The game of Wall Cops and Robbers*

Anthony Bonato[1] and Fionn Mc Inerney[1]

Ryerson University, Toronto, Canada

Abstract. Wall Cops and Robbers is a new vertex pursuit game played on graphs, inspired by both the games of Cops and Robbers and Conway's Angel Problem. In the game, the cops are free to move to any vertex and build a wall; once a vertex contains a wall, the robber may not move there. Otherwise, the robber moves from vertex-to-vertex along edges. The cops capture the robber if the robber is surrounded by walls. The *wall capture time* of a graph G, written $W_{c_t}(G)$, is the least number of moves it takes for one cop to capture the robber in G. In the present note, we focus on the wall capture time of certain infinite grids. We give upper bounds on the wall capture time for Cartesian, strong, and triangular grids, while giving the exact value for hexagonal grids. We conclude with open problems.

1 Introduction

Wall Cops and Robbers is a new vertex pursuit game played on graphs, inspired by the games of Cops and Robbers and the Angel Problem. In Wall Cops and Robbers, there are two players: a cop and a robber. The game starts with the cop building a "wall" on a vertex which blocks off that vertex so that the robber cannot occupy it. After that, the robber selects a vertex. The cop can build a wall on any vertex on his turn except for the vertex that the robber currently occupies. Hence, we may think of the cop as playing off the graph. The robber, however, can only move along an edge to an adjacent vertex each turn. The robber is also allowed to skip his turn, but to avoid complexities, he may not move back to the vertex he occupied on his previous turn (that is, the robber cannot backtrack).

The game of Cops and Robbers was first introduced by Quilliot [10] and independently by Nowakowski and Winkler [9]. For additional background on Cops and Robbers and its variants, see the book [5] and the surveys [1, 3, 4]. The Angel Problem was first introduced by John H. Conway [2]. The Angel Problem is a turn-based game played on either an infinite chessboard or an infinite 3-dimensional chessboard. The game is played by an Angel and the Devil. The Angel has power k, where k is a positive integer, which allows the Angel to move k spaces in any direction. We can think of this as the Angel having k moves on his turn where he can only move to an adjacent space on each move. The Devil has the same power as the cop in Wall Cops and Robbers, except that he eats squares in the Angel Problem while the cop builds walls on vertices. The objective of the game is the Angel trying to elude capture by the Devil on an infinite chessboard, which corresponds to a Cartesian product of infinite two-way paths. For more information on the Angel Problem, see [6].

Firefighting on graphs also has some similar aspects to Wall Cops and Robbers. Firefighter was introduced by Hartnell [8], and it is graph process where firefighters try to contain a spreading fire. The fire starts at some vertex, and then the firefighters place themselves on vertices making them *protected*. Then on the next turn the fire spreads to all adjacent vertices that are not protected, and the firefighters protect another set of vertices. The game continues in this manner, and one of the goals for the firefighters save as many vertices from the fire as possible. See the survey [7] for various desired outcomes of Firefighter.

The objective of Wall Cops and Robbers is for the cop is to build walls to block off all adjacent vertices to the robber so that the robber can no longer move on his next turn. The objective of the game for the robber is to evade capture by the cop for as long as possible. The *wall capture time*, written W_{c_t}, of a graph is the least number of moves it takes for the cop to capture the robber on the

* Supported by grants from NSERC and Ryerson University.

given graph given that the cop and robber have both played their best strategies. Note that Wall Cops and Robbers is equivalent to the Angel problem, where the Angel has power $k = 1$ (although Angels can backtrack).

For an elementary example, consider the game played on the graph G in Figure 1. We label the vertices 1, 2, 3, 4, 5, and 6 as in Figure 1 and the cop builds a wall on vertex 3. If the robber chooses 1 or 2, then he will be stuck in the left triangle and will lose on the next turn. Any of the vertices in the right triangle would be a good choice for the robber, so he chooses 5. The cop builds a wall on 4. The robber can either move to 6 or skip his turn and remain at 5; since both give the same result of him losing next turn, let us say he moves to 6. The cop builds a wall on 5 and captures the robber as he can no longer move. Since it takes exactly three moves for the cop to capture the robber with both sides playing at their best, we have that $W_{c_t}(G) = 3$.

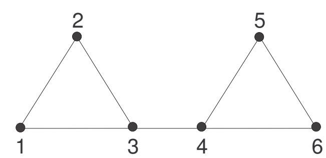

Fig. 1. A labelled graph G.

2 Grids

Given our limited space, we confine our discussion of the wall capture time to infinite grids and tilings of the plane in two dimensions. We study the wall capture time of infinite hexagonal grids, infinite Cartesian grids, infinite triangular grids, and infinite strong grids. Lastly, we study the wall capture time of n-layered infinite Cartesian grids which are certain subgraphs of three dimensional grids.

The infinite hexagonal grid, written H_∞, is a tiling of the plane by hexagons with vertices represented as vertices of the hexagons. We have the following result, whose proof is omitted for space considerations.

Theorem 1. $W_{c_t}(H_\infty) = 8$.

An infinite Cartesian grid, written $P_\infty \Box P_\infty$, is the Cartesian product of two infinite, two-way paths. A *trap* is two walls made by cops on a Cartesian grid such that they share the same x or y coordinate but not both, and are distance two apart. The vertex in between these two walls will be called the *middle vertex*. It is called a trap since if the robber moves onto the middle vertex, then the cop will close the trap by moving to the open vertex that is adjacent to the middle vertex that the robber did not just come from in the last move. The robber cannot move back to his previous vertex by the rules of the game, and then the cop will capture him by playing adjacent to him. Thus, moving into a trap guarantees that the robber will be captured in exactly two turns.

The proof of the following theorem, while elementary, involves the careful analysis of cases using traps and so is omitted here.

Theorem 2. $W_{c_t}(P_\infty \Box P_\infty) \leq 14$.

We next turn to the infinite triangular grid, written $\triangle_\infty$, which is a tiling of the plane by triangles with vertices represented as vertices of the triangles. We sketch the proof of this result below.

Theorem 3. $W_{c_t}(\triangle_\infty) \leq 138$.

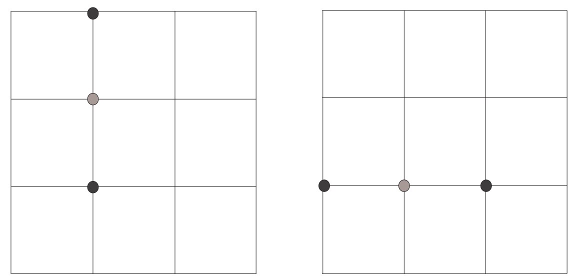

Fig. 2. The two possible traps the cop can build. The two walls of the trap are in black and the middle vertex is in grey.

Proof. The cop will first trap the robber in a hexagon with lengths of sides 3, 21, 21, 21, 21, and 3. The cop will build the hexagon, taking the robber's vertex as the centre of this hexagon so that he is distance 11 from each of its walls. The robber will not move backwards or skip his turn while the hexagon is being built as this will allow the cop to build a smaller hexagon and thus, use fewer moves to capture the robber. We will describe the corners of the hexagon as follows: TLC, TRC, LC, RC, BLC, BRC with T standing for top, B standing for bottom, R standing for right, L standing for left, and C standing for corner. The first 9 moves for the cop are as follows:

1. The first move of the cop is wasted as we are playing on an infinite grid, so the robber will just play so far away from the first cop that the wall he builds will be useless.
2. The cop plays one up and right of the LC.
3. The cop plays one down and right of the LC.
4. The cop plays one up and left of the RC.
5. The cop plays one down and left of the RC.
6. The cop plays on the TLC.
7. The cop plays on the TRC.
8. The cop plays on the BLC.
9. The cop plays on the BRC.

The robber must have moved towards one of the sides of the hexagon in these first 9 moves. If he moved up, then the cop's 10^{th} move is to play on the vertex two down and left of the TRC. If he moved down, then the cop's 10^{th} move is to play on the vertex two up and right of the BLC. If the robber went straight left or right without any diagonal movements then the cop would not have had to play there so the robber would not move like that. Otherwise, the cop could build a smaller hexagon. Then we know that the robber will move either up or down towards the TLC or TRC or the BLC or BRC. Now the robber is distance two away from a side (only one) and it is the cop's turn and thus, the cop can stop him getting on the sides of the hexagon.

As the robber runs along the sides of the hexagon, the cop will gain a move on the robber at the LC and RC due to the walls that were built at the start. The cop will use these two extra moves to build a wall on the vertex in between the TLC and TRC and a wall on the vertex in between the BLC and BRC in the order that the robber will approach these vertices.

The robber may be able to move toward the centre of the hexagon after most of the sides have been built by the cop in order to gain some extra moves. It is difficult to know exactly when this may happen and in which exact direction the robber would move at the start of the game and at this point in the game. Therefore, we will assume that the cop will build the entire hexagon even if the robber moves toward the centre of the hexagon early. Thus, the robber will force the cop to build a wall inside the hexagon on his 10^{th} move as described above. We will assume also that the robber can be anywhere inside this hexagon after it is built. The robber will be in the centre of the hexagon as if he is near any of the sides, it will allow the cop to use those sides to trap him in a smaller subgraph in this next phase.

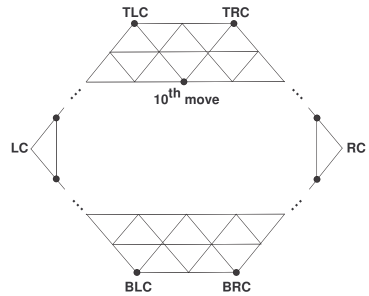

Fig. 3. The first 10 moves of the cop shown as black vertices, with the first move not shown as it is far away from the hexagon.

The cop will then confine the robber to a parallelogram with diagonals of length 23 and the other two sides of length five. We will assume, without loss of generality, that the diagonals go from the top left side of the hexagon to the bottom right side of the hexagon. Since both the top left side and bottom right side of the hexagon are distance 11 from the centre, the diagonal between them is of length 23. The two diagonals of length 23 will be built distance five apart. One will be built distance two from the robber on his left and the other will be built distance three from the robber on his right. This guarantees that the robber is always at least distance two away from either diagonal with it being the cop's turn. The first wall built on these diagonals will be built on the left one since it is only distance two away from the robber. Thus, the robber will never be able to occupy a vertex on either diagonal or leave the parallelogram. The two sides of length five are already built as they are part of the sides of the hexagon. Once this parallelogram is completely built, we will assume that the robber is in the best position possible. Therefore, he will be in the centre of this parallelogram.

The cop will then be building two sides parallel to the sides of the hexagon of length three to trap the robber in a 5×4 parallelogram. The first of these two sides will be built distance one away from the robber, down and to his right, since it is possible to stop him bypassing the side as there are only three open vertices to cover. Thus, the cop will play adjacent to the robber on this side he will be building as seen in Figure 4. The robber can move so that the side must actually be built as parallel to the sides of the parallelogram of length five. This does not matter as in either case the cop will trap the robber in a 5×4 parallelogram. From here, the other side will be built distance two away from the robber's starting position in the parallelogram on the opposite side of the first side built and parallel to the first side built. Now the robber is trapped in a 5×4 parallelogram. The robber will move back to the vertex he started on after the large parallelogram's sides were built. The cop will capture the robber in at most four moves inside this small parallelogram, no matter where the robber moves inside it as there are only six open vertices and the cop can cut out one of two vertices that are distance three apart.

Now we will sum up all the moves of the cop. The cop took one wasted move at the start. He then takes $21 \times 4 + 3 + 3 + 1 - 6 = 85$ moves (subtract six for counting six vertices twice) to build the sides of the hexagon and a wall on the vertex on his 10^{th} move. Followed by 42 moves to build the two diagonals of length 23 of the large parallelogram. Six moves to complete the sides of the 5×4 parallelogram. Then, four moves to capture the robber inside the 5×4 parallelogram. Thus, it took the cop $1 + 85 + 42 + 6 + 4 = 138$ moves to capture the robber. □

The infinite strong grid, written $P_\infty \boxtimes P_\infty$, is the strong product of two infinite, two-way paths. We will be labelling the vertices using Cartesian coordinates. We rely on the following fact from [2]. Note that in the Angel Problem, the Angel (that is robber in our case), unlike in Wall Cops and Robbers, is allowed to move to a vertex he occupied in the previous turn. However, such moves will only lengthen the play of the game. Hence, bounds on the length of the Angel Problem will also be bounds on the wall capture time played on the infinite strong grid.

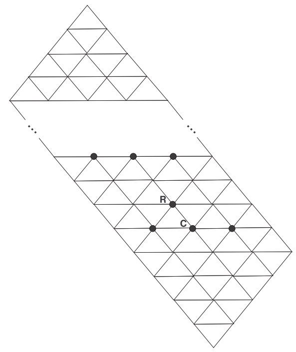

Fig. 4. The parallelogram.

Lemma 1 ([2]). *On the infinite strong grid, the cop can confine the robber to a 35×35 box by building three walls in each corner on his first 12 moves. The cop can confine the robber from there as he can always stop the robber reaching a side of this box if the robber is distance five from a side and it is the cop's turn.*

We now turn to the main result of this section.

Theorem 4. $W_{c_t}(P_\infty \boxtimes P_\infty) \leq 246$.

Proof. Without loss of generality, we will assume that after the cop plays his first turn, that the robber will place himself sufficiently far away that this first wall will play no part in capturing the robber. Thus, we will count this move only at the end in the total number of moves but will not count it in our ordered turns for the cop below. The cop's plan is to capture the robber in a 35×32 box first and then capture the robber inside that box. The robber's best strategy has to be to move in one direction such as up or right, and or diagonally up and right so as to make it difficult for the cop to trap him. Otherwise, as moving back in the opposite direction while this box is being set up will just waste the robber's turn and possibly allow the cop to build a smaller box to contain the robber. If the robber skips his turn several times, then it will allow the cop to either encroach the sides of the box or if they are already built, then it will allow the cop to encroach the vertical or horizontal walls that enclose the robber after that. Therefore, it is not a good idea for the robber to skip his turn until he knows he will be captured and there are few turns remaining because even doing it once will allow the cop to be one turn ahead of the robber which may also result in fewer moves needed.

The cop will start by building the corners of a 35×35 box as in Lemma 1 and once a few moves have been made it will be clear what direction, if any, the robber has chosen to move in which will allow the cop to encroach one of the sides in three steps making it a 35×32 box (or encroach two of the sides in for a total of three steps, also making it a 35×32 box). The box must start as a 35×35 box since the cop requires 12 moves before the robber can be at a distance of five moves from any side, thus, making it necessary that the robber start at a distance of 17 moves from any side which a 35×35 box ensures.

We will assume that the robber is moving diagonally opposite of the corner in which the cop first plays as this makes it most difficult for the cop since he will not be able to encroach the sides of the box further in than just three steps. If the robber moves in any other fashion at the beginning, then the same strategy can be used by the cop. It will be shown that the robber cannot escape the box if he is at a distance of five moves from any side of the box, there have been three walls built in each corner of the box, and it is the cop's turn as in Lemma 1.

Let T,B,L,R,C represent top, bottom, left, right, and corner, respectively, of the 35×35 box. The way the moves of the cop will be described should be interpreted as follows: two up from the BLC means that the bottom left corner is (0,0) and two up from that would be (0,2). We will play the first cop in the BLC and thus, we assume the robber is always moving diagonally toward the TRC but of course the strategy of the cop can be easily modified for the robber moving diagonally toward any corner. The robber by moving in the opposite direction of the second cop stops the cop from encroaching one or two of the sides by more than a total of three steps.

1. The cop plays four up from the BLC.
2. The cop plays two down from the TRC.
3. The cop plays one right of the TLC.
4. The cop plays two right of the TLC.
5. The cop plays four right of the TLC.
6. The cop plays one down from the TRC.
7. The cop plays four down from the TRC.
8. It is now clear that the robber is moving to the TRC and thus, the cop can encroach the bottom side of the box in three steps to make it a 35x32 box by playing five up from the BLC.
9. The cop plays seven up from the BLC.
10. The cop plays three up and one left of the BRC.
11. The cop plays three up and two left of the BRC.
12. The cop plays three up and four left of the BRC.

Now the next move the robber makes will bring him to within five moves of one or two sides. In the present case, the robber has moved to the TRC. If the robber moves up in any direction, then the cop's next move is to play on the side of the box directly above the robber. If the robber moves diagonally down and to the right or directly to the right then the cop's next move is to play on the side of the box directly to the right of the robber. From there in either case, we have the scenario where it is the robber's turn and he is distance five away from a side with a wall on the side directly in his path if he were to go straight at the side. We know that the cop can block the robber along this side now no matter which move he makes by Lemma 1 and Figure 5 below. With three walls built in each of the corners, the robber cannot run forever and must turn at the corners. We will assume the cop builds the entire box and that the robber can be anywhere inside that box afterwards. Therefore, it takes $35 + 35 + 30 + 30 = 130$ moves to build the walls of the box.

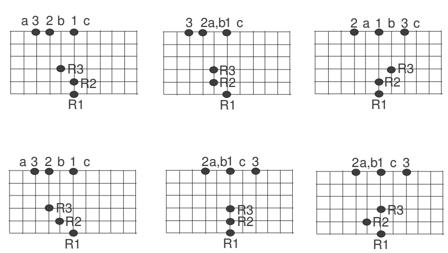

Fig. 5. The cop's strategy to stop robber reaching a side from a distance of five moves. In each of the figures, a, b, and c represent the fourth wall the cop would build if the robber moved diagonally up and to the left or straight up or diagonally up and to the right, respectively.

We assume the robber is in the centre of the 35×32 box. The cop will now confine the robber to a 32×12 box. The two sides of length 12 have already been built as they are part of the sides of the 35×32 box. The cop will build the two sides of length 32 that are distance 11 from each other. The robber can only ever be distance five from either of these two sides but not both. This ensures the cop can stop the robber reaching either of these sides. It is difficult to know exactly what the best position the robber can attain while still making the cop complete the 32×12 box. Thus, we will say that once that box is completely built, we will let the robber be in the middle of this smaller box as that is best for him with the knowledge of the cop's strategy. Therefore, up to this point including the first move for the cop, there have been 131 moves for building the box and the first move and 60 moves for the two sides of length 32 in the smaller box for a total of 191 moves.

The robber is in the middle of the 32×12 box with the sides of this area completely covered with walls. The cop will now confine the robber to a 12×12 box. Two of the sides of length 12 have already been built as they are part of the sides of the 32×12 box. The cop will then build the other two sides distance 11 from each other with the robber directly in the middle so that he cannot be distance five from both of them at the same time. These two sides take 20 moves in total to build. Thus, the number of moves is now at 211.

We will again assume the robber to be anywhere in this new box which is 12×12 with the sides of this area completely covered with walls. The cop's strategy is to build a 6×6 box directly in the middle of the 12×12 box. This 6×6 box takes 20 moves to build. Now the robber can be inside the 6×6 box or outside it or he could have been on one of the vertices of the sides of the 6×6 box. Let these be cases A through C, respectively.

Case A: The robber is inside the 6×6 box.

The cop builds a 4×4 cross such that the intersection of the cross occurs in one of the middle vertices. The cop will build his first wall on the intersection vertex and it is guaranteed to be open as there are four possible intersection vertices in the 6×6 box. If the robber allows the cross to be built which means not being on one of those vertices while the cop needed to build a wall there, then he will be captured in at most three moves after that as a 2×2 area is the largest compartment left. This would total 10 moves for capturing him in the 6×6 box. If the robber occupies one of the vertices of the cross the cop needs to build a wall on, then the robber would occupy one of the vertices that is adjacent to the 2×2 compartment that will be left open. But then the cop just builds a wall adjacent to the robber in this 2×2 compartment. No matter what the robber does now, he will be captured in at most 10 moves. Therefore, Case A results in 10 moves.

Case B: The robber is outside the 6×6 box.

Since all the sides are symmetric we will consider the robber to be in the bottom 12×3 part of the 12×12 box. Then the cop can build two vertical walls or one horizontal and one vertical wall distance two and three away from the robber to confine him to a 7×4 box. We can say that it will take a maximum of 8 moves for the cop to capture the robber in that 7×4 area since there are only 10 open vertices in it as the cop can choose the two open vertices after the 8 moves such that they will not be adjacent. Therefore, Case B results in at most 12 moves.

Case C: The robber is on one of the vertices of one of the sides of the 6×6 box after all the other vertices of the sides of this box have a wall built on them.

If the robber is not on one of the corners of this 6×6 box, then the cop will build a wall on the outside of the 6×6 box on the adjacent vertex that is closest to a corner. The cop will keep playing adjacent to the robber on the outside of the 6×6 box until he cannot anymore or the robber moves. If the robber moves inside the 6×6 box then he will be captured in 10 moves by Case A. If he moves outside of the 6×6 box then he will be captured in 12 moves by Case B but the walls being built while he is on one of the sides of the box count towards these 12 moves. Therefore, if the robber does not occupy a corner vertex of the 6×6 box, then he will be captured in at most 13 moves by skipping his turn until he cannot move outside of the 6×6 box at which point he will move into the 6×6 box. This is due to there being exactly three adjacent vertices outside of the 6×6 box. Note that the move where the cop builds the wall to complete the 6×6 box was already counted before.

If the robber is on one of the corners of the 6×6 box then there are five adjacent vertices outside of the 6×6 box. The cop will build a wall on each of these vertices and once he has done so the robber will move into the 6×6 box for the same reasons as above. This results in 15 moves for the cop. Therefore, Case C results in at most 15 moves.

Since Case C results in the most moves for the cop, then that would be the robber's strategy. Hence, we have proven the desired result that the total moves to capture the robber is $211+20+15 = 246$. $\square$

We also can say something about the wall capture time of layered Cartesian grids.

Theorem 5. *For n a positive integer, we have that*

$$W_{c_t}(P_\infty \square P_\infty \square P_n) \leq 44n^2 + 6n + 15.$$

3 Conclusion and Open Problems

The table below gives a summary of the bounds on the wall capture times for the various grids we considered.

Graph	W_{c_t}
H_∞	8
$P_\infty \square P_\infty$	≤ 14
$P_\infty \square P_\infty \square P_n$	$\leq 44n^2 + 6n + 15$
$P_\infty \square P_\infty \square P_\infty$	Open
$\triangle_\infty$	≤ 138
$P_\infty \boxtimes P_\infty$	≤ 246

We now present some open problems. For most of the graph classes we have studied, we only have an upper bound for the wall capture time. To improve on our results, one could find tight lower bounds.

For the three-dimensional infinite Cartesian grid, it is not known whether one cop can capture the robber on this graph. The same question is open for the infinite three-dimensional strong grid.

We have studied the Cartesian and strong graph products. There are 256 possible products, and some of the notable ones that we have not studied are the lexicographic, disjunction, and symmetric difference products; see [5].

References

1. W. Baird, A. Bonato, Meyniel's conjecture on the cop number: a survey, *Journal of Combinatorics* **3** (2012) 225–238.
2. E.R. Berlekamp, J.H. Conway, R.K. Guy, Winning Ways for your Mathematical Plays, vol. 2, Academic Press, New York, 1982.
3. A. Bonato, WHAT IS ... Cop Number? *Notices of the American Mathematical Society* **59** (2012) 1100–1101.
4. A. Bonato, Catch me if you can: Cops and Robbers on graphs, In: *Proceedings of the 6th International Conference on Mathematical and Computational Models (ICMCM'11)*, 2011.
5. A. Bonato, R.J. Nowakowski, *The Game of Cops and Robbers on Graphs*, American Mathematical Society, Providence, Rhode Island, 2011.
6. J.H. Conway, The Angel Problem, in: R. Nowakowski (editor) *Games of No Chance*, volume 29 of MSRI Publications, 1996, 3-12.
7. S. Finbow, G. MacGillivray, The Firefighter problem: a survey of results, directions and questions, *Australasian Journal of Combinatorics* **43** (2009) 57-77.
8. B. Hartnell, Firefighter! An application of domination. Presentation at the *25th Manitoba Conference on Combinatorial Mathematics and Computing*, University of Manitoba, Winnipeg, Canada, 1995.
9. R.J. Nowakowski, P. Winkler, Vertex-to-vertex pursuit in a graph, *Discrete Mathematics* **43** (1983) 235-239.
10. A. Quilliot, Jeux et pointes fixes sur les graphes, *Thèse de 3ème cycle*, Université de Paris VI, 1978, 131-145.