Domination parameters in random graphs

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Abstract. We consider domination parameters in random graphs \(G(n, p)\), where \(p\) is a fixed real number in \((0, 1)\). Several domination parameters have been widely studied, and can differ greatly from the domination number in deterministic graphs. We prove that for several well-known domination parameters, with probability tending to 1 as \(n \to \infty\), their asymptotic order is in fact equal to the domination number of \(G(n, p)\). The domination parameters we investigate include the independent, total, and \(k\)-domination numbers.

1 Introduction

Domination is an active subject in graph theory, and has numerous applications to distributed computing [3, 11], the web graph [8], and ad hoc networks [9, 18]. A dominating set \(S\) in a graph \(G\) has the property that each node not in \(S\) is joined to some node of \(S\). The minimum order of a dominating set is the domination number of \(G\), written \(\gamma(G)\). In real-world networks such as the web graph, domination models some task or service that must be provided to the nodes of the network. For time-efficiency, a direction connection is necessary; for cost-efficiency, we take \(S\) as small as possible. In wireless ad hoc networks, an important problem is to choose a set nodes as small as possible to form a backbone that supports routing. So called \(k\)-dominating sets, (where vertices not in in \(S\) are joined to at least \(k\) nodes of \(S\)) were proposed in [9] as a backbone to balance efficiency and fault tolerance. The books [12, 13] supply a comprehensive introduction to theoretical and applied facets of domination in graphs.

Relatively little work has been done on domination in random graph models. The domination number of Erdős, Rényi random graphs \(G(n, p)\) was determined by Wieland and Godbole in [19]. Define a probability space on graphs of a given order \(n \geq 1\) as follows. Fix a node set \(V\) consisting of \(n\) distinct elements, usually taken as \([n] = \{1, 2, \ldots, n\}\), and fix \(p \in (0, 1)\). Define the space of random graphs of order \(n\) with edge probability \(p\), written \(G(n, p)\), with sample space equalling the set of all \(2^{\binom{n}{2}}\) (labelled) graphs with node set \(V\), and

\[
P(G) = p^{|E(G)|} (1 - p)^{\binom{n}{2} - |E(G)|},
\]

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Informally, we may view $G(n, p)$ as graphs with node set $V$, so that two distinct nodes are joined independently with probability $p$. For background on $G(n, p)$ the reader is directed to [2, 4, 14].

The following concentration result was proved in [19]. We say that an event holds asymptotically almost surely (aas) if the probability that it holds tends to 1 as $n$ tends to infinity. We adopt the notation $\mathbb{L}n = \log_{1/(1-p)} n$.

**Theorem 1.** Aas $G \in G(n, p)$ has $\gamma(G)$ equaling one of

$$\lfloor \mathbb{L}n - \mathbb{L}((\mathbb{L}n)(\log n)) \rfloor + 1 \text{ or } \lfloor \mathbb{L}n - \mathbb{L}((\mathbb{L}n)(\log n)) \rfloor + 2.$$

Despite the fact that deterministic graphs can have large domination number (such as a path $P_n$ with $\gamma(G) = \lfloor n/2 \rfloor$ or co-clique $K_n$ with $\gamma(K_n) = n$) Theorem 1 demonstrates that aas $G \in G(n, p)$ has domination number equaling $\mathbb{L}n(1 + o(1)) = \Theta(\log n)$. Apart from the $G(n, p)$ random graph model, domination has been studied in other random graph models, such as in random regular graphs [10, 15] and in preferential attachment models for the web graph [8].

Many variants of the domination number have been studied. For example, a set $S$ is said to be an independent dominating set of $G$ if $S$ is both an independent and a dominating set of $G$ (in other words, $S$ is a maximal independent set). The independent domination number of $G$, written $\gamma_i(G)$, is the minimum order of an independent dominating set of $G$. It is straightforward to see that $\gamma(G) \leq \gamma_i(G)$. However, as proved in [20], for any nonnegative integer $m$, there exists a cubic graph $G$ such that $\gamma_i(G) - \gamma(G) \geq m$. In particular, for $k > 0$ an integer, $G$ has order $82k + 272$, with $m = k + 1$. Hence, the difference between $\gamma_i$ and $\gamma$ can have order $\Theta(n)$ even in cubic graphs.

Our main goal is to demonstrate that in $G(n, p)$ random graphs, asymptotically the order for several domination parameters equals the order of the domination number. Hence, the logarithmic order of domination number is ubiquitous in a wide range of domination parameters. This represents a kind of smoothing of the domination parameters in random graphs, as aas they are approximately equal. From a practical perspective, our work shows that for a wide variety of types of domination, as network size and traffic doubles, on average the order of a dominating set must increase by $\Theta(1)$ to meet demand.

All graphs in this article are finite, undirected, and simple. The complement of a graph is denoted by $\overline{G}$. If $A$ is an event in a probability space, then we write $\mathbb{P}(A)$ for the probability of $A$ in the space. We use the notation $\mathbb{E}(X)$ and $\text{Var}(X)$ for the expected value and variance of a random variable $X$ in the space, respectively. Throughout, all asymptotics are as $n \to \infty$, and $p \in (0, 1)$ is a fixed real number.

## 2 Main result

A graph parameter $f$ is good if the following properties hold.
For $n$ sufficiently large, there is a graph $G$ of order $n$ such that
\[ |f(G) - \gamma(G)| > cn, \]
where $0 < c < 1$ is a constant.

Aas for $G \in G(n, p)$, for every $\varepsilon > 0$,
\[ f(G) = (1 + \varepsilon)\gamma(G)(1 + o(1)). \]
In particular, $f(G) = \Theta(\gamma(G))$.

One example of a good parameter is the cop number of a graph. The game of Cops and Robber is a node pursuit game played on a (possibly infinite) graph $G$ first investigated in [1, 16, 17]. There are two players, a set of $k$ cops $C$, where $k > 0$ is a fixed integer, and the single robber $R$. The cops begin the game by occupying a set of $k$ nodes, and the cops and robber move in alternate rounds. More than one cop is allowed to occupy a node, and the players may pass; that is, remain on their current node. The cops win and the game ends if at least one of the cops can eventually occupy the same node as the robber; otherwise, $R$ wins. The minimum number of cops needed to capture the robber on a graph $G$ is known as the cop number, denoted by $c(G)$. Observe that $c(G) \leq \gamma(G)$ for all graphs $G$, as the cops can win in one round from a dominating set. As $\gamma(P_n) - c(P_n) = [\frac{n}{2}] - 1 > \frac{n}{3}$ for $n \geq 10$, (G1) holds for the cop number. The following concentration result in [6], demonstrates that (G2) holds for the cop number, and so the parameter $c(G)$ is good.

**Theorem 2.** For every fixed real $\varepsilon > 0$, aas for $G \in G(n, p)$
\[ (1 - \varepsilon)Ln \leq c(G) \leq (1 + \varepsilon)Ln. \]

We now recall some commonly studied domination parameters (see [12, 13], for example). A $k$-dominating set $S$ has the property that each node $u \notin S$ is joined to at least $k$ nodes in $S$. For a positive integer $k$, the $k$-domination number $\gamma_k(G)$ is the minimum cardinality of a $k$-dominating set of $G$. In the special case when $k = 1$, $\gamma_k$ is just $\gamma$. A total dominating set $S$ in a graph $G$ is a subset of $V$ satisfying that every $v \in V$ is joined to at least one node in $S$. A set $S \subseteq V$ is said to be a global dominating set if $S$ is a dominating set of $G$ and $\overline{G}$. A set $S \subseteq V$ is said to be a restrained dominating set, if $S$ is a dominating set of $G$ and for each $v \notin S$ there exists $u \notin S$ such that $uv \in E$. The total domination number $\gamma_t(G)$, the restrained domination number $\gamma_R(G)$, and the global domination number $\gamma_g(G)$ are defined analogously.

Similar to the independent domination number discussed in the introduction, each of the domination parameters defined above can differ significantly from the
domination number (and hence satisfy (G1) in the definition of a good parameter). Such examples are well-known (see [12, 13]); for completeness, we give explicit examples here.

It is not hard to see that \( \gamma_R(K_{1,n-1}) - \gamma(K_{1,n-1}) = n - 1 \) and \( \gamma_g(K_n) - \gamma(K_n) = n - 1 \). For total and \( k \)-domination, we define a family \( \mathcal{F} \) of graphs and the graphs \( H_s \). Let \( k \geq 2 \) be an integer. Denote by \( \Omega \) the set of graphs whose maximum degrees are less than or equal to \( k - 2 \). A graph \( G(F_1, \ldots, F_l) \) is obtained from the disjoint union of a single node \( w \) and \( l \) subgraphs \( F_1, \ldots, F_l \in \Omega \) by adding all possible edges \( wv \), where \( l \geq k \) and \( v \in \bigcup_{j=1}^l V(F_j) \). Let \( \mathcal{F} = \{ G(F_1, \ldots, F_l) : F_1, \ldots, F_l \in \Omega \} \).

Let \( H_s \) be the graph of order \( 4s \) obtained from the disjoint union of a path \( P = u_1u_2u_3\ldots u_{3s-2}u_{3s-1}u_{3s} \) and \( K_s \) by adding \( s - 1 \) edges so that the subgraph induced by \( S \) in \( H_s \) is a perfect matching, where \( S = \bigcup_{j=1}^s \{ u_{3j-1} \} \cup V(K_s) \). See Figure 2 for a drawing of \( H_s \).

\[ u_2 u_5 u_{3s-1} \]
\[ v_1 v_2 v_3 v_4 v_5 \]
\[ u_1 u_3 u_4 u_6 u_{3s-2} \]
\[ u_{3s} \]

Fig. 1. The graph \( H_s \).

**Theorem 3.** 1. \( \gamma_t(H_s) - \gamma(H_s) = \frac{n}{4} \).

2. \( \gamma_k(G) - \gamma(G) = n - 2 \) for every \( G \in \mathcal{F} \).

**Proof:** For (1), it is straightforward to see that \( \gamma(H_s) = s \). To see that \( \gamma_t(H_s) = 2s \), we first notice that every total dominating set must contain \( \{ u_2, u_5, \ldots, u_{3s-1} \} \). Moreover, for each \( 1 \leq j \leq s \), one of the nodes \( u_{3j-2}, v_j \) and \( u_{3j} \) must be in a total dominating set. Otherwise, there is a total dominating set containing no neighbours of \( u_{3j-1} \), which is a contradiction. Hence, \( \gamma_t(H_s) \geq 2s \). As \( \{ u_2, \ldots, u_{3s-1} \} \cup \{ v_1, \ldots, v_s \} \) is a total dominating set of \( H_s \), \( \gamma_t(H_s) = 2s \).

For (2), first note that \( \gamma(G) = 1 \). We observe that any \( k \)-dominating set must contain each node of degree at most \( k - 1 \). By the construction, we know that every node in \( G \) except \( w \) has degree at most \( k - 1 \). Hence, \( \gamma_k(G) \geq n - 1 \). Indeed, \( V \setminus \{ w \} \) is a \( k \)-dominating set of \( G \). Thus, \( \gamma_k(G) = n - 1 \). \( \square \)
As $\gamma(G) \leq f(G)$, where $f$ is any one of $\gamma_i, \gamma_k, \gamma_t, \gamma_R$, or $\gamma_g$, to show that these parameters are good we need suitable asymptotic upper bounds in $G(n, p)$. The main result we present is the following.

**Theorem 4.** As for $G \in G(n, p)$, we have the following inequalities.

1. $\gamma_i(G) \leq \lfloor Ln \rfloor$.
2. $\gamma_t(G) \leq \lfloor Ln - L((Ln)(\log n)) \rfloor + 2$.
3. $\gamma_t(G) \leq (1 + \varepsilon)[Ln]$, for all $\varepsilon > 0$.
4. $\gamma_R(G) \leq \lfloor Ln \rfloor$.
5. $\gamma_g(G) \leq \lfloor Ln \rfloor$, when $p > 1/2$.

In particular, each of the parameters $\gamma_i, \gamma_k, \gamma_t, \gamma_R$, and $\gamma_g$ (in the case $p > 1/2$) are good.

Theorem 4 shows that each of the parameters $f$ equalling $\gamma_i, \gamma_k, \gamma_t, \gamma_R$, and $\gamma_g$ concentrate on the domination number.

**Corollary 1.** As for $G \in G(n, p)$,

$$f(G) = \gamma(G)(1 + o(1)),$$

where $f$ is any one of $\gamma_i, \gamma_t, \gamma_R$, or $\gamma_g$.

The strategy of the proof is as follows. We deal with $\gamma_i$ first, since the techniques used are different than in the remaining parameters. We compute the asymptotic expected value of each domination parameter, then analyze the variance of the parameter. The second moment method (see [2], for example) allows us to complete the proof.

We note the following facts from [19]. For $r \geq 1$, let $X_r$ be the number of dominating sets of size $r$. Fix a $r$-set $S_1$. Denote by $S(j)$ the set of $r$-sets which intersect $S_1$ in $j$ elements. Let $I_1$ and $I^j$ be indicator random variables, where the events $I_1 = 1$ and $I^j = 1$ represent that $S_1 \in S$, and $S_j \in S(j)$ are dominating sets, respectively. Let

$$A = \binom{n}{r} \sum_{j=0}^{r-1} \binom{r}{j} \binom{n-r}{r-j} \mathbb{E}(I_1 I^j).$$

In [19], the following lemma was proved.

**Lemma 1.** 1. $\mathbb{E}(X_r) = \binom{n}{r} (1 - (1 - p)^r)^{n-r}$.
2. For $r \geq \lfloor Ln - L((Ln)(\log n)) \rfloor + 2$, $\mathbb{E}(X_r) \to \infty$.
3. For $r \geq \lfloor Ln - L((Ln)(\log n)) \rfloor + 2$,

$$A \leq \mathbb{E}^2(X_r) \left( 1 + 2r(1 - p)^r - \frac{r^2}{n} \right) (1 + o(1)) + rg(1) \binom{n}{r},$$

where

$$g(1) = \frac{2rn^{r-1}}{(r-1)!} \exp \left( n(1-p)^{2r-1} - 2(1-p)^r \right).$$ (1)
3 Independent domination number

In this section, we prove Theorem 4 (1).

**Theorem 5.** Aas for $G \in G(n, p)$,

$$\gamma_i(G) \leq \lfloor L_n \rfloor.$$

To prove Theorem 5, we need the following two lemmas, whose proofs are omitted. For $r \geq 1$, let $X_{r}^I$ be the random variable which denotes the number of independent dominating sets of size $r$.

**Lemma 2.** For all $r \geq 1$

$$\mathbb{E}(X_{r}^I) = \binom{n}{r} (1 - (1 - p)^r)^{n-r} (1 - p)^{\binom{r}{2}}.$$

**Lemma 3.** Let $\lambda \in (\frac{1}{2}, 1)$ be fixed. For $\lfloor L_n \rfloor + 1 \leq r \leq \lfloor 2\lambda L_n \rfloor$, as $n \to \infty$ we have that $\mathbb{E}(X_{r}^I) \to \infty$.

Our final lemma of the section estimates the variance of $X_{r}^I$.

**Lemma 4.** Let $p \in (0, 1)$ and $\lambda \in (\frac{1}{2}, 1)$ be fixed. For $\lfloor L_n \rfloor + 1 \leq r \leq \lfloor 2\lambda L_n \rfloor$,

$$\text{Var}(X_{r}^I) = \mathbb{E}(X_{r}^I)^2 O \left( \frac{(\log n)^4}{n^{1-\lambda}} \right).$$

With these lemmas at our disposal, we may prove the main result.

**Proof of Theorem 4 (1):** We use the second moment method. By Chebychev’s inequality, Lemmas 3 and 4, we have that

$$\mathbb{P}(\gamma_i(G) > r) = \mathbb{P}(X_{r}^I = 0) \leq \mathbb{P}(\left| X_{r}^I - \mathbb{E}(X_{r}^I) \right| \geq \mathbb{E}(X_{r}^I))$$

$$\leq \frac{\text{Var}(X_{r}^I)}{\mathbb{E}^2(X_{r}^I)} = o(1).$$

The rest of the section is devoted to the proof of Lemma 4.

**Proof of Lemma 4:** We denote by $\mathbb{E}\left( (X_{r}^I)^2 \right)$ the expectation of the number of ordered pairs of independent domination sets of size $r$ in $G \in G(n, p)$. The expectation satisfies

$$\mathbb{E}\left( (X_{r}^I)^2 \right) = \sum_j \binom{n}{r} \binom{r}{j} \binom{n-r}{r-j} (1 - (1 - p)^r)^{2(n-2r+j)}$$

$$\times (1 - (1 - p)^{r-j})^{2(r-j)} (1 - p)^{2\binom{j}{2}}. \quad (2)$$
The explanations for the terms in the equation (2) are as follows. The nodes of the first independent dominating set $S_1$ may be chosen in $\binom{n}{r}$ ways. The independent dominating sets $S_1$ and $S_2$ may have $j$ elements in common. These nodes may be chosen in $\binom{r}{j}$ ways. The rest of $r-j$ nodes of $S_2$ may have to be chosen from $V(G)\backslash S_1$, which gives the $\binom{n-r}{r-j}$ term. Every node not in $S_1 \cup S_2$ must be joined to one of $S_1$ and one of $S_2$, and so we obtain the term $(1-(1-p)^r)^{2(n-2r+j)}$. Every node in $S_1 \backslash S_2$ must be joined to one of $S_2 \backslash S_1$, and every node in $S_2 \backslash S_1$ must be joined to one of $S_1 \backslash S_2$, and so we have the term $(1-(1-p)^{r-j})^{2(r-j)}$. Both sets $S_1$ and $S_2$ are independent, which supplies the last term.

Observe that $(1-p)^{r-j} \geq (1-p)^r$. Hence, by (2) and Lemma 2, we have that

$$\mathbb{E}\left((X^t_r)^2\right) \leq \mathbb{E}\left(X^t_r\right) \cdot \frac{1}{\binom{n}{r}} \times \left(\binom{n-r}{r} + r\binom{n-r}{r-1} + \sum_{j=2}^{r} \binom{r}{j} \binom{n-r}{r-j} (1-p)^{-\binom{j}{2}}\right). \quad (3)$$

By the choice of $r$ it may be shown that

$$\frac{1}{\binom{n}{r}} \left(\binom{n-r}{r} + r\binom{n-r}{r-1}\right) = \left(1 - \frac{r^2}{n}\right) \left(1 + O\left(\frac{\log n}{n^2}\right)\right)$$

$$+ \frac{r^2}{n} + O\left(\frac{\log n}{n^2}\right)$$

$$= 1 + O\left(\frac{\log n}{n^2}\right), \quad (4)$$

and

$$\frac{1}{\binom{n}{r}} \sum_{j=2}^{r} \binom{r}{j} \binom{n-r}{r-j} (1-p)^{-\binom{j}{2}} = O\left(\frac{\log n}{n^{1-\lambda}}\right). \quad (5)$$

By (3), (4), and (5) the result follows. \qed

4 The remaining domination parameters

In this section, we investigate the total domination number, the restrained domination number, the $k$-domination number, and global domination number in the random graph $G(n,p)$. As 1-domination is the same as domination, we assume that $k \geq 2$ is an integer throughout the rest of paper.

For $r \geq 1$, denote by $X^t_r$, $X^k_r$, $X^R_r$ and $X^g_r$ the numbers of total dominating sets of size $r$, $k$-dominating sets of size $r$, restrained dominating sets of size $r$, and
global dominating sets of size $r$, respectively. We first show that for each of
the random variables $X$ above, $\mathbb{E}(X) = (1 + o(1))\mathbb{E}(X_r)$ for suitable values of $r$. As
in the proof of Theorem 5 (1) we use the second moment method, by proving that
$\mathbb{E}(X) \to \infty$ as $n \to \infty$, and $\text{Var}(X) = o(\mathbb{E}^2(X))$.

The following result follows from the definitions, and so is omitted.

Lemma 5. 1. $\mathbb{E}(X^t_r) = \binom{n}{r} (1 - (1 - p)^r)^{n-r} (1 - (1 - p)^{r-1})^r$.
2. $\mathbb{E}(X^k_r) = \binom{n}{r} \left(1 - \sum_{j=0}^{k-1} \binom{j}{r} p^j (1 - p)^{r-j}\right)^{n-r}$.
3. $\mathbb{E}(X^R_r) = \binom{n}{r} (1 - (1 - p)^r)^{n-r} (1 - (1 - p)^{n-r-1})^{n-r}$.
4. $\mathbb{E}(X^g_r) = \binom{n}{r} (1 - (1 - p)^r - p^r)^{n-r}$.

Lemma 6. If $X$ is $X^t_r$, $X^k_r$, $X^R_r$, and $X^g_r$, then we have that $\mathbb{E}(X) = (1 + o(1))\mathbb{E}(X_r)$ for the following values of $r$.

1. For $X^t_r$, $r = \lfloor \ln - \ln((\ln n)(\log n)) \rfloor + 2$.
2. For $X^k_r$, $r = (1 + \varepsilon)\lfloor \ln n \rfloor$, for all $\varepsilon > 0$.
3. For $X^R_r$, $r = \lfloor \ln n \rfloor$.
4. For $X^g_r$, $r = \lfloor \ln n \rfloor$ and $p > 0.5$.

Proof. We only prove item 1, as items 2, 3, and 4 can be proved similarly. By
Lemma 5 1 and Lemma 1 1, we have that

$$
\frac{\mathbb{E}(X^t_r)}{\mathbb{E}(X_r)} = (1 - (1 - p)^{r-1})^r.
$$

Then

$$
\lim_{n \to \infty} \frac{\mathbb{E}(X^t_r)}{\mathbb{E}(X_r)} = \lim_{n \to \infty} (1 - (1 - p)^{r-1})^r = 1,
$$

when $r = \lfloor \ln - \ln((\ln n)(\log n)) \rfloor + 2$.

By Lemma 1 2 and Lemma 6, the following result is immediate.

Lemma 7. If $X$ is $X^t_r$, $X^k_r$, $X^R_r$, and $X^g_r$, then we have that $\mathbb{E}(X) \to \infty$ as $n \to \infty$ for the following values of $r$.

1. For $X^t_r$, $r = \lfloor \ln - \ln((\ln n)(\log n)) \rfloor + 2$.
2. For $X^k_r$, $r = (1 + \varepsilon)\lfloor \ln n \rfloor$, for all $\varepsilon > 0$.
3. For $X^R_r$, $r = \lfloor \ln n \rfloor$.
4. For $X^g_r$, $r = \lfloor \ln n \rfloor$, and $p > 0.5$.

We now analyze the variances of the random variables $X^t_r$, $X^k_r$, $X^R_r$, and $X^g_r$.

Lemma 8. If $X$ is $X^t_r$, $X^k_r$, $X^R_r$, and $X^g_r$, then we have that $\text{Var}(X) = o(\mathbb{E}^2(X))$ for the following values of $r$. 

1. For $X^t_r$, $r = \lfloor \ln - \ln((\ln)(\log n)) \rfloor + 2$.
2. For $X^k_r$, $r = (1 + \varepsilon)\lfloor \ln \rfloor$, for all $\varepsilon > 0$.
3. For $X^R_r$, $r = \lfloor \ln \rfloor$.
4. For $X^g_r$, $r = \lfloor \ln \rfloor$, and $p > 0.5$.

Proof of Theorem 4: 2, 3, 4, 5: The proof follows by Lemmas 7 and 8, and Chebychev’s inequality.

We now turn to the proof of Lemma 8.

Proof of Lemma 8: We only prove item 1. The proofs of items 2, 3, and 4 will appear in the full version of the paper.

For $r \geq 1$, recall the random variable $X^t_r$ which denotes the number of total dominating sets of size $r$. For $1 \leq j \leq (\binom{n}{r})$, let $I^j$ be the corresponding indicator random variables. Hence,

$$X^t_r = \sum_{j=1}^{\binom{n}{r}} I^j.$$  

By the linearity of expectation, we have that

$$E((X^t_r)^2) = \sum_{j=1}^{\binom{n}{r}} E(I^j)^2 + 2 \sum_{j \neq i} E(I^i I^j)$$

$$= E(X^t_r) + 2 \sum_{j \neq i} E(I^i I^j).$$  

(6)

We fix a $r$-set $S_1$. For $0 \leq j \leq r - 1$, denote by $S(j)$ the set of $r$-sets which intersect $S_1$ in $j$ elements. Let $I^t_1$ and $I^t_j$ be the indicator random variables, where the events $I^t_1 = 1$ and $I^t_j = 1$ represent that $S_1$ and $S_j \in S(j)$ are total dominating sets, respectively. Then

$$2 \sum_{j \neq i} E(I^t_i I^j) = \binom{n}{r} \sum_{j=0}^{r-1} \binom{r}{j} \binom{n-r}{r-j} E(I^t_1 I^j).$$

Together with (6), we obtain that

$$E((X^t_r)^2) = E(X^t_r) + \binom{n}{r} \sum_{j=0}^{r-1} \binom{r}{j} \binom{n-r}{r-j} E(I^t_1 I^j)$$

$$= E(X^t_r) + A^t,$$  

(7)
where \( A^t = \binom{n}{r} \sum_{j=0}^{r-1} \binom{n-r}{r-j} \mathbb{E}(I_1 I_j) \). As each total dominating set is a dominating set, \( A^t \leq A \). By Lemma 1 (3) and Lemma 6, we therefore have that

\[
A^t \leq \frac{rg(1)}{E^2(X^t_r)} \left( 1 + \frac{2r(1-p)r - \frac{r^2}{n}}{2} \right) (1 + o(1)),
\]

where \( g(1) \) is given in (1).

By (7) and (8) we have that

\[
\frac{Var(X^t_r)}{E^2(X^t_r)} \leq \frac{1}{E(X^t_r)} + \left( 2r(1-p)r - \frac{r^2}{n} \right) (1 + o(1))
\]

\[
+ \frac{rg(1)}{E^2(X^t_r)}.
\]

To show that \( Var(X^t_r) = o(E^2(X^t_r)) \), it suffices by Lemma 7 to show that

\[
\frac{rg(1)}{E^2(X^t_r)} = o(1).
\]

For sufficiently large \( n \) we have that

\[
\frac{rg(1)}{E^2(X^t_r)} = \frac{r2r \frac{n^{r-1}}{(r-1)!} \exp(n((1-p)^{2r-1} - 2(1-p)^r))}{\binom{n}{r} \left( (1 - (1 - p)r)^{n-r} (1 - (1 - p)^r-1)^r \right)^2}
\]

\[
\leq \frac{3r^3}{n} \frac{(1 - 2(1-p)^r + (1-p)^{2r-1})^n}{(1 - (1 - p)^r-1)^{2r}}
\]

\[
\leq \frac{3r^3}{n} \left( 1 + \frac{p(1-p)^{2r-1}}{(1 - (1 - p)^r)^2} \right)^{n-r} \left( 1 + \frac{2p(1-p)^{r-1}}{(1 - (1 - p)^{r-1})^2} \right)^r,
\]

where the first equality follows by (1) and since \( \exp(x) \sim 1 + x \) if \( x \) is close to 0.

Since \( 1 + x \leq \exp(x) \), we obtain that

\[
\frac{rg(1)}{E^2(X^t_r)} \leq \frac{3r^3}{n} \exp \left( \frac{(n-r)p(1-p)^{2r-1}}{(1 - (1 - p)^r)^2} + \frac{2rp(1-p)^{r-1}}{(1 - (1 - p)^{r-1})^2} \right)
\]

\[
\leq \frac{3r^3}{n} \exp \left( p \left( \frac{\ln((\ln n))}{n} \right)^2 \right) = o(1),
\]

as \( r = [\ln n - \ln((\ln n)(\log n))] + 2. \)
5 Conclusion

We determined the asymptotic order of several domination parameters in the $G(n,p)$ random graph, such as the independent domination number, the total domination number, the $k$-domination number, the restrained domination number, and the global domination number. In each case, the parameter can differ from the domination number in deterministic graphs by as much as $\Theta(n)$. However, we proved that for each parameter $f$, $\text{aas } f(G) = \Theta(\gamma(G))$ for $G \in G(n,p)$. Our work is suggestive of the ubiquity of the asymptotic logarithmic order of domination parameters. For real-world network tasks modelled by some type of domination, our results suggest that as network size and traffic doubles, on average the order of a dominating set must increase by $\Theta(1)$ to meet demand.

An important type of domination parameter we have not yet considered are irredundance numbers. A set $S$ of nodes in $G$ is irredundant if for all $x \in S$, $x$ is isolated in the subgraph induced by $S$, or there exists a unique vertex $y$ not in $S$ joined to $S$. The minimum cardinality of a maximal irredundant set is the irredundance number of a graph, written $ir(G)$; the maximum cardinality of a maximal irredundant set is the upper irredundance number, written $IR(G)$. Note that $ir(G) \leq \gamma(G) \leq IR(G)$. We will consider the asymptotic orders of $ir(G)$ and $IR(G)$ for $G \in G(n,p)$ in future work. Another case which we will consider in the sequel is the global domination number if $p \leq \frac{1}{2}$.

In [5], we considered the cop number in the random power law graphs of Chung and Lu; see [7]. An intriguing open problem is to determine asymptotic order of domination parameters in random power law graphs, and in preferential attachments models for the web graph. For example, results of [8] show that graphs generated by a preferential attachment process have fairly large dominating sets which are linear in the order of the graph. We expect a similar behaviour for the domination parameters studied in this paper.

References

6. A. Bonato, G. Hahn, C. Wang, The cop density of a graph, accepted to *Contributions to Discrete Mathematics*.