

THE GAME OF FLIPPING COINS

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Abstract

We consider FLIPPING COINS, a partizan version of the impartial game TURNING TURTLES, played on lines of coins. We show the values of this game are numbers, and these are found by first applying a reduction, then decomposing the position into an iterated ordinal sum. This is unusual since moves in the middle of the line do not eliminate the rest of the line. Moreover, when G is decomposed into lines H and K , then $G = (H : K^R)$. This is in contrast to HACKENBUSH STRINGS where $G = (H : K)$.

KEYWORDS: Combinatorial Game Theory, ordinal sum, FLIPPING COINS

1. Introduction

In Winning Ways Volume 3 [2], Berlekamp, Conway, and Guy introduced TURNING TURTLES and considered many variants. Each game involves a finite row of turtles, either on feet or backs, and a move is to

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turn one turtle over onto its back, with the option of flipping a number of other turtles, to the left, each to the opposite of its current state (feet or back). The number depends on the rules of the specific game. The authors moved to playing with coins as playing with turtles is cruel.

These games can be solved using the Sprague-Grundy theory for impartial games [2], but the structure and strategies of some variants are interesting. The strategy for: MOEBIUS (flip up to 5 coins) played with 18 coins, involves *Möbius transformations*; MOGUL (flip up to 7 coins) on 24 coins, involves the *miracle octad generator* developed by R. Curtis in his work on the Mathieu group M_{24} and the Leech lattice, [5, 6]; TERNUPS (flip three equally spaced coins) requires ternary expansions; and TURNING CORNERS, a two-dimensional version where the corners of a rectangle are flipped, needs nim-multiplication.

We consider a simple partizan version of TURNING TURTLES, also played with coins. We give a complete solution and show that it involves ordinal sums. This is somewhat surprising since moves in the middle of the line do not eliminate moves at the end. Compare this with HACKENBUSH STRINGS [2], and DOMINO SHAVE [4].

We will denote heads by 0 and tails by 1. Our partizan version will be played with a line of coins, represented by a 0-1 sequence, $d_1d_2\dots d_n$, where $d_i \in \{0, 1\}$. To this position, we associate the binary number $\sum_{i=1}^n d_i 2^{i-1}$. Left moves by choosing some pair of coins d_i, d_j , $i < j$, where $d_i = d_j = 1$ and flips them over so that both coins are 0s. Right also chooses a pair d_k, d_ℓ , $k < \ell$, with $d_k = 0$ and $d_\ell = 1$, and flips them over. If j is the greatest index such that $d_j = 1$, then d_k , $k > j$, will be deleted. For example,

$$1011 = \{0001, 001, 1 \mid 1101, 111\}.$$

The game eventually ends since the associated binary number decreases with every move. We call this game FLIPPING COINS.

The game is biased to Left. If there are a non-zero even number of 1s in a position, then Left always has a move; that is, she will win. Left also wins any non-trivial position starting with 1. However, there are positions that Right wins. The two-part method to find the outcomes and values of the remaining positions can be applied to all positions. First, apply a modification to the position (unless it is all 1s), which

reduces the number of consecutive 1s to at most three. After this reduction, build an iterated ordinal sum, by successively deleting everything after the third last 1, this deleted position determines the value of the next term in the ordinal sum. As a consequence, the original position is a Right win, if the position remaining at the end is of the form $0 \dots 01$, and the value is given by the ordinal sum.

Necessary background for numbers is in Section 2. Section 3 contains results about outcomes, and it also includes our main results. First, we show that the values are numbers in Theorem 3.2. Next, an algorithm to find the value of a position is presented, and Theorem 3.3 states that the value given by the algorithm is correct.

The actual analysis is in Section 4. It starts by identifying the best moves for both players in Theorem 4.2. This leads directly to the core result Lemma 4.5, which shows that the value of a position is an ordinal sum. The ordinal sum decomposition of G is found as follows. Let G^L be the position after the Left move that removes the rightmost 1s. Let H be the string $G \setminus G^L$; that is, the substring eliminated by Left's move. Let H^R be the result of Right's best move in H . Now, we have that $G = G^L : H^R$. In contrast, the ordinal sums for HACKENBUSH STRINGS and DOMINO SHAVE [4], involve the value of H not H^R .

The proof of Theorem 3.3 is given in Section 4.1. The final section includes a brief discussion of open problems.

Finally we pose a question for the reader, which we answer at the end of Section 4.1: Who wins $0101011111 + 1101100111 + 0110110110111$ and how?

2. Numbers

All the values in this paper are numbers and this section contains all the necessary background to make the paper self-contained. For further details, consult [1, 7]. Positions are written in terms of their options, that is $G = \{G^L \mid G^R\}$.

Definition 2.1. [1, 2, 7] *Let G be a number whose options are numbers and let G^L, G^R be the Left and Right options of the canonical form of*

G .

1. If there is an integer k , $G^L < k < G^R$, or if either G^L or G^R does not exist, then G is the integer, say n , closest to zero that satisfies $G^L < n < G^R$.
2. If both G^L and G^R exist and the previous case does not apply, then $G = \frac{p}{2^q}$, where q is the least non-negative integer such that there is an odd integer p satisfying $G^L < \frac{p}{2^q} < G^R$.

The properties of numbers required for this paper are contained in the next two theorems.

Theorem 2.2. [1, 2, 7] *Let G be a number whose options are numbers and let G^L , G^R be the Left and Right options of the canonical form of G . If G' and G'' are any Left and Right options respectively, then*

$$G' \leq G^L < G < G^R \leq G''.$$

Theorem 2.2 shows that if we know that the property holds, we need to only consider the best move for both players in a number.

We include the following examples.

- (a) $0 = \{ | \} = \{-9 | \} = \{-\frac{1}{2} | \frac{7}{4}\};$
- (b) $-2 = \{ | -1\} = \{-\frac{5}{2} | -\frac{31}{16}\};$
- (c) $1 = \{0 | \} = \{0 | 100\};$
- (d) $\frac{1}{2} = \{0 | 1\} = \{\frac{3}{8} | \frac{17}{32}\}.$

For games G and H , to show that $G \geq H$, we need to show that $G - H \geq 0$. Meaning, we need to show that $G - H$ is a Left win moving second. For more information, see Sections 5.1, 5.8, and 6.3 of the second edition of [1].

Let G and H be games. The *ordinal sum* of G , the *base*, and H , the *exponent*, is

$$G : H = \{G^L, G : H^L | G^R, G : H^R\}.$$

Intuitively, playing in G eliminates H but playing in H does not affect G . For ease of reading, if an ordinal sum is a term in an expression, then we enclose it in brackets.

Note that $x : 0 = x = 0 : x$ since neither player has a move in 0. We demonstrate how to calculate the values of other positions with the following examples.

$$(a) \ 1 : 1 = \{1 \mid \} = 2;$$

$$(b) \ 1 : -1 = \{0 \mid 1\} = \frac{1}{2};$$

$$(c) \ 1 : \frac{1}{2} = \{0, (1 : 0) \mid (1 : 1)\} = \{0, 1 \mid \{1 \mid \}\} = \{1 \mid 2\} = \frac{3}{2};$$

$$(d) \ \frac{1}{2} : 1 = \{0, (\frac{1}{2} : 0) \mid 1\} = \{0, \frac{1}{2} \mid 1\} = \{\frac{1}{2} \mid 1\} = \frac{3}{4};$$

$$(e) \ (1 : -1) : \frac{1}{2} = (\frac{1}{2} : \frac{1}{2}) = \{0, (\frac{1}{2} : 0) \mid 1, (\frac{1}{2} : 1)\} = \{0, \frac{1}{2} \mid 1, \frac{3}{4}\} = \{\frac{1}{2} \mid \frac{3}{4}\} = \frac{5}{8}.$$

Note that in all cases, players prefer to play in the exponent rather than the base. This is true in all cases, but in this paper all the exponents will be positive.

Theorem 2.3. *Let G be a number all of whose options are numbers, and let $H \geq 0$ be a number.*

$$1. \ G^L < (G : H^L) < (G : H) < (G : H^R) < G^R.$$

$$2. \ \text{If } H = 0, \text{ then } G : H = G. \text{ If } H > 0, \text{ then } (G : H) > G.$$

Proof. For item (1), by definition, G^L is a Left option of $G : H^L$ and both are Left options of G . Thus $G^L < (G : H^L) < G$ by Theorem 2.2. The proof is similar for the Right options.

For item (2), if $H = 0$, then $G : 0 = \{G^L \mid G^R\}$ and this is G . Suppose $H > 0$. Let Right move first in $(G : H) - G$. If Right moves to $G^R - G$ or $(G : H) - G^L$, then Left responds to $G^R - G^R$ and $G^L - G^L$ respectively and wins. If Right moves to $(G : H^R) - G$, then since $H^R > 0$, $(G : H^R) - G > 0$ by induction on the options. \square

To prove that all the positions are numbers, we use results from [3]. A set of positions from a ruleset is called a *hereditary closed set of positions of a ruleset* if it is closed under taking options. This game satisfies ruleset properties introduced in [3]. In particular, the properties are called the *F1 property* and the *F2 property* and are defined formally as follows.

Definition 2.4. [3] *Let S be a hereditary closed ruleset. Given a position $G \in S$, the pair $(G^L, G^R) \in G^L \times G^R$ satisfies the F1 property if there is a $G^{RL} \in G^{RL}$ such that $G^{RL} \geq G^L$ or there is a $G^{LR} \in G^{LR}$ such that $G^{LR} \leq G^R$.*

Definition 2.5. [3] *Let S be a hereditary closed ruleset. Given a position $G \in S$, the pair $(G^L, G^R) \in G^L \times G^R$ satisfies the F2 property if there is a $G^{RL} \in G^{RL}$ and there is a $G^{LR} \in G^{LR}$ such that $G^{LR} \leq G^{RL}$.*

As proven in [3], if all positions of the ruleset satisfy one of these properties, the value of the position is a number.

Theorem 2.6. [3] *Let S be a hereditary closed ruleset. Every position in S satisfies either the F1 or the F2 property if and only if every position is a number.*

3. Main results

Before considering the values and associated strategies, we consider the outcomes; that is, we partially answer the question: “Who wins the game?” The full answer requires an analogous analysis to finding the values.

Theorem 3.1. *Let $G = d_1 d_2 \dots d_n$. If $d_1 d_2 \dots d_n$ contains an even number of 1s, or if $d_1 = 1$ and there are least two 1s, then Left wins G .*

Proof. A Right move does not decrease the number of 1s in the position. Thus, if in G , Left has a move, then she still has a move after any Right move in G . Consequently, regardless of d_1 , if there are an even number of 1s in G , it will be Left who reduces the game to all 0s. Similarly, if $d_1 = 1$ and there are an odd number of 1s, Left will eventually reduce G to a position with a single 1; that is, to $d_1 = 1$ and $d_i = 0$ for $i > 1$. In this case, Right has no move and loses. \square

The remaining case, $d_1 = 0$ and an odd number of 1s, is more involved. The analysis of this case is the subject of the remainder of the paper. We first prove the following.

Theorem 3.2. *All FLIPPING COINS positions are numbers.*

Proof. Let G be a FLIPPING COINS position. If only one player has a move, then the game is an integer. Otherwise, let L be the Left move to change (d_i, d_j) from $(1, 1)$ to $(0, 0)$. Let R be the Right move to change (d_k, d_ℓ) from $(0, 1)$ to $(1, 0)$. No other digits are changed. If all four indices are distinct, then both L and R can be played in either order. In this case $G^{LR} = G^{RL}$. Thus, the F2 property holds. If there are only three distinct indices, then two of the bits are ones. If Left moves first, then $d_i = d_j = d_k = 0$. If Right moves first, then there are still two ones remaining after his move. After Left moves, the position becomes $G(0, 0, 0)$ and hence, $G^L = G^{RL}$. The F1 property holds.

There are no more cases since there must be at least three distinct indices. Since every position satisfies either the F1 or F2 property then, by Theorem 2.6, every position is a number. \square

Given a position G , the following algorithm returns a value.

Algorithm: Let G be a FLIPPING COINS position. Let $G_0 = G$.

1. Set $i = 0$.
2. Reductions: Let α and β be strings of coins, and either can be empty.
 - (a) If $G_0 = \alpha 01^{3+j}\beta$, $j \geq 1$, then set $G_0 = \alpha 101^j\beta$.
 - (b) If $G_0 = \alpha 01^3\beta$, and β contains an even number of 1s, then set $G_0 = \alpha 10\beta$.
 - (c) Repeat until neither case applies, then go to Step 3.
3. If G_i is $0^r 1$, $r \geq 0$ or $1^a 0^{p_i} 10^{q_i} 1$, $a \geq 0$ and $p_i + q_i \geq 0$; then go to Step 5.

Otherwise, $G_i = \alpha 01^a 0^{p_i} 10^{q_i} 1$, $p_i + q_i \geq 1$, $a > 0$ and some α . Set

$$\begin{aligned} Q_i &= 0^{p_i} 10^{q_i} 1, \\ G_{i+1} &= \alpha 01^a. \end{aligned}$$

Go to Step 4.

4. Set $i = i + 1$. Go to Step 3.

5. If $G_i = 0^r 1$, then set $v_i = -r$. If $G_i = 1^a 0^{p_i} 10^{q_i} 1$, then set $v_i = \lfloor \frac{a}{2} \rfloor + \frac{1}{2^{2p_i+q_i}}$. Go to Step 6.
6. For j from $i - 1$ down to 0, set $v_j = v_{j+1} : \frac{1}{2^{2p_j+q_j-1}}$.
7. Return the number v_0 .

The algorithm implicitly returns two different results:

1. For Step 3, the substrings, $Q_0, Q_1, \dots, Q_{i-1}, G_i$, partition the reduced version of G ;
2. The value v_0 .

First we illustrate the algorithm with the following example. Let $G = 10011110110110111011110011$. The reductions, applied to the underlined digits, give that:

$$\begin{aligned}
10011110110110111011110011 &= 10011110110110111011110011 \\
&= 100111101101101111010011 \\
&= 10011110110111101010011 \\
&= 100111101111001010011 \\
&= 100111110001010011 \\
&= 1010110001010011.
\end{aligned}$$

Step 3 partitions the last expression into $101(011)(000101)(0011)$ so that the ordinal sum is given by

$$\begin{aligned}
v_0 &= \left(\left(\left(\frac{1}{2} : \frac{1}{2} \right) : \frac{1}{64} \right) : \frac{1}{8} \right) \\
&= \frac{10257}{16348}.
\end{aligned}$$

Now let $H = 01001110110111011101$. The reductions give that:

$$\begin{aligned}
01001110110111011101 &= 01001110110111011101 \\
&= 0100111011110011101 \\
&= 0100111100011101 \\
&= 01010100011101.
\end{aligned}$$

The last expression partitions into 01(0101)(00011)(101) so that

$$\begin{aligned} v_0 &= \left(\left(-1 : \frac{1}{4} \right) : \frac{1}{32} \right) : 1 \\ &= -\frac{893}{1024}. \end{aligned}$$

The next theorem is the main result of the paper.

Theorem 3.3 (Value Theorem). *Let G be a FLIPPING COINS position. If v_0 is the value obtained by the algorithm applied to G , then $G = v_0$.*

In the next section, we derive several results that will be used to prove Theorem 3.3. The proof of Theorem 3.3 will appear in Section 4.1.

4. Best moves and Reductions

The proofs in this section use induction on the options. An alternate but equivalent approach is to regard the techniques as induction on the associated binary number of the positions. The proofs require detailed examination of the positions and we will use notation suitable to the case being considered. Often, a typical position will be written as a combination of generic strings and the substring under consideration. For example, 111011000110101 might be parsed as (11101)(100011)(0101), and written $\alpha 100011\beta$ or more compactly as $\alpha 10^3 1^2 \beta$.

We require several results before being able to prove Theorem 3.3. We begin by proving a simplifying reduction, followed by the best moves for each player, and then the remaining reductions used in the algorithm.

As an immediate consequence of Theorems 3.2 and 2.2 we have the following.

Corollary 4.1. *Let α , β , and γ be arbitrary substrings of coins. We then have that $\alpha 1\beta 0\gamma > \alpha 0\beta 1\gamma$. Moreover, for an integer $r \geq 0$ we have that $\beta 10^r 1 > \beta$.*

Proof. Recall that by Theorem 3.2 all FLIPPING COINS positions are numbers. Thus, Theorem 2.2 applies.

A Right option of $\alpha 0 \beta 1 \gamma$ is $\alpha 1 \beta 0 \gamma$ and so we have that $\alpha 1 \beta 0 \gamma > \alpha 0 \beta 1 \gamma$. Similarly, a Left option of $\beta 1 0^r 1$ is β we have that $\beta 1 0^r 1 > \beta$. \square

Next we prove the best moves for each player. Right wants to play the zero furthest to the right and the 1 adjacent to it. Left wants to play the two ones furthest to the right.

Theorem 4.2. *Let G be a FLIPPING COINS position, where in G , k and n are the greatest indices such that the associated coin is 1. Let i be the greatest index such that $d_i = 0$. Left's best move is to play (d_k, d_n) , and Right's best move is to play (d_i, d_{i+1}) .*

Proof. We prove this theorem by induction on the options. Note that we use the equivalent binary representation of the game position. If there are three or fewer coins, then by exhaustive analysis the theorem is true.

Let G be $d_1 d_2 \dots d_n$. We begin by proving Left's best moves. Let k and n be the two largest indices, where $d_k = d_n = 1$, and let i and j , $i < j$, be two indices with $d_i = d_j = 1$. We use the notation $G(d_i, d_j, d_k, d_n)$ to highlight the salient coins. The claimed best Left move is from $G(1, 1, 1, 1)$ to $G(1, 1, 0, 0)$. This must be compared to any other Left move, represented by moving from $G(1, 1, 1, 1)$ to $G(0, 0, 1, 1)$. That is, we need to show that $G(1, 1, 0, 0) - G(0, 0, 1, 1) \geq 0$. For the moves to be different, at least three of i, j, k, n are distinct. Further, by the choice of the indices, we now know that $i < k$.

We first assume the four indices are distinct. In this case, we have that $i < j < k < n$. By applying Corollary 4.1 twice, we have that

$$G(1, 1, 0, 0) > G(1, 0, 0, 1) > G(0, 0, 1, 1).$$

We may assume then, without loss of generality, that $j = k$ or $j = n$. Now consider $G(d_i, d_k, d_n) = G(1, 1, 1)$. By Corollary 4.1, we have that if $j = k$, $G(1, 0, 0) > G(0, 0, 1)$, and if $j = n$, $G(1, 0, 0) > G(0, 1, 0)$.

We now prove Right's best move. There are more cases to consider. Let k be the largest index such that $d_k = 0$ and $d_{k+1} = 1$ and also let

$i, j, i < j$ be indices with $d_i = 0$ and $d_j = 1$. The claimed best move is d_k, d_{k+1} and this must be compared to the arbitrary Right move d_i, d_j . The original position is

$$G(d_i, d_j, d_k, d_{k+1}) = G(0, 1, 0, 1)$$

and we need to show that $D = G(1, 0, 0, 1) - G(0, 1, 1, 0) \geq 0$. For the moves to be different, there must be at least three distinct indices.

Suppose Right plays in the first summand of D . Note that, by induction, the best moves of Left and Right are known.

(1) First, suppose $j < k$. By induction, Right's best move in the first summand of D , is to $D' = G(1, 0, 1, 0) - G(0, 1, 1, 0)$. Since $i < j$, then $G(1, 0, 1, 0)$ is a Right option of $G(0, 1, 1, 0)$ and thus, D' is non-negative by Corollary 4.1.

(2) If $j = k + 1$, then there are only three distinct indices. The original game is $G(d_i, d_k, d_{k+1}) = G(0, 0, 1)$ and $D = G(1, 0, 0) - G(0, 1, 0)$. Since $G(1, 0, 0)$ is a Right option of $G(0, 1, 0)$, then D is non-negative by Corollary 4.1.

(3) If $j > k + 1$, then the original game, with the indices in increasing order, is $G(d_i, d_k, d_{k+1}, d_j) = G(0, 0, 1, 1)$ and $D = G(1, 0, 1, 0) - G(0, 1, 0, 1)$. Right's best move is to $G(1, 1, 0, 0) - G(0, 1, 0, 1)$ and Left responds to $G(1, 1, 0, 0) - G(1, 1, 0, 0) = 0$, and Left wins.

In all cases, if Right moves in the first summand of D , then Left wins. Next, we consider Right moving in the second summand of $D = G(1, 0, 0, 1) - G(0, 1, 1, 0)$. Note that by the choices of the subscripts, $d_\ell = 1$ if $n \geq \ell \geq k + 1$.

(1) If $n > k + 2$, then Right's best move in the second summand is to change d_{n-1}, d_n from $(1, 1)$ to $(0, 0)$. Left copies this move in the first summand and the resulting difference game is non-negative by induction.

(2) Suppose $n = k + 2$.

(2i) If $j < k + 1$, then $G(d_i, d_j, d_k, d_{k+1}, d_{k+2}) = G(0, 1, 0, 1, 1)$ and $D = G(1, 0, 0, 1, 1) - G(0, 1, 1, 0, 1)$. Right's best move is to $G(1, 0, 0, 1, 1) - G(0, 1, 0, 0, 0)$. Left moves to $G(1, 0, 0, 0, 0) - G(0, 1, 0, 0, 0)$. This is non-negative by Corollary 4.1 and Left wins.

For the next two sub-cases, exactly two 1s will occupy two of the four indexed positions. Since Right is moving in the second summand, he is changing two 1s to two 0s. Thus, Left's best response for each case is to move in the first summand, bringing the game to $G(0, 0, 0, 0) - G(0, 0, 0, 0) = 0$, and she wins. For these cases, we only list the original position. The strategy for both cases is as just described.

(2ii) If $j = k + 1$, then $G(d_i, d_k, d_{k+1}, d_{k+2}) = G(0, 0, 1, 1)$ and $D = G(1, 0, 0, 1) - G(0, 1, 0, 1)$.

(2iii) If $j = k + 2$, then $G(d_i, d_k, d_{k+1}, d_{k+2}) = G(0, 0, 1, 1)$ and $D = G(1, 0, 1, 0) - G(0, 1, 0, 1)$.

(3) Now suppose $n = k + 1$.

(3i) If $j < k + 1$, then let $\ell < k + 1$ be the largest index such that $d_\ell = 1$.

If $j < \ell$, then we have $G(d_i, d_j, d_\ell, d_k, d_{k+1}) = G(0, 1, 1, 0, 1)$ and $D = G(1, 0, 1, 0, 1) - G(0, 1, 1, 1, 0)$. Right's best move is to $G(1, 0, 1, 0, 1) - G(0, 1, 0, 0, 0)$. Left moves to $G(1, 0, 0, 0, 0) - G(0, 1, 0, 0, 0)$ which is non-negative since $G(1, 0, 0, 0, 0)$ is a Right option of $G(0, 1, 0, 0, 0)$.

If $j = \ell$, then $G(d_i, d_j, d_k, d_{k+1}) = G(0, 1, 0, 1)$ and $D = G(1, 0, 0, 1) - G(0, 1, 1, 0)$. Right's best move is to $G(1, 0, 0, 1) - G(0, 0, 0, 0)$. Left moves to $G(0, 0, 0, 0) - G(0, 0, 0, 0) = 0$, and Left wins.

(3ii) If $j = k + 1$, then $G(d_i, d_k, d_{k+1}) = G(0, 0, 1)$ and $D = G(1, 0, 0) - G(0, 1, 0)$. This is non-negative by Corollary 4.1.

In all cases, Left wins D , proving the result. □

Suppose in a position that the coins of the best Right move are

different from those of the best Left move. The next lemma essentially says that the position before and after one move by each player are equal. It is phrased in a way that is useful for reducing the length of the position. For the Algorithm, it suffices to prove the result for β being empty. However, it is useful, certainly for a human, to reduce the length of the position as much as possible.

Lemma 4.3. *If $\alpha \geq 0$, then we have that $\alpha 01111^a = \alpha 101^a$.*

Proof. Let $H = \alpha 01111^a - \alpha 101^a$. We need to show $H = 0$. We have several cases to consider.

(1) If $a \geq 2$, then playing the same move in the other summand are good responses. After two such moves we have either

$$\alpha 01111^{a-2} - \alpha 101^{a-2} = 0, \quad \text{by induction,}$$

or

$$\alpha 10111^a - \alpha 1101^{a-1} = \alpha 1101^{a-1} - \alpha 1101^{a-1} = 0, \quad \text{by induction.}$$

(2) If $a = 1$, then $H = \alpha 01111 - \alpha 101$. The cases are:

- i. Left plays in the first summand to $\alpha 011 - \alpha 101$, then Right moves to $\alpha 101 - \alpha 101 = 0$.
- ii. Right plays in the second summand to $\alpha 01111 - \alpha$, then Left moves to $\alpha 011 - \alpha$. Since $(\alpha 011)^L = \alpha$, then $\alpha 011 > \alpha$.
- iii. Right plays in the first summand to $\alpha 10111 - \alpha 101$, then Left responds to $\alpha 101 - \alpha 101 = 0$.
- iv. Left plays in the second summand to $\alpha 01111 - \alpha 11$, then Right moves to $\alpha 10111 - \alpha 11 = \alpha 11 - \alpha 11 = 0$, by induction.

(3) If $a = 0$, then $H = \alpha 01111 - \alpha 1$. There are several cases to consider.

- i. If Left or Right play in the first summand, then the response is in the first summand giving $\alpha 1 - \alpha 1 = 0$.

- ii. If Left plays in the second summand, then since there is a Left move, then $\alpha = \beta 01^b$, $b \geq 0$. We have that $\beta 01^b 0111 - \beta 01^b 1$ and Left moves to $\beta 01^b 01^3 - \beta 101^b$. Here, Right responds to $\beta 101^{b-1} 01^3 - \beta 101^b$, which by induction is equal to $\beta 101^{b-1} 1 - \beta 101^b = 0$.
- iii. Right plays in the second summand. For a Right move to exist, then $\alpha = \beta 10^a$, $a \geq 0$. Thus, $H = \beta 10^a 0111 - \beta 10^a 1$, and Right moves to $\beta 10^a 0111 - \beta$. Left responds by moving to $\beta 00^a 011 - \beta$. We then have that $(\beta 00^a 011)^L = \beta$; thus, $\beta 00^a 011 > \beta$. Hence, we find that $\beta 00^a 011 - \beta > 0$.

In all cases, the second player wins H thereby proving the result. \square

There are reductions that can be applied to the middle of the position, but extra conditions are needed.

Lemma 4.4. *Let α and β be arbitrary binary strings where either (a) β starts with a 1, or (b) β starts with 0 and has an even number of 1s. We then have that*

$$\alpha 0111\beta = \alpha 10\beta.$$

Proof. Let $H = \alpha 0111\beta - \alpha 10\beta$. We need to show that $H = 0$. We have several cases to consider.

1. If β is empty or $\beta = 1^a$, then $H = 0$ by Lemma 4.3. Therefore, we may assume that β has at least one 1 and one 0.
2. If $\beta = 1\gamma 1$ (β must end in a 1), then the best moves, in both summands, are pairs of coins in β and $-\beta$. If each player copies the opponent's move in the other summand, then this leads to

$$\alpha 0111\beta - \alpha 10\beta \rightarrow \alpha 0111\beta' - \alpha 10\beta'$$

and the latter expression is equal to 0, by induction.

3. If $\beta \neq 1\gamma 1$, then $\beta = 0\gamma 1$ and $\gamma 1$ has at least two 1's. The best moves are in β and $-\beta$ and are the best responses to each other. We then derive that

$$\alpha 0111\beta - \alpha 10\beta \rightarrow \alpha 0111\beta' - \alpha 10\beta' = 0, \quad \text{by induction.}$$

In all cases $H = 0$, and this concludes the proof. \square

In Lemma 4.4, the conditions are necessary. An example is:

$$3/8 = 011101 \neq 1001 = 1/4.$$

Here, β starts with a 0 and has an odd number of 1s.

These reduction lemmas are important in evaluating a position. The reduced positions will end in 011 or 01. By considering the exact end of the string, specifically, if there are at least two 0s (in one special case three 0s), then we can find an ordinal sum decomposition. The decomposition is determined by where the third topmost 1 is situated.

The next result is the start of the ordinal sum decomposition of a position. The exponent is the value of the Right option of the substring being removed.

Lemma 4.5. *If $a \geq 1$ and p and q are non-negative integers such that $p + q \geq 1$, then*

$$\alpha 01^a 0^p 10^q 1 = \alpha 01^a : \frac{1}{2^{2p+q-1}}.$$

Proof. We prove that

$$\alpha 01^a 0^p 10^q 1 - \left(\alpha 01^a : \frac{1}{2^{2p+q-1}} \right) = 0.$$

Note that in Theorem 2.3 we have that playing in the base of $\alpha 01^a : \frac{1}{2^{2p+q-1}}$ is worse than playing in the exponent. We have two cases to consider.

Case 1: Left plays in the first summand and Right in the second. Right has a move in the exponent (moves to 0) since $2p + q - 1 \geq 0$.

In either order, the final position is given by:

$$\alpha 01^a - (\alpha 01^a : 0) = \alpha 01^a - \alpha 01^a = 0.$$

Case 2: Right plays in the first summand and Left plays in the second summand.

Consider

$$\alpha 01^a 0^p 10^q 1 - \left(\alpha 01^a : \frac{1}{2^{2p+q-1}} \right).$$

We have two sub-cases.

(a) Assume $2p + q - 1 \neq 0$. After the two moves we have the position

$$\alpha 01^a 0^r 10^s 1 - \left(\alpha 01^a : \frac{1}{2^{2p+q-2}} \right), \quad \text{where } 2r + s = 2p + q - 1.$$

By induction, we have that

$$\begin{aligned} \alpha 01^a 0^r 10^s 1 &= \alpha 01^a : \frac{1}{2^{2r+s-1}} \\ &= \alpha 01^a : \frac{1}{2^{2p+q-2}}. \end{aligned}$$

$$\text{Thus, } \alpha 01^a 0^r 10^s 1 - \left(\alpha 01^a : \frac{1}{2^{2p+q-2}} \right) = 0.$$

(b) Assume $2p + q - 1 = 0$, that is, $q = 1, p = 0$. The original position is

$$\alpha 01^a 101 - (\alpha 01^a : 1).$$

After the two moves we have the position $\alpha 01^a 11 - \alpha 101^{a-1}$ (note that Left has no move in the exponent). By Lemma 4.3, $\alpha 01^a 11 = \alpha 101^{a-1}$. Hence, we have that $\alpha 01^a 11 - \alpha 101^{a-1} = 0$ and the result follows. \square

The values of the positions not covered by Lemma 4.5 are given next.

Lemma 4.6. *Let $a, p,$ and q be non-negative integers. We then have that*

$$0^p 1 = -p, \quad \text{and} \quad 1^a 0^p 10^q 1 = \left\lfloor \frac{a}{2} \right\rfloor + \frac{1}{2^{2p+q}}.$$

Proof. Let $G = 0^p 1$. Left has no moves and Right has p . Note that in 1^a , Left has $\lfloor \frac{a}{2} \rfloor$ moves and Right has none.

Now, let $G = 1^a 0^p 10^q 1$. We proceed by induction on $p + q$. In all cases, Left's move is to 1^a , that is, to $\lfloor \frac{a}{2} \rfloor$. If $p = 0$ and $q = 0$ then $G = 1^a 11$, which has value $\lfloor \frac{a}{2} \rfloor + \frac{1}{2^0} = \lfloor \frac{a}{2} \rfloor + 1$. Assume that $p + q = k$,

$k > 0$. If $q > 0$, then $G = \{\lfloor \frac{a}{2} \rfloor \mid 1^a 0^p 10^{q-1} 1\}$. By induction, we have that

$$G = \left\{ \left\lfloor \frac{a}{2} \right\rfloor \mid \left\lfloor \frac{a}{2} \right\rfloor + \frac{1}{2^{2p+q-1}} \right\} = \left\lfloor \frac{a}{2} \right\rfloor + \frac{1}{2^{2p+q}}.$$

If $q = 0$, then $G = \{\lfloor \frac{a}{2} \rfloor \mid 1^a 0^{p-1} 10^1 1\}$. By induction, we have that

$$G = \left\{ \left\lfloor \frac{a}{2} \right\rfloor \mid \left\lfloor \frac{a}{2} \right\rfloor + \frac{1}{2^{2(p-1)+1}} \right\} = \left\lfloor \frac{a}{2} \right\rfloor + \frac{1}{2^{2p}},$$

and the result follows. \square

4.1. Proof of the Value Theorem

We now have all of the tools to prove Theorem 3.3.

Proof of Theorem 3.3. Let G be a FLIPPING COINS position. Step 2 reduces the string of coins. The reductions in Step 2(a) are those of Lemma 4.3 and Lemma 4.4 part(a). The reductions in Step 2(b) are those of Lemma 4.4 part(b). In all cases, these lemmas show that each new reduced position is equal to G .

In Step 3, we claim $G_i \neq \beta 1^3$ for any β . This is true for $i = 0$ by Lemma 4.3. If $i > 0$, then at each iteration of Step 3, the last two 1s are removed from G_{i-1} . Now, the original reduced position would be $G_0 = \beta 1^3 \gamma$, where γ has an even number of 1s. Lemma 4.4 part(b) would apply eliminating the three consecutive 1s. Now either G_i is one of $0^r 1$, $r \geq 0$ or $1^a 0^{p_i} 10^{q_i} 1$, $a \geq 0$ and $p_i + q_i \geq 0$, or $G_i = \alpha 0 1^a 0^{p_i} 10^{q_i} 1$, $p_i + q_i \geq 1$, $a > 0$. In the latter case, the index is incremented and the algorithm goes back to Step 3.

Step 5 applies when Step 3 no longer applies, i.e., G_i is one of $0^r 1$, $r \geq 0$ or $1^a 0^{p_i} 10^{q_i} 1$, $a \geq 0$ and $p_i + q_i \geq 0$. Now, v_i is the value of G_i , as given in Lemma 4.6.

Lemma 4.5 shows that for each $j < i$, $G_j = G_{j+1} : \frac{1}{2^{2p_j+q_j}}$, the evaluation in Step 6. Thus, the value of G is v_0 , and the theorem follows. \square

The question: “Who wins $0101011111+1101100111+0110110110111$ and how?” from Section 1 can now be answered.

First, we have that

$$\begin{aligned}
0101011111 &= 01011011 = \left(01011 : \frac{1}{2}\right) = \left(\left(01 : \frac{1}{2}\right) : \frac{1}{2}\right) \\
&= \left(\left(-1 : \frac{1}{2}\right) : \frac{1}{2}\right) = -\frac{11}{16} \\
1101100111 &= 1101101 = (1101 : 1) = \left(\frac{1}{2} : 1\right) = \frac{3}{4} \\
0110110110111 &= 0110110111 = 0110111 = 0111 = 0.
\end{aligned}$$

Thus, we have that

$$0101011111 + 1101100111 + 0110110110111 = -\frac{11}{16} + \frac{3}{4} + 0 = \frac{1}{16}.$$

Left's only winning move is to

$$01010111 + 1101100111 + 0110110110111 = -\frac{3}{4} + \frac{3}{4} + 0 = 0.$$

Her best moves in the second position gives a sum of $-\frac{11}{16} + \frac{5}{8} + 0 = -\frac{1}{16}$, and in the third yields $-\frac{11}{16} + \frac{3}{4} - \frac{1}{8} = -\frac{1}{16}$. Left loses both times.

5. Open Problems

Natural variants of FLIPPING COINS involve increasing the number of coins that can be flipped from two to three or more. A brief computer search suggests that the only version where the values are numbers is the game in which Left flips a subsequence of all 1s and Right a subsequence of 0s ended by a 1. We conjecture that a similar ordinal sum structure will arise in these variants. Other variants have values that include *switches*, *tinies*, *minies*, and other three-stop games. However, some variants, when the reduced canonical values are considered, only seem to consist of numbers and switches. A more thorough investigation should shed light on their structures.

Instead of playing on a string, we can play on a directed acyclic graph (or DAG). We know that the impartial game TWINS, played on a DAG,

reduces to a simple multi-heap game. Of course, instead of coins, other objects could be used. In a line of dice, for example, the end dice would be turned to a lower number.

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