{C}$ . Then with probability 1,  $G$  is  $(\mathcal{C}, m)$ -e.c.

*Proof.* Fix disjoint finite subsets  $A$  and  $B$  of  $V(G)$  so that  $|A| = m$  and  $G[A] \in \mathcal{C}$ . Let  $A = \{v_{i_1}, \dots, v_{i_m}\}$ , where the vertex  $v_{i_j}$  was born before  $v_{i_{j+1}}$  for all  $j$ . Let  $t_0$  be an integer greater than the height of  $A \cup B$ . For each  $t \geq t_0$ , let  $V_t$  be the event that  $v_t$  is pointing to exactly all vertices in  $A$ . Note that  $v_t$  has out-degree  $m$  when it is born, so that if  $V_t$  occurs, then there are no edges between  $v_t$  and any vertex of  $B$ . Then the probability that  $V_t$  occurs, written  $\mathbb{P}(V_t)$ , equals  $2^{-(i_1 + \dots + i_m)} / N_t$ , where  $N_t$  is the normalizing factor defined above.

Note that

$$\begin{aligned} N_t &\leq \sum_{1 \leq j_1 < j_2 < \dots < j_m \leq t+m-1} 2^{-(j_1 + j_2 + \dots + j_m)} \\ &\leq \left( \sum_{j=1}^{t+m-1} 2^{-j} \right)^m \\ &\leq 1, \end{aligned}$$

for all  $t$ . Therefore, for all  $t \geq t_0$ ,

$$\mathbb{P}(V_t) \geq 2^{-(i_1 + \dots + i_m)} \geq 2^{-mt_0}.$$

Hence, the probability that there exists no vertex in  $G$  that is joined to all vertices in  $A$  and none of  $B$  is at most

$$\begin{aligned} \mathbb{P} \left( \bigcap_{t=t_0}^{\infty} \overline{V}_t \right) &= \prod_{t=t_0}^{\infty} (1 - \mathbb{P}(V_t)) \\ &\leq \lim_{t \geq t_0} (1 - 2^{-mt_0}) = 0. \end{aligned}$$

As there are only countably many finite subsets  $A$  and  $B$  and a countable union of measure 0 events is a measure 0 event, the proof follows.

The following corollary follows immediately from Theorems 1 and 2. It supplies an analogue of the Erdős and Rényi isomorphism result for  $R$ .

**Corollary 2.** With probability 1, a limit graph generated by ADP with parameters  $m$ ,  $H$ , and  $\mathcal{C}$  is isomorphic to  $R_{m,H}(\mathcal{C})$ .

## 2.2 Model-theoretic properties of $R_{m,H}(\mathcal{C})$

We investigate some of the first-order model-theoretic properties of semi-directed graphs. We work within the *first-order language of graphs*  $\mathcal{L}_G$ , which contains one binary relation symbol  $E$ . We consider a digraph as an  $\mathcal{L}_G$ -structure, where  $E$  is taken as irreflexive binary relation on vertices. The (first-order)  $\mathcal{L}_G$ -sentences, or *sentences*, are defined in the usual way using  $E$ ,  $=$ , and the standard logical connectives and quantifiers. An  $\mathcal{L}_G$ -theory is a set of  $\mathcal{L}_G$ -sentences; the *theory* of an  $\mathcal{L}_G$ -structure  $M$ , written  $Th(M)$ , is the set of  $\mathcal{L}_G$ -sentences satisfied by  $M$ . For background in first-order logic and its model theory, see [18].

An  $\mathcal{L}_G$ -theory  $T$  is  $\aleph_0$ -categorical if every two countable models of  $T$  are isomorphic. If we consider the first-order theory of undirected graphs where  $E$  is interpreted as a symmetric binary relation, then  $Th(R)$  is  $\aleph_0$ -categorical, and is *axiomatized* by the e.c. property (that is, the models of  $Th(R)$  are exactly the e.c. graphs).

An easy (and so omitted) Compactness argument demonstrates that the on-line property (item 1 in the definition of semi-directedness) is not first-order expressible in  $\mathcal{L}_G$ . However, for semi-directed graphs we consider the following  $\mathcal{L}_G$ -theory  $\Phi$ . The set  $\Phi$  contains the digraph axioms, and the following sentences. Fix  $m > 0$  an integer, and choose  $\mathcal{C}$  a good class so it is first-order axiomatizable by finitely many sentences (for example, as is the case for the class of all digraphs). For simplicity, take  $H$  to be the digraph with  $m$  vertices and no edges.

1. There are exactly  $m$  vertices of out-degree 0 which form an independent set. All other vertices have out-degree  $m$ .
2. The out-neighbour set of each vertex induces a subdigraph in  $\mathcal{C}$ .
3. There are no directed cycles.
4. The  $(\mathcal{C}, m)$ -e.c. property holds.

It is straightforward to verify that the sentences in  $\Phi$  are indeed first-order expressible in  $\mathcal{L}_G$ . We remark that an analogous theory was presented in [19] for the case  $m = 2$ . By its construction, the digraph  $R_{m,H}(\mathcal{C})$  satisfies  $\Phi$ . However, unlike  $Th(R)$ , the theory  $\Phi$  is not  $\aleph_0$ -categorical. For example, if we let  $R_{m,H}(\mathcal{C})_n$  denote the disjoint union of  $n$  copies of  $R_{m,H}(\mathcal{C})$ , where  $2 \leq n \leq \aleph_0$ , then  $R_{m,H}(\mathcal{C})_n$  satisfies  $\Phi$  and none of these graphs are pairwise isomorphic.

We next observe that  $R_{m,H}(\mathcal{C})$  is a (*algebraically*) *prime model* for  $\Phi$ : each model of  $\Phi$  contains an isomorphic copy of  $R_{m,H}(\mathcal{C})$ . Theorem 3 is an interesting contrast to the universality result of Corollary 1.

**Theorem 3.** *The digraph  $R_{m,H}(\mathcal{C})$  is a prime model for the theory  $\Phi$ .*

*Proof.* Suppose that  $G$  is a model of  $\Phi$ . By the  $(\mathcal{C}, m)$ -e.c. property,  $H$  is isomorphic to an induced subdigraph of  $G$ . (More explicitly, we may build a copy of  $H$  in  $G$  inductively, by extending a given independent set  $X$  by a vertex joined to nothing in  $X$ .) Let  $f_0$  isomorphically map  $H$  at time 0 in  $R_{m,H}(\mathcal{C})$  to an arbitrary but fixed copy  $X_0$  of  $H$  in  $G$ .

For  $t \geq 0$ , suppose that  $f_t$  is an isomorphism with domain  $R_t = V(R_{m,H}(\mathcal{C}))$  whose range is an induced subdigraph  $X_t$  of  $G$ . Fix a set of vertices  $S$  of  $R_t$  that has  $m$  vertices and which induces a subdigraph in  $\mathcal{C}$ ; by the construction of  $R_{m,H}(\mathcal{C})$  the vertex  $x_S$  is joined to each vertex of  $S$ . By the  $(\mathcal{C}, m)$ -e.c. property for  $G$ , there is a vertex  $y_{f_t(S)}$  whose out-neighbour set is exactly  $f_t(S)$ , and such that there are no directed edges between  $X_t \setminus f_t(S)$  and  $y_{f_t(S)}$ .

We extend  $f_t$  to the isomorphism  $f_{t+1}$  which maps  $x_S$  to  $y_{f_t(S)}$  for all choices of  $S$  in  $R_t$ . It is straightforward to see this is an isomorphism of  $R_{t+1}$  onto an induced subdigraph of  $G$ . Define

$$F = \bigcup_{t \rightarrow \infty} f_t.$$

Then  $F$  witnesses that  $R_{m,H}(\mathcal{C}) \leq G$ .

### 3 THE GROUP OF $R_{M,H}(\mathcal{C})$

The infinite random graph  $R$  possesses a rich group of symmetries. In particular, the graph  $R$  is *homogeneous*: isomorphisms between finite induced subgraphs extend to automorphisms. The homogeneous graphs were characterized in [22], while the homogeneous digraphs were characterized in [11]. The graph  $R_{m,H}(\mathcal{C})$  is not homogeneous; it is not even vertex-transitive: two vertices with different heights are in different orbits of  $\text{Aut}(R_{m,H}(\mathcal{C}))$ . Hence, the symmetries exhibited by  $R$  and  $R_{m,H}(\mathcal{C})$  are quite different.

Henson [17] proved that  $\text{Aut}(R)$  embeds (that is, contains subgroups isomorphic to) all countable groups. We now prove that the group  $\text{Aut}(R_{m,H}(\mathcal{C}))$  shares this property with  $R$ . Given a set  $X$ , we use the notation  $\text{Sym}(X)$  for the group of permutations of  $X$ . For a set  $S$  of vertices and automorphism  $f$ ,  $f(S)$  is the image of  $S$  under  $f$ .

**Theorem 4.** *The group  $\text{Sym}(X)$  embeds in  $\text{Aut}(R_{m,H}(\mathcal{C}))$ , where  $X$  is countably infinite. In particular, each countable group embeds in  $\text{Aut}(R_{m,H}(\mathcal{C}))$ .*

Theorem 4 is, after Corollary 1, our second universality result for  $R_{m,H}(\mathcal{C})$ . Before we prove Theorem 4 we need the following lemma. The

graph  $R_{m,H}(\mathcal{C})'$  is defined analogously to  $R_{m,H}(\mathcal{C})$ , but at each time-step  $R_{t+1}$ , *infinitely many* vertices  $x_S$  are joined to each induced subdigraph of order  $m$  from  $\mathcal{C}$  in  $R_t$ .

**Lemma 1.** *The digraph  $R_{m,H}(\mathcal{C})'$  is isomorphic to  $R_{m,H}(\mathcal{C})$ .*

*Proof.* It is sufficient to prove that  $R_{m,H}(\mathcal{C})'$  satisfies the hypotheses of Theorem 1. By its construction, the graph  $R_{m,H}(\mathcal{C})'$  is semi-directed with initial graph  $H$  and constant out-degree  $m$ . Further, each vertex has its out-neighbour set inducing an  $m$ -vertex subdigraph in  $\mathcal{C}$ . To see that  $R_{m,H}(\mathcal{C})'$  satisfies the  $(\mathcal{C}, m)$ -e.c. property, suppose we are given  $A$  a set of  $m$ -vertices in  $R_{m,H}(\mathcal{C})'$  which induces a graph in  $\mathcal{C}$ , and a finite set  $B$  of vertices in  $R_{m,H}(\mathcal{C})'$  disjoint from  $A$ . Let  $t_0$  be the maximum time a vertex of  $A \cup B$  was born. A vertex joined to  $A$  and not to  $B$  may be found in  $R_{t_0+1}$ .

*Proof.* [Proof of Theorem 4.] Without loss of generality, by Lemma 1 we will work with  $R_{m,H}(\mathcal{C})'$  for the remainder of the proof. By Cayley's theorem, it is sufficient to prove that  $\text{Sym}(X)$  embeds in  $\text{Aut}(R_{m,H}(\mathcal{C})')$ .

We first observe that  $\text{Sym}(X)$  embeds in  $\text{Aut}(R_1)$ . To see this, label the vertices of  $V(R_1) \setminus V(R_0)$  as  $X = \{x_i : i \in \mathbb{N}\}$ . Fix a bijective mapping  $f : X \rightarrow X$ . Define  $F : R_1 \rightarrow R_1$  which acts as the identity on  $H$ , and otherwise acts as  $f$  on  $X$ . As the  $x_i$  have the same out-neighbours in  $R_1$ , it follows that  $F$  is an automorphism of  $R_1$ . Define  $\beta : \text{Sym}(X) \rightarrow \text{Aut}(R_1)$  by  $\beta(f) = F$ . It is straightforward to check that  $\beta$  is an injective group homomorphism.

We next prove that there exists an injective group homomorphism

$$\alpha : \text{Aut}(R_1) \rightarrow \text{Aut}(R_{m,H}(\mathcal{C})').$$

Once this is established, then  $\alpha\beta : \text{Sym}(X) \rightarrow \text{Aut}(R_{m,H}(\mathcal{C})')$  supplies an embedding of  $\text{Sym}(X)$  into the automorphism group  $\text{Aut}(R_{m,H}(\mathcal{C})')$ , and the assertion will follow.

Fix  $j$  an automorphism of  $R_1$ . Let  $J_1 = j$ . For  $t \geq 1$ , assume that  $J_t$  is an automorphism of  $R_t$ , and the restriction of  $J_t$  to  $R_1$  equals  $J_1$ . Let  $N^+(z)$  be the set of out-neighbours of a vertex  $z$ . Define  $J_{t+1}$  by

$$J_{t+1}(z) = \begin{cases} J_t(z) & \text{if } z \in V(R_t); \\ x_{J_t(S)} & \text{if } z = x_S \text{ and } S = N^+(z). \end{cases}$$

From the definition of  $R_{t+1}$  and the fact that  $J_t \in \text{Aut}(R_t)$ , it follows that  $J_{t+1}$  is an automorphism of  $R_{t+1}$ . Note that  $J_{t+1}$  restricted to  $R_t$  equals  $J_t$ .

The map  $J = \bigcup_{t \in \mathbb{N}} J_t$  is an automorphism of  $\text{Aut}(R_{m,H}(\mathcal{C})')$ . Hence, the function  $\alpha : \text{Aut}(R_1) \rightarrow \text{Aut}(R_{m,H}(\mathcal{C})')$  defined by  $\alpha(j) = J$  is well-defined.

It is straightforward to see that  $\alpha$  is injective, and that  $\alpha$  preserves the identity automorphism.

Now fix  $f, g \in \text{Aut}(R_1)$  and  $z \in V(R_H)$ . We prove by induction on time  $t$  that the vertex  $z$  was born that

$$\alpha(fg)(z) = \alpha(f)\alpha(g)(z). \quad (1)$$

Equation (1) will establish that  $\alpha$  is an embedding of groups, and is immediate if  $t = 0$ . Fix  $t \geq 1$ . Suppose that  $z$  was born at time  $t + 1$  and so  $z$  is of the form  $x_S$ , where  $S = N^+(z) \subseteq V(R_t)$ . Then

$$\begin{aligned} \alpha(fg)(z) &= x_{\alpha(fg)(S)} \\ &= x_{\alpha(f)\alpha(g)(S)} \\ &= \alpha(f)\alpha(g)(z). \end{aligned}$$

The second equality follows since the times that vertices of  $S$  were born are all strictly less than  $t + 1$ , and by induction hypothesis.

The property of extending automorphisms of  $R_1$  to automorphisms of all of  $\text{Aut}(R_{m,H}(C))$  in the proof of Theorem 4 clearly generalizes to any  $R_t$  with  $t \geq 0$ . In particular,  $j \in \text{Aut}(R_t)$  extends to  $J \in \text{Aut}(R_{m,H}(C))$ , and the map  $\alpha_t : \text{Aut}(R_t) \rightarrow \text{Aut}(R_{m,H}(C))$  defined by  $\alpha_t(j) = J$  is an injective group embedding. Although  $\text{Aut}(R_{m,H}(C))$  is not homogeneous, we may refer to the above property as *temporal homogeneity*: symmetries of the graphs  $R_t$  at time  $t$  lift to symmetries of the entire limit graph.

We consider some computational consequences of Theorem 4. We refer the reader to Hodges [18] for any terms not explicitly defined. The *language of groups*  $\mathcal{L}$  contains a binary function symbol  $\cdot$ , a unary function  $^{-1}$ , and a constant symbol  $1$ . A group is considered as an  $\mathcal{L}$ -structure in the usual way, and satisfies the group axioms. A *universal sentence* is the smallest class of  $\mathcal{L}$ -formulas which contains the quantifier-free formulas and is closed under conjunction and disjunction, and adding universal quantifiers at the front. The *universal theory of a group*  $G$  is the set of universal  $\mathcal{L}$ -sentences satisfied by  $G$ , and the *universal theory of all groups* is the set of universal sentences satisfied by all groups.

**Corollary 3.** *The group  $\text{Aut}(R_{m,H}(C))$  does not satisfy any non-trivial group identity. In particular,  $\text{Aut}(R_{m,H}(C))$  generates the variety of all groups.*

*Proof.* Since every countable group embeds into  $\text{Aut}(R_{m,H}(C))$  by Theorem 4, so does the free group on a countable set of generators, written  $F(X)$ .

If there were an equation  $s = t$  in  $\mathcal{L}$  that is not a consequence of the groups axioms, and satisfied by  $\text{Aut}(R_{m,H}(C))$ , then  $s = t$  would be satisfied by  $F(X)$ , which is a contradiction.

A theory  $T$  in the language of groups is *decidable* if there is an effective procedure that, given an arbitrary formula in the of the theory, decides whether the formula is a member of the theory or not.

**Corollary 4.** *The universal theory of  $\text{Aut}(R_{m,H}(C))$  is undecidable.*

*Proof.* We first note that the universal theory of  $\text{Aut}(R_{m,H}(C))$  equals the universal theory of all groups. This follows since every countable group embeds into  $\text{Aut}(R_{m,H}(C))$  by Theorem 4, every universal sentence true in  $\text{Aut}(R_{m,H}(C))$  will be true in all countable groups and, by the Löwenheim-Skolem Theorem (see [18]), in all groups.

It is well-known that the universal theory of groups is undecidable. This fact follows this by the existence of a group with an undecidable word problem; see [9, 23]. Hence, the universal theory of  $\text{Aut}(R_{m,H}(C))$  is undecidable.

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