NEW CONSTRUCTIONS OF MEYNIEL EXTREMAL FAMILIES OF GRAPHS

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Abstract. We provide new constructions of Meyniel extremal graphs, which are families of graphs with the conjectured largest asymptotic cop number. Using spanning subgraphs, we prove that there are an exponential number of new Meyniel extremal families with specified degrees. Using linear programming on hypergraphs, we explore the degrees in families that are not Meyniel extremal. We give the best known upper bound on the cop number of vertex-transitive graphs with a prescribed degree. We find new Meyniel extremal families of regular graphs with large chromatic number, large diameter, and explore the connection between Meyniel extremal graphs and bipartite graphs. Conjectures are presented relating Meyniel extremal families to maximum and average degrees in their graphs.

1. Introduction

One of the most challenging directions in the study of the game of Cops and Robbers played on graphs is understanding how large the cop number can be as a function of the number of vertices of the graph. The definition of the game is given at the end of the introduction, and we denote the cop number of a graph G by $c(G)$. *Meyniel's conjecture* states that there is a constant $D > 0$ such that for all connected graphs of order n, $c(G) \leq D\sqrt{n}$. Despite the known sublinear bounds on the cop number [12, 17, 20], the conjecture remains wide open; it is unknown if the *soft Meyniel's conjecture* is true: there is a constant $D > 0$ such that the cop number is bounded above by $Dn^{1-\epsilon}$, where $\epsilon > 0$. For more background on Meyniel's conjecture, see [2] and the books [4, 6].

Families of graphs realizing the conjectured asymptotic upper bound are of interest in their own right. Let I be an infinite set of positive integers, and let $\{G_n\}_{n\in I}$ be a family of graphs, where G_n has order n. Note that I may be a proper subset of the positive integers, such as the set of prime power integers. We say that $\{G_n\}_{n\in I}$ is a *Meyniel extremal* family if there exists a positive constant d such that $c(G_n) \geq d\sqrt{n}$ for all $n \in I$. We sometimes abuse notation and refer to Meyniel extremal graphs.

The incidence graphs of projective planes form the earliest known example of a Meyniel extremal family, where in this case I is the set of prime powers [11]. Several other families of incidence graphs of combinatorial designs are known to be Meyniel extremal [5], such as the incidence graphs of partial affine planes [2]. Other Meyniel extremal families include polarity graphs [5], t-orbit graphs [5], and certain families of Cayley graphs [7, 13].

In the present work, our main goal is to provide several results that diversify the kind and quantity of known Meyniel extremal families. Our first approach uses spanning subgraphs in Section 2, and we apply those results to give an exponential number of new Meyniel extremal families with specified degrees. For these theorems, we use a new lower bound on the cop

2020 *Mathematics Subject Classification.* 05C57, 05C35.
Key words and phrases. graphs, cop number, Cops and Robbers, Meyniel's conjecture, hypergraphs.
The first author was funded by NSERC.

number of a graph. In Section 3, we use linear programming on hypergraphs to explore vertex degrees within families assuming Meyniel's conjecture is false. In Corollary 16, the best known upper bound on the cop number of vertex-transitive graphs with a prescribed degree is given. New constructions are given of Meyniel extremal families with various prescribed properties in Section 4; we find Meyniel extremal families of regular graphs with large chromatic number and large diameter. We consider bipartite double covers to explore the connection between Meyniel extremal families and bipartite graphs. The final section contains conjectures and open problems relating graphs in Meyniel extremal families to their degrees and cycle subgraphs.

To finish this introduction, we give a brief overview of Cops and Robbers for those readers unfamiliar with the game. *Cops and Robbers* is a game played on a graph G. There are two players consisting of a set of *cops* and a single *robber*. The game is played over a sequence of discrete time-steps or *rounds* indexed by nonnegative integers, with the cops going first in round 0. The cops and robber occupy vertices; for simplicity, we often identify the player with the vertex they occupy. We refer to the set of cops as C and the robber as R. When a player is ready to move in a round, they must move to a neighboring vertex. Players can *pass*, or remain on their own vertex. Any subset of C may move in a given round.

The cops win the game if, after a finite number of rounds, one of them can occupy the same vertex as the robber. This situation is called a *capture*. The robber wins if they can evade capture indefinitely. The minimum number of cops required to win is a well-defined positive integer, called the *cop number* of G, written $c(G)$. For additional background on the cop number of a graph, see the book [6].

All graphs considered are simple, finite, and undirected. We only consider connected graphs, unless otherwise stated. The diameter of a graph G is denoted by $\mathrm{diam}(G)$. For a graph H, a graph is *H-free* if it does not contain H as a subgraph. In the case of $H = K_3$, we say the graph is *triangle-free*. For a positive integer k and a vertex x in a graph G, let $N_k(x)$ be the set of vertices of distance k to x. We denote $N_1(x) = N(x)$, and refer to vertices in $N(x)$ as *neighbors* of x. For a set S of vertices, $N(S)$ is the set of neighbors of vertices in S. For a vertex x in a graph G, we denote the degree of x by $\deg_G(x)$; we drop the subscript G if it is clear from context. For a graph G, let $\delta(G)$ and $\Delta(G)$ be the minimum and maximum degrees in G, respectively. For background on graph theory, see [21]. All logarithms are in base 2, unless otherwise stated.

2. Meyniel extremal families and spanning subgraphs

Many of the known constructions that result in Meyniel extremal families contain vertices with degrees in $\Theta(\sqrt{n})$ and forbid $K_{2,t}$ for integer constant $t \geq 1$. We therefore arrive at the following definition.

A graph family $\{G_n\}_{n\in I}$ is *elementary* if the following properties hold for all $n \in I$.

(1) The graph G_n is $K_{2,t}$-free for $t \geq 1$ an integer constant.
(2) All vertices of G_n have degree in $\Theta(\sqrt{n})$.

In this section, we consider new Meyniel extremal families that are generated by considering subgraphs of members of elementary Meyniel extremal families. This will result in many families that are not elementary Meyniel extremal and many families whose members are nonisomorphic. To make this precise, we define two families $\{G_n\}_{n\in I}$ and $\{J_n\}_{n\in I}$ with the same index set I to be *nonisomorphic* if for each $n \in I$, there is no isomorphism between G_n

and J_n. Further, we say that the family $\{J_n\}_{n\in I}$ is a *spanning family* of $\{G_n\}_{n\in I}$ if J_n is a spanning subgraph of G_n for all $n \in I$.

We first give a useful generalization of the lower bound provided by Bonato and Burgess [5]. The following lemma will be useful later in this section.

Lemma 1. *Let n be a positive integer and k a nonnegative integer with $n \geq k$. If G is $K_{2,t}$-free for $t \geq 1$ an integer and has $n-k$ vertices of degree at least D and k vertices of degree less than D, with $D > k$, then*

$$c(G) \geq \frac{D-k}{t}.$$

Proof. Define G_D to be the subgraph induced by the $n-k$ vertices of degree at least D, and let G_k be the subgraph induced by the remaining vertices. Suppose that C is the set of cops, and suppose that $|C| < \frac{D-k}{t}$.

Consider a robber strategy where the robber only moves on vertices in G_D, and on any given robber move, will move to any neighboring vertex that is not adjacent to a cop. Suppose the robber is on vertex u on the robber's turn, which we assume does not contain a cop. There are at most k neighbors of u in G_k that the robber refuses to move to, leaving at least $D-k$ neighbors that the robber can move to. Each cop can be adjacent to at most $t-1$ vertices among these $D-k$ vertices and can be on another one vertex among these, and so there are at most $|C|t$ vertices among the $D-k$ vertices that the robber cannot move to. As $|C|t < D-k$, then there is always some vertex in the neighborhood of u that is both in G_D and is not adjacent to any cop.

This shows that the robber can escape capture, given that the robber is not captured in the first round. To see the later, after the cops have chosen their starting positions, let u' be any vertex of G_D that does not contain a cop. By the above analysis, u' has a neighbor in G_D that is not adjacent to a cop, and the robber starts on such a vertex. □

We have the following consequence of Lemma 1, first proven in [5].

Corollary 2. *Let $t \geq 1$ be an integer. If G is $K_{2,t}$-free, then $c(G) \geq \delta(G)/t$.*

For a positive integer r, an *r-factor* of a graph G is a spanning r-regular subgraph of G, and an *r-factorization* of G partitions the edges of G into disjoint r-factors.

Lemma 3. *Fix $0 < \varepsilon < 1$ and let k be a positive integer. Let $\{G_n\}_{n\in I}$ be a k-regular elementary Meyniel extremal family with an r-factorization, where $1 \leq r < k$. We then have that for each $1 \leq i \leq \lfloor \varepsilon k/r \rfloor$ there exists a family of Meyniel extremal graphs $\{G_{i,n}\}_{n\in I}$ such that $G_{i,n}$ is $(k-ri)$-regular.*

Proof. Fix an r-factorization $F_1, F_2, \ldots, F_{k/r}$ of G_n. For all $1 \leq i \leq \lfloor \varepsilon k/r \rfloor$, define $G_{i,n}$ to have the vertex set $V(G)$ and the edge set

$$E(G_{i,n}) = E(G_n) \setminus \left(\bigcup_{j=1}^{i} E(F_i) \right).$$

Each vertex of G_n loses ri adjacent edges, so $G_{i,n}$ is $(k-ri)$-regular and is $K_{2,t}$-free. Further, we have that

$$k - ri \geq k(1-\epsilon) \geq d_1\sqrt{n},$$

for some constant d_1. Hence, the family $\{G_{i,n}\}_{n\geq 1}$ is elementary Meyniel extremal by Corollary 2 and its members are $(k-ri)$-regular. □

Lemma 3 can be applied to families of regular bipartite graphs and families of regular graphs with even degree, which contain 1- and 2-factorizations, respectively. For example, the family consisting of the incidence graphs of the projective plane for each prime power can be used as a starting elementary Meyniel extremal family.

Lemma 1 allows us to find families of Meyniel extremal graphs using spanning subgraphs.

Lemma 4. *Let $\{G_n\}_{n\in I}$ be an elementary Meyniel extremal family, let k be a nonnegative integer, and let J_n be a connected spanning subgraph of G_n that contains at least $n-k$ vertices of degree at least D where $D-k=\Theta(\sqrt{n})$. We then have that $\{J_n\}_{n\in I}$ is a Meyniel extremal family.*

Proof. Since $\{G_n\}_{n\geq 1}$ is an elementary Meyniel extremal family, we may apply Lemma 1 to obtain that

$$c(J_n) \geq (D-k)/t \geq d\sqrt{n}/t$$

for some constant d (recall by the definition of an elementary Meyniel extremal family, t is constant). □

We have the following corollary.

Corollary 5. *Fix $0<\varepsilon<1$. Let $\{G_n\}_{n\in I}$ be a family of C_4-free graphs and let the degrees of G_n be $\Theta(\sqrt{n})$ and $\delta_n=\delta(G_n)$. For each $\mathbf{x}=(x_i)_{i=1}^{\lceil\varepsilon\delta_n\rceil}$, there exists a Meyniel extremal family $\{G_{\mathbf{x},n}\}_{n\in I}$ with the following properties:*

1. *$G_{\mathbf{x},n}$ is a spanning subgraph of G_n;*
2. *$G_{\mathbf{x},n}$ contains vertices $v_i\in V(G_n)$ with $0\leq x_i\leq \deg_{G_n}(v_i)-3$; and*
3. *$G_{\mathbf{x},n}$ has $\deg_{G_{\mathbf{x},n}}(v_i)=\deg_{G_n}(v_i)-x_i$ for $1\leq i\leq\lceil\varepsilon\delta_n\rceil$.*

Proof. Fix $\mathbf{x}=(x_i)_{i=1}^{\lceil\varepsilon\delta_n\rceil}$. Choose a vertex $v\in V(G_n)$ of minimum degree. Note that since G_n is C_4-free, every pair of vertices in $N_1(v)$ have no common neighbors in $N_2(v)$. To see this, if two vertices $x,y\in N_1(v)$ have a common neighbor z in $N_2(v)$, then we obtain a subgraph X isomorphic to C_4 with edges vx, xz, zy, and yv. In addition, each vertex $x\in N_1(v)$ has at most one neighbor in $N_1(v)$. Otherwise, if $y,z\in N_1(v)$ are both neighbors of x, then X violates the C_4-free property. Hence, each of the δ_n vertices in $N_1(v)$ has at least $\deg_{G_n}(v_i)-2$ unique neighbors in $N_2(v)$.

Now choose $\lceil\varepsilon\delta_n\rceil$ vertices from $S\subseteq N_1(v)$ and label them $v_1,v_2,\ldots,v_{\lceil\varepsilon\delta_n\rceil}$ and obtain $G_{\mathbf{x},n}$ from G_n by deleting x_i edges incident with v_i and in $N_2(v)$. The graph $G_{\mathbf{x},n}$ is connected since the only edges removed are between S and $N_2(v)$ and each vertex in S has degree at least 1 into $N_2(v)$. Thus, $\{G_{\mathbf{x},n}\}_{n\in I}$ is Meyniel extremal by Lemma 4 and contains vertices of the desired degree. □

Notice that Corollary 5 applies to any elementary Meyniel extremal graphs that have graphs with girth at least 5. One application of Lemma 4 is to construct large families of nonisomorphic, Meyniel extremal, spanning families using a similar approach to the proof of Corollary 5. We will require the following lemma on spanning subgraphs.

Lemma 6. *Let $J=(A,B)$ be a bipartite graph with $|A|=a$. Suppose that each of the vertices of A has degree at least $d\leq a$, and no two vertices of A have common neighbors. Let X count the number of distinct spanning subgraphs of J. We then have that*

$$X \geq \binom{a+d-1}{d-1}.$$

If $a(n) = \Theta(d(n))$, then for $H(x) = -x\log_2 x-(1-x)\log_2(1-x)$ the binary entropy function, we have that

$$X \sim 2^{(a+d-1)H((d-1)/(a+d-1))}.$$

Proof. Label the vertices of A as $\{v_1, v_2, \ldots, v_a\}$. For each $v_i \in A$, delete edges incident to v_i until it has exactly d neighbors. Define for each vector $\mathbf{x} = (x_i)_{i=1}^a \in \{0, 1, 2, \ldots d-1\}^a$ the graph $J_{\mathbf{x}}$ obtained by deleting x_i edges incident with v_i.

Define $g_i(\mathbf{x})$ be the number of coordinates of $\mathbf{x}$ that contain i, for $0 \le i \le d-1$. For two vectors $\mathbf{x}, \mathbf{y}$ if there exists i such that $g_i(\mathbf{x}) \neq g_i(\mathbf{y})$, then $J_{\mathbf{x}}$ and $J_{\mathbf{y}}$ are nonisomorphic since they will have different degree sequences. Hence, the number of nonisomorphic spanning subgraphs of J is at least

$$\binom{a+d-1}{d-1}$$

since this is the number of different ways to distribute a coordinates among the d variables $g_0, g_1, \ldots, g_{d-1}$.

Since $a = \Theta(d)$, we may estimate this binomial coefficient using Stirling's approximation to achieve:

$$\binom{a+d-1}{d-1} \sim \sqrt{\frac{a+d-1}{2\pi(d-1)a}} \left(\frac{a+d-1}{d-1}\right)^{d-1} \left(\frac{a+d-1}{a}\right)^{a}.$$

Taking the logarithm yields

$$\log_2 \binom{a+d-1}{d-1} \sim H\left(\frac{d-1}{a+d-1}\right)(a+d-1),$$

where $H(x) = -x\log_2 x - (1-x)\log_2(1-x)$ for $x \in (0, 1)$. Notice that since

$$\beta(n) = \frac{d-1}{a+d-1} = 1 - \frac{1}{1+\frac{d}{a}-\frac{1}{a}}$$

has $0 < \beta \le 1/2$, we have that

$$\binom{a+d-1}{d-1} \sim 2^{(a+d-1)H((d-1)/(a+d-1))}.$$

The proof follows. □

Theorem 7. *Fix $0 < \varepsilon < 1$. Let G_n be a graph on n vertices with girth at least 5 that has degrees in $\Theta(\sqrt{n})$ and let $\delta_n = \delta(G_n)$. We then have that there exists r pairwise nonisomorphic, Meyniel extremal, spanning families of $\{G_n\}_{n\in I}$, where*

$$r \geq \binom{(1+\varepsilon)\delta_n - 2}{\delta_n - 2} \sim 2^{((1+\varepsilon)\delta_n-2)H(1/(1+\varepsilon))}$$

and $H(x)$ is the binary entropy function.

Proof. Let v be a vertex of minimum degree in G_n and choose $\lceil \varepsilon\delta_n \rceil$ vertices $S \subseteq N_1(v)$ and label them $v_1, \ldots, v_{\lceil \varepsilon\delta_n \rceil}$. Notice that the v_i have at least $\delta_n - 1$ unique neighbors in $N_2(v)$. Ignoring the edges in $N_2(v)$ gives us the bipartite graph J with parts S and $N_2(v)$

in Lemma 6 with parameters $d = \delta_n - 1$ and $a = \lceil \varepsilon\delta_n \rceil$. Hence, define $\{J_{k,n}\}_{k=1}^{r}$ as the r spanning subgraphs of G_n guaranteed by Lemma 6. In addition, we have that r is at least

$$\binom{(1+\varepsilon)\delta_n - 2}{\delta_n - 2} \sim 2^{((1+\varepsilon)\delta_n - 2)H(1/(1+\varepsilon))}$$

since

$$\frac{d-1}{a+d-1} \sim \frac{1}{1+\varepsilon}.$$

Furthermore, since each subgraph is connected and has at most $\lceil \varepsilon\delta_n \rceil$ many vertices with degree $o(\sqrt{n})$ and $\delta_n - \varepsilon\delta_n = \Theta(\sqrt{n})$, Lemma 4 implies that $\{J_{k,n}\}_{n\in I}$ is a Meyniel extremal family for each $1 \le k \le r$. □

Theorem 8. *Fix $0 < \varepsilon < 1$. If G_n is a C_4-free graph with degrees in $\Theta(\sqrt{n})$ and $\delta_n = \delta(G_n)$, then there exists r pairwise nonisomorphic, Meyniel extremal, spanning families of $\{G_n\}_{n\in I}$, where*

$$r \ge \binom{(1+\varepsilon)\delta_n - 3}{\delta_n - 3} \sim 2^{((1+\varepsilon)\delta_n - 3)H(1/(1+\varepsilon))}$$

and $H(x)$ is the binary entropy function.

Proof. Let v be a vertex of maximum degree in G_n and label $N_1(v) = \{v_1, \dots, v_{\lceil \varepsilon\delta_n \rceil}\}$. As in the proof of Corollary 5, we may make $N_1(v)$ an independent set, choose the $\lceil \varepsilon\delta_n \rceil$ from $N_1(v)$ and label them v_i. Each v_i have at least $\delta_n - 2$ unique neighbors in $N_2(v)$. We then use Lemma 6 with parameters $d = \delta_n - 2$ and $a = \lceil \varepsilon\delta_n \rceil$. Following a similar application of Lemma 4 as in the proof of Theorem 7, we have that G_n contains spanning subgraphs $\{J_{k,n}\}_{k=1}^{r}$ where

$$r \ge \binom{(1+\varepsilon)\delta_n - 3}{\delta_n - 3} \sim 2^{((1+\varepsilon)\delta_n - 3)H(1/(1+\varepsilon))}$$

and each family $\{J_{k,n}\}_{n\in I}$ is a Meyniel extremal family. As in Theorem 7, we have that

$$\frac{d-1}{a+d-1} \sim \frac{1}{1+\varepsilon}.$$

The proof follows. □

In both of these proofs, we use Lemma 6 to focus solely on edges between the first and second neighborhoods of a vertex of minimum degree. Although this is a local approach, it allows us to conclude that if two starting families are nonisomorphic due to some global structure, such as the presence of triangles, then all of the $2r$ families guaranteed by Theorem 7 and 8 will be pairwise nonisomorphic. More generally, if two families are nonisomorphic due to some structure that avoids the first and second neighborhood of a minimum degree vertex (suitably chosen for each corresponding member), then the $2r$ resulting families will be pairwise nonisomorphic.

For example, take some C_4-free, bipartite, Meyniel extremal family $\{G_n\}_{n\in I}$. Each of the families $\{G_{i,n}\}_{n\in I}$ given by Lemma 3 are nonisomorphic due to their graphs being d-regular for different d. In addition, each family satisfies the conditions of Theorem 8. Thus, all of the families generated by taking $\{G_{i,n}\}_{n\in I}$ as a starting family will be pairwise nonisomorphic.

These observations apply to the following two corollaries, whose starting families are nonisomorphic over the presence of triangles.

Corollary 9. *Fix $0 < \varepsilon < 1$ and let I be the set of prime powers. If G_q is the incidence graph of the projective plane on $2(q^2+q+1)$ vertices for a prime power q, then there exists r pairwise nonisomorphic, Meyniel extremal families of spanning subgraphs of $\{G_q\}_{q\in I}$, where $r \geq (1+o(1))2^{((1+\varepsilon)(q+1)-2)H(1/(1+\varepsilon))}$ for $H(x)$ the binary entropy function.*

Proof. The graph G_q is regular with vertices of degree $q+1$. Therefore, Theorem 7 implies the result. □

Corollary 10. *Fix $0 < \varepsilon < 1$ and let I be the set of prime powers. Let G_q be a graph that has order q^2+q+1, each vertex has degree $q+1$ or q, is C_4-free, and has diameter 2. There then exists r pairwise nonisomorphic, Meyniel extremal, spanning families of $\{G_q\}_{q\in I}$, where*

$$r \geq (1+o(1))2^{((1+\varepsilon)q-3)H(1/(1+\varepsilon))}$$

for $H(x)$ the binary entropy function.

Proof. By Theorem 3.1 in [5], the family $\{G_q\}_q$ is Meyniel extremal. Theorem 8 yields the result. □

In light of the observations above, the following corollary allows us to take a starting family and generate nonisomorphic, families of spanning subgraphs that have a specified number of triangles. This is done such that families with different number of triangles will produce pairwise nonisomorphic families.

Corollary 11. *Fix $0 < \varepsilon < 1$. Let G_n be a C_4-free graph with t_n triangles and degrees in $\Theta(\sqrt{n})$. Let $\delta_n = \delta(G_n)$. There then exists r pairwise nonisomorphic, Meyniel extremal, spanning families of $\{G_n\}_{n\in I}$, where*

$$r \geq \binom{(1+\varepsilon)\delta_n - 3}{\delta_n - 3} \sim 2^{((1+\varepsilon)\delta_n-3)H(1/(1+\varepsilon))},$$

the function $H(x)$ is the binary entropy function, and each member of the new families has exactly t'_n triangles, with $0 \leq t'_n \leq t_n$.

Proof. Denote the set of triangles in G_n as $\{T_i\}_{i=1}^{t_n}$ where T_i is on $x < y < z$. For the vector $\mathbf{x} \in \{0,1,2\}^{t'_n}$, let $G_{\mathbf{x},n}$ be obtained by removing one edge from T_i for $1 \leq i \leq t'_n$ in the following way. If $\mathbf{x} = (a_i)_{i=1}^{t'_n}$ and $a_i = 0$, then remove the edge xy from T_i. If $a_i = 1$, then remove the edge yz; if $a_i = 2$, then remove xz. Notice that each vertex has degree at least $\delta_n/2 = \Theta(\sqrt{n})$. For this, consider a vertex v. Each edge in G_n not contained in a triangle will remain in $G_{\mathbf{x},n}$, so v loses at most one edge for each triangle containing v. As all the triangles in G_n are edge-disjoint, there are at most $\deg_{G_n}(v)/2$ triangles containing v.

Now take a vertex of minimum degree v in G_n. Apply the above deletion procedure to each G_n until we have t'_n triangles remaining. Observe that all edges in G_n within $N_1(v)$ are contained in triangles in G_n containing v, so we may choose $\mathbf{x}$ such that only the edges in $N_1(v)$ are removed. In addition, we may choose $\mathbf{x}$ in such a way that all edges from $N_1(v)$ to $N_2(v)$ remain. Define a family of bipartite graphs $\{J_n\}_{n\in I}$ such that one bipartition consists of $\lceil \varepsilon\delta_n \rceil$ vertices of $N_1(v)$ in G_n and the other is $N_2(v)$.

We may apply Lemma 6 to J_n (ignoring the edges in $N_2(v)$) to obtain nonisomorphic families $\{J_{k,n}\}_{n\in I}$ for $1 \leq k \leq r$ with

$$r \geq \binom{(1+\varepsilon)\delta_n - 3}{\delta_n - 3} \sim 2^{((1+\varepsilon)\delta_n-3)H(1/(1+\varepsilon))},$$

as

$$\frac{d-1}{a+d-1} \sim \frac{1}{1+\varepsilon}.$$

Note that $J_{k,n}$ is a spanning subgraph of J_n (and thus, of G_n). In addition, since each subgraph is connected and has at most $\varepsilon\delta_n$ many vertices with degree $o(\sqrt{n})$, Lemma 4 implies that each of these spanning subgraphs are Meyniel extremal. Since $J_{k,n}$ also contains exactly t'_n triangles, the result follows. □

3. Techniques from hypergraphs

A *hypergraph* is a discrete structure with vertices and *hyperedges*, which consists of sets of vertices. Graphs are special cases of hypergraphs, where each hyperedge has cardinality two. A *blocking set* of a hypergraph (V, E) is a subset of its vertices such that each edge contains some vertex from the subset of vertices. Define the indicator variable x_v to be 1 if v is in the blocking set. The condition

$$\sum_{v \in e} x_v \geq 1,$$

holds for each edge e in the hypergraph. We can then think of finding a minimum cardinality of a blocking set as an IP problem, with an objective function

$$\sum_{v \in V} x_v,$$

which is being minimized. The minimum value of this objective function will be denoted τ. We may relax x_v to be a nonnegative real value, in which case the IP problem becomes an LP one. The resulting minimum value of the objective function is written τ^*, and the solution is known as a *fractional solution to the blocking problem.*

Theorem 12 (Lovász [16]). *For a hypergraph (V, E), let τ denote the cardinality of a minimum cardinality blocking set, τ^* denote the minimum value of a fractional solution of the blocking set, and d the maximum degree of a vertex. We then have that*

$$\tau < \tau^*(1 + \log d).$$

3.1. Domination number.

Theorem 12 can be used to prove results on graph domination, and the aim of this subsection is to use it to prove the following.

Theorem 13. *Let $\omega = \omega(n)$ be a nondecreasing, integer-valued function tending to infinity, and let $\{G_n\}_{n\in I}$ be a family of graphs, where G_n is of order n. If G_n has at most $O(\sqrt{n})$ vertices of degree $o(\sqrt{n})$, then G_n has domination number $O(\omega\sqrt{n}\log n)$.*

We have the following corollary.

Corollary 14. *If Meyniel's conjecture is false, then either any family of graphs has cop number $O(\sqrt{n}\log n)$, or there is a family of graphs $\{G_n\}_{n\in I}$, where G_n has order n, such that G_n has cop number $\omega(\sqrt{n}\log n)$, and $\omega(\sqrt{n})$ vertices of degree $o(\sqrt{n})$.*

By Corollary 14, we know that a family of graphs that violates Meyniel's conjecture may have a nontrivial number of vertices with small degree. In the search for graphs with the asymptotically largest cop number (in particular, if the aim is to find graphs with cop number $\omega(\sqrt{n})$), it may therefore be important to consider those graphs with some vertices of a smaller degree. This is at odds with the current constructions of graph families with high cop number, as these typically have all vertices of approximately the same degree, and each vertex has a relatively large degree.

Another consequence of Theorem 13 is that if the soft Meyniel's conjecture is true, and there exists a graph family with cop number $\Theta(n^{1-\alpha})$ for some $\alpha > 0$, then we must have the minimum degree of a graph of the graph class to be at most $n^{\alpha}(1 + \log \Delta)$. This may be asymptotically much smaller than $\Theta(n^{1/2})$, although so far we only require one vertex to have such a small degree. Theorem 13 therefore strengthens this result, showing that such a family of graphs requires a significant number of such vertices.

The idea underlying our proof of Theorem 13 is to construct a certain hypergraph, where each hyperedge is formed as the closed neighborhood of a given vertex of the graph. The set of vertices that form a blocking set of the hypergraph is also a dominating set of the original graph. We note that this is a modification of an idea by Beke [3] to determine an upper bound on the metric dimension of incidence graphs of Möbius planes.

Proof of Theorem 13. For convenience, assume k defined as $(\omega(n))^k = n^{1/2}$ is an integer. (The proof is straightforwardly modified, otherwise.) Consider G, a graph in the family of graphs, that has n vertices. Suppose V_i is the subset of vertices u such that $(\omega(n))^i < \deg(u) \leq (\omega(n))^{i+1}$ for $1 \leq i \leq k-1$, V_0 is the subset of vertices u such that $1 \leq \deg(u) \leq \omega(n)$, and V_k is the subset of vertices u such that $n^{1/2} < \deg(u)$. Note that by assumption, $\sum_{i=0}^{k-2} |V_i| = O(\sqrt{n})$.

Define a hypergraph with vertices $V(G)$ and with hyperedge $h_w = N_G(w)$ for each $w \in V(G)$. Define the function $s(u) = \min_{v \in N(u)} \deg(v)$. It follows that the vertices u that satisfy

$$(\omega(n))^{-(i+1)} \leq \frac{1}{s(u)} < (\omega(n))^{-i},$$

for $0 \leq i \leq k-2$, are contained in a subset of $N(V_i)$, and so there are at most $|V_i|(\omega(n))^{i+1}$ such vertices. There are also at most n vertices u that satisfy

$$(\omega(n))^{-k} \leq \frac{1}{s(u)} < (\omega(n))^{-k+1} = n^{-1/2}\omega(n),$$

and at most n vertices u that have $\frac{1}{s(u)} < n^{-1/2}$, which are subsets of $N(V_{k-1})$ and $N(V_k)$, respectively.

We now define the variables in the corresponding LP problem. For each vertex v in the hypergraph, we may define $x_v = \frac{1}{s(u)}$. It follows that $\sum_{v \in h_w} x_v \geq 1$ for each $w \in V(G)$. We then have that

$$\begin{aligned}
\tau^* &\leq \sum_{i=0}^{k} \sum_{v \in V_i} \frac{1}{s(u)} \\
&< |V_{k-1}| n^{-1/2} \omega(n) + |V_k| n^{-1/2} + \sum_{i=0}^{k-2} |V_i| (\omega(n))^{i+1} (\omega(n))^{-i} \\
&= n^{1/2} + n^{1/2}\omega(n) + \sum_{i=0}^{k-1} O(n^{1/2})\omega(n) \\
&= O(n^{1/2}\omega(n)).
\end{aligned}$$

The proof now follows by Theorem 12. □

3.2. **Cop number.** Frankl [11] proved that for a graph G, $c(G) \leq (1+o(1))\frac{n \log\log n}{\log n}$, which was improved to $O(\frac{n}{\log n})$ in [9]. The results of [12, 17, 20] reduced this upper bound down further to

$$c(n) \leq O\left(\frac{n}{2^{(1-o(1))\sqrt{\log_2 n}}}\right). \tag{1}$$

We give new upper bounds on the cop number, improving (1) in some cases, using the hypergraph techniques in this section.

As first defined in [9], a *minimum distance caterpillar* (or *mdc*) in a graph G is an induced subgraph of G whose vertices consist of a shortest path P between two vertices of G, along with a subset of the neighbors of vertices in P. We say the *length* of an mdc is the length of P. As shown in [9], five cops may *guard* an mdc, in the sense that after some number of rounds, the robber is captured if they enter it. Define a *diameter length caterpillar* (or *DLC*) as an mdc of length $\mathrm{diam}(G)$. In particular, five cops may guard a DLC.

A graph is *vertex-transitive* if for every two vertices x and y, there is an automorphism mapping x to y. A vertex-transitive graph is m-regular for some nonnegative integer m, and we refer to m as its *degree*. The following theorem bounds the cop number of a vertex-transitive graph by a function of its degree and diameter.

Theorem 15. *Let G be a vertex-transitive graph G with degree m, and let $d = m \cdot \mathrm{diam}(G)$. We then have that*

$$c(G) \leq \frac{3n \log d}{d} = O\left(\frac{n \log d}{d}\right).$$

Proof. Let V be the vertex set of G. Let μ_1 be the number of DLCs that any one vertex is in (note that by the vertex-transitivity of G, this value is the same for any choice of vertex). Let μ_2 be the minimum number of vertices in any DLC. The maximum cardinality of a DLC is at most d, so $\mu_2 \leq d$.

Define a new set of vertices W, where each $w \in W$ is associated with a DLC, m_w, of G. Define a hypergraph $\mathcal{H} = (W, H)$ on the vertex set W by including the hyperedge $h_v = \{w \in W : v \in m_w\}$ in the hyperedge set H, for each $v \in V$. In particular, a hyperedge $h_v \in H$ contains vertex $w \in W$ exactly when the vertex $v \in V$ is contained in the DLC m_w of G. Let $x_w = \frac{1}{\mu_1}$. We then have that $\sum_{w \in h} x_w = |h|/\mu_1 = 1$ for each hyperedge h, so the conditions of the LP are satisfied.

We now find an upper bound for $|W|$. Consider the double count of the pair $\{(w, v) \in W \times V : w \in h_v\}$, which gives

$$\sum_{v \in V} |h_v| = \sum_{w \in W} |m_w|.$$

The left-hand side is $\mu_1 n$. As $|m_w| \geq \mu_2$, the right side is greater than $|W|\mu_2$. We then have that $|W| \leq \frac{\mu_1 n}{\mu_2}$.

As a result,

$$\tau^* = \sum_{w \in W} \frac{1}{\mu_1} \leq |W| \frac{1}{\mu_1} = \frac{n}{\mu_2}.$$

By Theorem 12, this gives $\tau < \frac{n \log d}{\mu_2}$.

The set of vertices in a blocking set of τ vertices corresponds to a set of DLCs in G such that every vertex of G is covered by some DLC. Since five cops may guard each of these DLCs, 5τ cops is sufficient to capture the robber in G.

Let P be the path of length $\operatorname{diam}(G)$ associated with some DLC, and let P' be a subset of the vertices of P such that any pair of vertices in P' have distance at least 2 from each other. Note that P' contains $\lceil \operatorname{diam}(G)/3 \rceil \geq \operatorname{diam}(G)/3$ vertices. Now $u, v \in P'$ are also vertices in the DLC, and have distance at least two in G. As such, $N(u) \cap N(v) = \emptyset$, and so the set of all neighbors of vertices in P' is equal to $m \cdot P' \geq m \cdot \operatorname{diam}(G)/3$. This implies that the number μ_2 must be larger than $m \cdot \operatorname{diam}(G)/3 = d/3$, and so we have that $\tau < \frac{n \log d}{d/3}$, and the result follows. □

The upper bound in Theorem 15 provides the best known bounds on the cop number for vertex-transitive graphs when the degree is not too small, improving the one in (1).

Corollary 16. *Suppose that $\omega = \omega(n)$ is a nondecreasing, integer-valued function tending to infinity, and G is a vertex-transitive graph with degree m and with*

$$m \cdot \operatorname{diam}(G) \geq 3\omega 2^{(1-o(1))\sqrt{\log_2 n}} \sqrt{\log_2 n}.$$

The following inequalities then hold for n sufficiently large.

(1) *If $\omega = \Omega\left(2^{\sqrt{\log_2 n}}\right)$, then*

$$c(G) \leq \frac{n}{2^{(1-o(1))\sqrt{\log_2 n}}} \frac{3 \log \omega}{3\omega \sqrt{\log_2 n}} = o\left(\frac{n}{2^{(1-o(1))\sqrt{\log_2 n}}}\right).$$

(2) *If $\omega = o\left(2^{\sqrt{\log_2 n}}\right)$, then*

$$c(G) \leq \frac{n}{2^{(1-o(1))\sqrt{\log_2 n}}} \frac{1 - o(1)}{3\omega} = o\left(\frac{n}{2^{(1-o(1))\sqrt{\log_2 n}}}\right).$$

We also have that Meyniel's soft conjecture holds up to a logarithmic factor for vertex-transitive graphs of order n when the degree is a fractional power or linear in n.

Corollary 17. *If G is a vertex-transitive graph with degree $m = \Theta(n^{1-\varepsilon})$ for a constant $0 \leq \varepsilon < 1$, then*

$$c(G) = O(n^{1-\varepsilon} \log n).$$

The proof technique of Theorem 15 can be generalized for a larger class of graphs than vertex-transitive graphs. In a graph G, define μ_1 to be the maximum number of DLCs that contain any one vertex. We define two parameters σ and e_1 such that most vertices are contained in between μ_1/σ and μ_1 DLCs, with exactly e_1 exceptional vertices that are contained in less than μ_1/σ DLCs. These exceptional vertices can be covered by placing a cop on each of them, and so can be ignored for the proof. By allowing vertices to be contained in different numbers of DLCs, the upper bound changes to $c(G) \leq O\left(\frac{n\sigma \log d}{\mu_2}\right)$. We also define e_2 and μ_2 such that all DLCs contain μ_2 vertices not in the e_1 exceptions mentioned previously, except for exactly e_2 exceptional DLCs that have less than μ_2 such vertices. This modifies Theorem 15 by allowing μ_2 to be potentially much higher, as the small number of exceptions can each be covered by a set of five cops. As long as a number of conditions still hold for e_1 and e_2, we will have that $c(G) \leq O\left(\frac{n\sigma \log d}{\mu_2}\right)$.

4. New families of Meyniel extremal graphs with prescribed properties

This section provides new constructions of Meyniel extremal families whose graphs are regular with large chromatic number, and ones with diameter at least some fixed constant. We give a construction for creating bipartite Meyniel extremal families from any given Meyniel extremal family.

4.1. Regular with large chromatic number.

We consider a method using graph products to construct new Meyniel extremal families consisting of regular graphs with various properties such as large chromatic number or clique number. For more on graph products, the reader is directed to [15]. For graphs G and H, define their *lexicographic product*, written $G \bullet H$, to have vertices $V(G) \times V(H)$, and (u, v) is adjacent to (x, y) if u is adjacent to x in G, or $u = x$, and v is adjacent to y in H. We may think of $G \bullet H$ as replacing each vertex x of G with a copy of H labeled as H_x, such that if $xy \in E(G)$, then all edges are present between H_x and H_y. Note that the order of $G \bullet H$, is $|V(G)||V(H)|$. Schröder [19] proved that if $c(G) \geq 2$, then $c(G \bullet H) = c(G)$.

Theorem 18. *Suppose that $\{G_n\}_{n\in I}$ is a Meyniel extremal family and H is a fixed graph. We then have that $\{G_n \bullet H\}_{n\in I}$ is a Meyniel extremal family.*

Proof. Suppose that $|V(G_n)| = n$, $|V(H)| = m$, and let $D > 0$ be a constant such that $c(G_n) \geq D\sqrt{n}$ for all n. We then have that $c(G_n \bullet H) \geq D\sqrt{n}$ and the order of $G_n \bullet H$ is nm.

Hence, $c(G_n \bullet H) \geq D'\sqrt{|V(G_n \bullet H)|}$, with $D' = D/\sqrt{m}$. □

Corollary 19. *For an integer $t \geq 1$, there exist Meyniel extremal families containing graphs that are regular and with clique and chromatic number at least t.*

Proof. Apply Theorem 18 with $H = K_t$, the complete graph of order t, and $\{G_n\}_{n\in I}$ the family of incidence graphs of projective planes. As H is an induced subgraph of $G_n \bullet H$, the result follows. □

With our approach, we may also find Meyniel extremal graphs with bounded clique number and chromatic number at least a fixed constant: choose H to have sufficiently large girth and chromatic number. By taking H to be a graph with no edges, we may also find Meyniel extremal families of regular graphs with independence number larger than any fixed constant.

4.2. Large diameter.

In most cases of Meyniel extremal families, the graphs are of small diameter. For example, the incidence graphs of projective planes have diameter 3, while polarity graphs have diameter 2. In this subsection, we find Meyniel extremal families of graphs with any constant even diameter by using a variation of incidence graphs of projective planes.

A *projective plane* consists of a set of points and lines (or blocks) satisfying the following axioms:

(1) There is exactly one line incident with every pair of distinct points.
(2) There is exactly one point incident with every pair of distinct lines.
(3) There are four points such that no line is incident with more than two of them.

We only consider finite projective planes. It can be shown that projective planes have q^2+q+1 many points and lines for an integer q called its *order*, each point is on $(q+1)$-many lines, and each line contains $(q+1)$-many points. The only orders where projective planes are known to exist are prime powers, and it is conjectured that these are the only orders for which they exist. For these items and further background on projective planes, see [8].

Let q be a prime power and m a positive integer. Consider a projective plane of order q with points X and lines $\mathcal{B}$. For each line $B \in \mathcal{B}$ of cardinality $q+1$, we define B', B'', each of cardinality $\frac{q+1}{2}$, such that $B = B' \cup B''$. Write the corresponding sets of lines as $\mathcal{B}'$ and $\mathcal{B}''$.

We define a tripartite graph on vertex set $X \cup \mathcal{B}' \cup \mathcal{B}''$, and where an edge connects $u \in X$ to $v \in \mathcal{B}' \cup \mathcal{B}''$ if $u \in v$. Orient a cycle C_m such that each vertex has in-degree and out-degree 1. Construct a *blow up* of C_m, say H, where each vertex v of C_m is replaced with q^2+q+1 vertices $\{(v, B) : B \in \mathcal{B}\}$ and each edge e of C_m is replaced with q^2+q+1 vertices $\{(e, x) : x \in X\}$. Vertices (u, B) and (e, x) of H are adjacent if e is an out-edge of u, $B \in \mathcal{B}'$ and $x \in B$; or if e is an in-edge of u, $B \in \mathcal{B}''$ and $x \in B$. The resulting graph is denoted by $\mathrm{BF}(q, m)$.

Theorem 20. *Let q be a prime power and m a positive integer. The graph $\mathrm{BF}(q, m)$ is a bipartite graph with the following properties:*

1. *order $2(q^2 + q + 1)m$ and $(q^2 + q + 1)(q + 1)m$ edges;*
2. *C_4-free;*
3. *diameter $2m$;*
4. *$(q + 1)$-regular; and*
5. *cop number at least $q + 1$.*

Proof. For each vertex and each edge in C_m, $q^2 + q + 1$ vertices are created, giving $(q^2 + q + 1)(|V(G)| + |E(G)|)$ vertices in the blow up. A copy of the tripartite graph replaces each edge in G, and each tripartite graph has $2(q^2 + q + 1)$ line vertices, each of degree $(q + 1)/2$, giving $(q^2 + q + 1)(q + 1)m$ edges in the constructed graph. Hence, item (1) holds.

The tripartite graph that replaces each edge is C_4-free, since the incidence graph of the projective plane was C_4-free. If a C_4 existed between two tripartite graphs that replaced two adjacent edges, then the incidence graph of the projective plane would contain a C_4, giving a contradiction. Item (2) holds.

Each vertex that was the result of blowing up a vertex of C_m has degree $2\left(\frac{q+1}{2}\right) = q + 1$, and each vertex that was the result of blowing up an edge of C_m has degree $q + 1$. The diameter of the graph is twice that of C_m. Items (3) and (4) hold. Item (5) follows by Theorem 3 from [1], and the theorem follows. □

The following corollary follows from Theorem 20.

Corollary 21. *For every positive integer $d \geq 3$ there exists a Meyniel extremal family whose graphs are regular and have diameter at least d.*

4.3. **Bipartite Meyniel extremal graphs.** Let G be graph that is not bipartite. The *categorical product* $G \times K_2$ has vertex set $\{(v, a) : v \in V(G), a \in V(K_2)\}$ and an edge between (v, a) and (v', a') when $vv' \in E(G)$ and $aa' \in E(K_2)$. The graph $G \times K_2$ is called the *bipartite double cover* or *Kronecker cover* of G, and is denoted by $B(G)$. Note that $B(G)$ has twice the number of vertices as G. The degree of vertex (v, a) in $B(G)$ is the same as the degree of v in G. Further, $B(G)$ is connected as G contains at least one cycle of odd length (since it is not bipartite), and is bipartite.

The following theorem provides bounds on the cop number of a bipartite double cover.

Theorem 22. *For a graph G, we have that*

$$c(G) \leq c(B(G)) \leq 2c(G).$$

Proof. We begin by showing the lower bound. When playing Cops and Robbers on $B(G)$, the robber can apply the mapping $f : B(G) \to V(G)$ defined as $f(v, a) = v$ to the location of each cop to convert the game to a game on G. The robber can then use the strategy defined on G to avoid the cops. Label the vertices of K_2 by 0 and 1. If this strategy tells the robber to move from vertex u to v in G, then the robber moves from (u, a) to $(v, a + 1) \pmod 2$ in $B(G)$. The robber in $B(G)$ is captured only if the robber in G is captured. Therefore, we have that $c(B(G)) \geq c(G)$.

Now we show the upper bound. Suppose that cops $C_1, C_2, \ldots, C_k$ capture the robber on G. To each cop on G, we associate two cops playing on $c(B(G))$. During each round and for $1 \leq i \leq k$, the two cops $C_{i,0}, C_{i,1}$ associated to C_i play on the vertices $(C_i, 0)$ and $(C_i, 1)$. The robber is captured by the cops $C_{i,0}, C_{i,1}$ on $c(B(G))$ when the robber is captured on G by the cops C_i. □

We have the following immediate corollary, which gives a strong case to focus on the cop number of bipartite graphs.

Corollary 23. *If there exists a family of connected graphs $\{G_n\}_{n \in I}$ with $c(G_n) = \Theta(n^{1-\epsilon})$, where $0 \leq \epsilon < 1$, and $|V(G_n)| = \Theta(n)$, then there exists a family of bipartite connected graphs $\{G'_n\}_{n \in I}$ with $c(G'_n) = \Theta(n^{1-\epsilon})$ and $|V(G'_n)| = \Theta(n)$.*

In particular, Corollary 23 shows that every Meyniel extremal family gives rise to a Meyniel extremal family whose members are bipartite. Further, if there is a family of connected graphs that violate Meyniel's conjecture, then we can find a family of connected bipartite graphs that violate Meyniel's conjecture.

We note that the number of C_4's in $B(G)$ will be twice the number in G. The number of C_6's in $B(G)$ will be the number of triangles in G plus twice the number of C_6's in G. Hence, if G is C_4-free, then so is $B(G)$, and if G is triangle-free and C_6-free, then $B(G)$ is C_6-free.

Our approach using the bipartite double cover also yields another short proof of the result first proven in [5].

Corollary 24. *If G is a C_4-free graph, then $c(G) \geq \delta(G)/2$.*

Proof. If G is C_4-free, then $B(G)$ has girth at least 6, and so has cop number $c(B(G)) \geq \delta(B(G)) = \delta(G)$. By Theorem 22, this gives that $\delta(G) \leq 2c(G)$, and the result follows. □

5. Further directions

All known Meyniel extremal families, such as the incidence graphs of projective planes and polarity graphs, rely on properties of degrees and on forbidding subgraphs. We demonstrated that the minimum degree of graphs in Meyniel extremal families may vary widely: Corollary 5 shows that there exist classes of Meyniel extremal families containing graphs on n vertices with $\Theta(\sqrt{n})$ vertices of constant degree. In our constructions of Meyniel extremal families in Sections 2 and 5, both the maximum degree and the average degree remain high. In the recent paper [14], it was shown that for every $\varepsilon > 0$, there exists a graph with maximum degree 3 on n vertices with cop number $\Omega(n^{1/2-\varepsilon})$. The proof use a degree-reducing technique that seems to break down when trying to obtain a Meyniel extremal family.

Our discussion suggests the following.

Maximum degree conjecture: Every Meyniel extremal family contains graphs with maximum degree $\omega(1)$.

If the Maximum Degree Conjecture is true, then there is no Meyniel extremal family of subcubic graphs (that is, graphs whose degrees are at most 3). Corollary 5 as well as Lemma 1 suggests that the number of vertices of certain degrees must be controlled to maintain Meyniel extremality. We also conjecture that the average degree must be unbounded, which would imply the previous conjecture.

Average degree conjecture: Every Meyniel extremal family contains graphs with average degree $\omega(1)$.

Many of the known examples do not deviate from an average degree of $\Theta(\sqrt{n})$, and our current efforts seem to suggest that this is an important threshold. The analysis of the cop number of the binomial random graph $G(n,p)$ in [18] suggests that there are examples with lower average degree: their results may be extended to when the average degree is $n^{1/(2k)+o(1)}$ for natural numbers k and achieve a cop number of $\Theta(\sqrt{n})$, with the possible additional factor of $\log^{O(1)} n$. An open problem is to find a Meyniel extremal family containing graphs with average degree $o(\sqrt{n})$.

Forbidding too many short cycles is not possible if the degree is too large. A Moore bound argument gives that if a graph G has n vertices and $\delta(G) = \Theta(\sqrt{n})$, then G has girth at most 6. We must therefore search for graphs with smaller minimum degree to find Meyniel extremal families whose graphs do not contain any C_6. Finding a Meyniel extremal family whose members are C_6-free remains an open problem. Interestingly, one of the best known examples of a Meyniel extremal family, the incidence graphs of projective planes, has the largest possible number of C_6's in a C_4-free, balanced bipartite graph; see [10].

6. Acknowledgements

The authors were supported by NSERC.

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