Burning a graph as a model of social contagion

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Abstract. We introduce a new graph parameter called the burning number, inspired by contact processes on graphs such as graph bootstrap percolation, and graph searching paradigms such as Firefighter. The burning number measures the speed of the spread of contagion in a graph; the lower the burning number, the faster the contagion spreads. We provide a number of properties of the burning number, including characterizations and bounds. The burning number is computed for several graph classes, and is derived for the graphs generated by the Iterated Local Transitivity model for social networks.

1 Introduction

The spread of social influence is an active topic in social network analysis; see, for example, [3, 8, 13, 14, 18, 19]. A recent study on the spread of emotional contagion in Facebook [16] has highlighted the fact that the underlying network is an essential factor; in particular, in-person interaction and nonverbal cues are not necessary for the spread of the contagion. Hence, agents in the network spread the contagion to their friends or followers, and the contagion propagates over time. If the goal was to minimize the time it took for the contagion to reach the entire network, then which agents would you target with the contagion, and in which order?

As a simplified, deterministic approach to these questions, we consider a new approach involving a graph process which we call burning. Burning is inspired by graph theoretic processes like Firefighting [4, 7, 10], graph cleaning [1], and graph bootstrap percolation [2]. Graph burning may be viewed as a one-player game, where the player attempts to burn all the nodes as quickly as possible. There are discrete time-steps or rounds. Each node is either burned or unburned; if a node is burned, then it remains in that state until the end of the game. Every round, the player chooses a node to burn. Once a node is burned in round $t$, in round $t + 1$, each of its unburned neighbours becomes burned. In every round,
the player chooses one additional unburned node to burn (if such a node is available). The game ends when all nodes are burned. The burning number of a graph $G$, written by $b(G)$, is the minimum number of rounds needed for the game to end. For example, it is straightforward to see that $b(K_n) = 2$. However, even for a relatively simple graph such as the path $P_n$ on $n$ nodes, computing the burning number is more complex; in fact, $b(P_n) = \lceil n^{1/2} \rceil$ as stated below in Theorem 3 (and proven in [6]).

Burning may be viewed as a simplified model for the spread of social contagion in a social network such as Facebook or Twitter. The lower the value of $b(G)$, the easier it is to spread such contagion in the graph $G$. Suppose that in the process of burning a graph $G$, we eventually burned the whole graph $G$ in $k$ steps, and for each $i$, $1 \leq i \leq k$, we denote the node that we burn in the $i$-th step by $x_i$. We call such a node simply a source of fire. The sequence $(x_1, x_2, \ldots, x_k)$ is called a burning sequence for $G$. With this notation, the burning number of $G$ is the length of a shortest burning sequence for $G$; such a burning sequence is referred to as optimal. For example, for the path $P_4$ with nodes $v_1, v_2, v_3, v_4$, the sequence $(v_2, v_4)$ is an optimal burning sequence. See Figure 1. Note that for a $G$ with at least two nodes, we have that $b(G) \geq 2$.

![Fig. 1. Burning the path $P_4$. The open circles represent burned nodes.](image)

The goal of the current paper is to introduce the burning number and explore its core properties. A characterization of burning number via a decomposition into trees is given in Theorem 1. As proven in [6], computing the burning number of a graph is $NP$-complete, even for planar, disconnected, or bipartite graphs. As such, we provide sharp bounds on the burning number for connected graphs, which are useful in many cases when computing the burning number. See Theorem 2.2 for bounds on the burning number. We compute the burning number on the Iterated Local Transitivity model for social networks (introduced in [5]) and grids; see Theorem 8 and Theorem 9, respectively. In the final section, we summarize our results and present open problem for future work.
2 Properties of the burning number

In this section, we collect a number of results on the burning number, ranging from characterizations, bounds, to computing the burning number on certain kinds of graphs. We first make the following observation. Suppose that \((x_1, x_2, \ldots, x_k)\), where \(k \geq 3\), is a burning sequence for a given graph \(G\). For \(1 \leq i \leq k\), the fire spread from \(x_i\) will burn only all the nodes within distance \(k - i\) from \(x_i\) by the end of the \(k\)-th step. On the other hand, every node \(v \in V(G)\) must be either a source of fire, or burned from at least one of the sources of fire by the end of the \(k\)-th step. In other words, any node of \(G\) that is not a source of fire must be an element of \(N_{k-i}[x_i]\), for some \(1 \leq i \leq k\). Therefore, we can see that \((x_1, x_2, \ldots, x_k)\) forms a burning sequence for \(G\) if and only if the following set equation holds:

\[
N_{k-1}[x_1] \cup N_{k-2}[x_2] \cup \ldots \cup N_0[x_k] = V(G). \tag{1}
\]

2.1 Characterizations of burning number via trees

The following theorem provides an alternative characterization of the burning number. We first need some terminology. If \(G\) is a graph and \(v\) is a node of \(G\), then the eccentricity of \(v\) is defined as \(\max\{d(v, u) : u \in G\}\). The radius of \(G\) is the minimum eccentricity over the set of all nodes in \(G\). The center of \(G\) consists of the nodes in \(G\) with minimum eccentricity. Given a positive integer \(k\), the \(k\)-th closed neighborhood of \(v\) is defined to be the set \(\{u \in V(G) : d(u, v) \leq k\}\) and is denoted by \(N_k[v]\); we denote \(N_1[v]\) simply by \(N[v]\). The depth of a node in a rooted tree is the number of edges in a shortest path from the node to the tree’s root. The height of a rooted tree \(T\) is the greatest depth in \(T\).

A rooted tree partition of \(G\) is a collection of rooted trees which are subgraphs of \(G\), with the property that the node sets of the trees partition \(V(G)\).

**Theorem 1.** Burning a graph \(G\) in \(k\) steps is equivalent to finding a rooted tree partition into \(k\) trees \(T_1, T_2, \ldots, T_k\), with height at most \((k - 1), (k - 2), \ldots, 0\), respectively.

**Proof.** Assume that \((x_1, x_2, \ldots, x_k)\) is the burning sequence. For all \(1 \leq i \leq k\), after \(x_i\) is burned, in each round \(t > i\) those unburned nodes of \(G\) in the \((t - i)\)-neighborhood of \(x_i\) will burn. Let \(v\) be a fixed node of \(G\). The node \(v\) is either a source of fire, is burned by receiving fire directly from some \(x_i\), or via a shortest path of burned nodes from \(x_i\).
Hence, we may define a surjective function $f : V(G) \to \{x_1, x_2, \ldots, x_k\}$, with $f(v) = x_i$ if $v$ receives fire from $x_i$, where $i$ is chosen with the smallest index. Now $\{f^{-1}(x_1), f^{-1}(x_2), \ldots, f^{-1}(x_k)\}$ forms a partition of $V(G)$ such that $G'[f^{-1}(x_i)]$ (the subgraph induced by $f^{-1}(x_i)$) forms a connected subgraph of $G$. Since every node $v$ in $f^{-1}(x_i)$ receives the fire spread from $x_i$ through a shortest path between $x_i$ and $v$, by deleting extra edges in $G[f^{-1}(x_i)]$ we can make a rooted subtree of $G$, called $T_i$ with root $x_i$. Since every node is burned after $k$ steps, the distance between each node on $T_i$ and $x_i$ is at most $k - i$. Therefore, the height of $T_i$ is at most $k - i$.

Conversely, suppose that we have a decomposition of the nodes of $G$ into $k$ rooted subtrees $T_1, T_2, \ldots, T_k$, such that for each $1 \leq i \leq k$, $T_i$ is of height at most $k - i$. Assume that $x_1, x_2, \ldots, x_k$ are the roots of $T_1, T_2, \ldots, T_k$, respectively. Then $(x_1, x_2, \ldots, x_k)$ is a burning sequence for $G$, since the distance between any node in $T_i$ and $x_i$ is at most $k - i$. Thus, after $k$ steps the graph $G$ will be burned.

Figure 2 illustrates Theorem 1. The burning sequence is $(x_1, x_2, x_3)$. We have shown the decomposition of $G$ into subgraphs $T_1, T_2,$ and $T_3$ based on this burning sequence by drawing dashed curves around the corresponding subgraphs. Each node has been indexed by a number corresponding to the step that it is burned.

![Fig. 2. A rooted tree partition.](image)

The following corollary is useful for determining the burning number of a graph, as it reduces the problem of burning a graph to burning its spanning trees. First, for a spanning subgraph $H$ of $G$, it is evident that $b(G) \leq b(H)$ (since every burning sequence for $H$ is also a burning sequence for $G$). However, this inequality may fail for non-spanning sub-
graphs. For example, let $H$ be a path of order 5, and form $G$ by adding a universal node to $H$. Then $b(H) = 3$, but $b(G) = 2$.

**Corollary 1.** For a graph $G$ we have that

$$b(G) = \min\{b(T) : T \text{ is a spanning subtree of } G\}.$$  

**Proof.** By Theorem 1, we assume that $T_1, T_2, \ldots, T_k$ is a rooted tree partition of $G$, where $k = b(G)$, derived from an optimal burning sequence for $G$. If we take $T$ to be a spanning subtree of $G$ obtained by adding edges sequentially between the $T_i$’s which do not induce a cycle in $G$, then $b(T) \leq k = b(G) \leq b(T)$, where the second inequality holds since $T$ is a spanning subgraph of $G$. \hfill \Box

### 2.2 Bounds

A subgraph $H$ of a graph $G$ is called an *isometric subgraph* if for every pair of nodes $u, v$ in $H$, we have that $d_H(u, v) = d_G(u, v)$. For example, a subtree of a tree is an isometric subgraph. As another example, if $G$ is a connected graph and $P$ is a shortest path connecting two nodes of $G$, then $P$ is an isometric subgraph of $G$. The following theorem (with proof omitted) shows that the burning number is monotonic on isometric subgraphs.

**Theorem 2.** For any isometric subgraph $H$ of a graph $G$, we have that $b(H) \leq b(G)$.

The following corollary is an immediate consequence of Theorem 2.

**Corollary 2.** If $T$ is a tree and $H$ is a subtree of $T$, then we have that $b(H) \leq b(T)$.

The burning number of paths is derived in the following result (with proof omitted).

**Theorem 3.** For a path $P_n$ on $n$ nodes, we have that $b(P_n) = \lceil n^{1/2} \rceil$.

We have the following immediate corollaries.

**Corollary 3.** 1. For a cycle $C_n$, we have that $b(C_n) = \lceil n^{1/2} \rceil$.

2. For a graph $G$ of order $n$ with a Hamiltonian (that is, spanning) path, we have that $b(G) \leq \lceil n^{1/2} \rceil$. 
The following theorem gives sharp bounds on the burning number. For \( s \geq 3 \), let \( K_{1,s} \) denotes a star; that is, a complete bipartite graph with parts of order 1 and \( s \). We call a graph obtained by a sequence of subdivisions starting from \( K_{1,s} \) a spider graph. In a spider graph \( G \), any path which connects a leaf to the node with maximum degree is called an arm of \( G \). If all the arms of a spider graph with maximum degree \( s \) are of the same length \( r \), we denote such a spider graph by \( SP(s,r) \).

**Lemma 1.** For any graph \( G \) with radius \( r \) and diameter \( d \), we have that
\[
\lceil (d + 1)^{1/2} \rceil \leq b(G) \leq r + 1.
\]

*Proof.* Assume that \( c \) is a central node of \( G \) with eccentricity \( r \). Since every node in \( G \) is within distance \( r \) from \( c \), the fire will spread to all nodes after \( r + 1 \) steps. Hence, \( r + 1 \) is an upper bound for \( b(G) \).

Now, let \( P \) be a path connecting two nodes \( u \) and \( v \) in \( G \) with \( d(u,v) = d \). Since \( P \) is an isometric subgraph of \( G \), and \(|P| = d + 1\), by Theorem 2 and Theorem 3 we conclude that \( b(G) \geq b(P) = \lceil (d + 1)^{1/2} \rceil \). □

As proven in [6], the lower bound is achieved by paths, and the right side bound is achieved by spider graphs \( SP(r,r) \). Note that when proving \( b(G) \leq r + 1 \) in Theorem 1, we viewed \( G \) as covered by a ball with radius \( r \), with a central node chosen as a center of the ball. Hence, by burning a central node, after \( r + 1 \) steps every node in \( G \) will be burned. A covering of \( G \) is a set of subsets of the nodes of \( G \) whose union is \( V(G) \). We may generalize this idea to the case that there is a covering of \( G \) by a collection of balls with a specified radius.

**Theorem 4.** Let \( \{C_1,C_2,\ldots,C_t\} \) be a covering of the nodes of a graph \( G \), in which each \( C_i \) is a connected subgraph of radius at most \( k \). Then we have that \( b(G) \leq t + k \).

Let \( G \) be a connected graph, and \((x_1,x_2,\ldots,x_k)\) be an optimal burning sequence for \( G \). Then for each pair \( i \) and \( j \), with \( 1 \leq i < j \leq k \), \( d(x_i,x_j) \geq j - i \). Since, otherwise, if \( d(x_i,x_j) = l < j - i \), then \( x_j \) will be burned at stage \( l + i \) (\(< j \)) which is a contradiction. Hence, we have the following.

**Corollary 4.** Suppose that \((x_1,x_2,\ldots,x_k)\) is a burning sequence for a connected graph \( G \). If for some node \( x \in V(G) \setminus \{x_1,\ldots,x_k\} \) and for all \( 1 \leq j \leq k - 1 \), we have that \( N[x] \subseteq N[x_j] \), and for every \( i \neq j \), \( d(x,x_i) \geq |i - j| \), then \((x_1,\ldots,x_{j-1},x,x_{j+1},\ldots,x_k)\) is also a burning sequence for \( G \).
We finish this section by providing some bounds on the burning number in terms of certain domination numbers. A \( k \)-distance dominating set like \( D_k \) for \( G \) is a subset of nodes such that for every node \( u \in V(G) \setminus D_k \), there exists a node \( v \in D_k \), with \( d(u, v) \leq k \). The number of the nodes in a minimum \( k \)-distance dominating set of \( G \) is denoted by \( \gamma_k(G) \) and we call it the \( k \)-distance domination number of \( G \).

We have the following result (proof omitted).

**Theorem 5.** For any graph \( G \) with burning number \( k \) we have, \( \gamma_{k-1}(G) \leq k \).

We now give bounds on the burning number in terms of distance domination numbers.

**Theorem 6.** If \( G \) is a connected graph, then we have that

\[
\frac{1}{2} \min_{i \geq 1} \{ \gamma_i(G) + i \} \leq b(G) \leq \min_{i \geq 1} \{ \gamma_i(G) + i \}.
\]

*Proof.* The upper bound is an immediate corollary of Theorem 4. For the lower bound, let \( k = b(G) \), and let \( (x_1, \ldots, x_k) \) be a burning sequence. Then we have that

\[
V(G) \subseteq N_1[x_1] \cup \ldots \cup N_k[x_k] \\
\subseteq N_k[x_1] \cup \ldots \cup N_k[x_k].
\]

Hence, \( \{x_1, \ldots, x_k\} \) is a \( k \)-distance dominating set of \( G \). Since by Theorem 5 we have that \( \gamma_{k-1}(G) \leq k \), and \( \gamma_{k-1}(G) + (k - 1) \leq 2k - 1 = 2b(G) - 1 \), we derive that \( \min_{i \geq 1} \{ \gamma_i(G) + i \} \leq \gamma_{k-1}(G) + (k - 1) \leq 2b(G) - 1 \).

We have the following fact about the \( k \)-distance domination number of graphs.

**Theorem 7.** [17] If \( G \) is a connected graph of order \( n \) with \( n \geq k + 1 \), then we have that

\[
\gamma_k(G) \leq \frac{n}{k+1}.
\]

Now we use the bound in Theorem 7 for \( k \)-distance domination number which provides another upper bound for the burning number.

**Corollary 5.** If \( G \) is a connected graph of order \( n \), then we have that

\[
b(G) \leq 2n^{1/2} - 1.
\]

We conjecture that for any connected graph \( G \) of order \( n \), \( b(G) \leq \lceil n^{1/2} \rceil \).
3 Burning in the ILT Model

The Iterated Local Transitivity (ILT) model [5], simulates on-line social networks (or OSNs). The central idea behind the ILT model is what sociologists call transitivity: if \( u \) is a friend of \( v \), and \( v \) is a friend of \( w \), then \( u \) is a friend of \( w \). In its simplest form, transitivity gives rise to the notion of cloning, where \( u \) is joined to all of the neighbors of \( v \). In the ILT model, given some initial graph as a starting point, nodes are repeatedly added over time which clone each node, so that the new nodes form an independent set. The only parameter of the model is the initial graph \( G_0 \), which is any fixed finite connected graph. Assume that for a fixed \( t \geq 0 \), the graph \( G_t \) has been constructed. To form \( G_{t+1} \), for each node \( x \in V(G_t) \), add its clone \( x' \), such that \( x' \) is joined to \( x \) and all of its neighbors at time \( t \). Note that the set of new nodes at time \( t+1 \) form an independent set of cardinality \( |V(G_t)| \).

The ILT model shares many properties with OSNs such as low average distance, high clustering coefficient densification, and bad spectral expansion; see [5]. The ILT model has also been studied from the viewpoint of competitive diffusion which is one model of the spread of influence; see [20].

We have the following theorem about the burning number of graphs obtained based on ILT model. Even though the graphs generated by the ILT model grow exponentially in order with \( t \), we see that the burning number of such networks remains constant.

**Theorem 8.** Let \( G_t \) be the graph generated at time \( t \geq 1 \) based on the ILT model with initial graph \( G_0 \). If \( G_0 \) has an optimal burning sequence \((x_1, \ldots, x_k)\) in which \( x_k \) has a neighbor that is burned in the \((k-1)\)-th step, then \( b(G_t) = b(G_0) \). Otherwise, \( b(G_t) = b(G_0) + 1 \).

**Proof.** It is straightforward to see that \( G_0 \) is an isometric subgraph of \( G_t \). Therefore, by Theorem 2, \( b(G_t) \geq b(G_0) \). On the other hand, assume that \((x_1, \ldots, x_k)\) is an optimal burning sequence for \( G_0 \). Since every node \( x' \in V(G_t) \setminus V(G_0) \) is adjacent to a node in \( G_0 \), we have that \((x_1, \ldots, x_k)\) is a burning sequence for the subgraph of \( G_t \) induced by \( V(G_t) \setminus (N_{G_t}[x_k] \setminus N_{G_0}[x_k]) \). Thus, \( b(G_t) \leq b(G_0) + 1 \). Hence, we conclude that always either we have that \( b(G_t) = b(G_0) \), or \( b(G_t) = b(G_0) + 1 \).

Now, suppose that for every optimal burning sequence of \( G_0 \) all the neighbors of \( x_k \) are burned in the \( k \)-th step. We claim that \( b(G_1) = b(G_0) + 1 \). Assume not; that is, \( b(G_1) = b(G_0) \). If \((y_1, y_2, \ldots, y_k)\) is a optimal burning sequence for \( G_1 \), then we have two possibilities; either
\( y_k = x \) or \( y_k = x' \in V(G_1) \setminus V(G_0) \), for some \( x \in V(G_0) \). If the latter holds, then again we have two possibilities, either \( x = y_{k-1} \) or \( x \neq y_{k-1} \).

Note that all the neighbours of \( x \) must be burned either in the \((k - 1)\)-th step or the \(k\)-th step; Otherwise, \( y_k \) is burned before the \(k\)-th step, which is a contradiction. If \( x \neq y_{k-1} \), then when burning \( x \) by the \(k\)-th step, one of the neighbours of \( x \) must be burned in an earlier stage. But then in this case, \((y_1, \ldots, y_{k-1}, x_k)\) forms a optimal burning sequence for \( G_0 \) in which one of the neighbours of \( x \) is burned in the \((k - 1)\)-th step which is a contradiction with the assumption. Thus, \( x = y_{k-1} \).

If all the neighbours of \( x \), including \( y \), are burned in the \((k - 1)\)-th step, then \((y_1, \ldots, y_{k-1}, y, x)\) forms a optimal burning sequence for \( G_0 \). But this is a contradiction with the assumption. If at least one of the neighbours of \( x \) like \( y \) is burned at the \(k\)-th step, then \((y_1, \ldots, y_{k-2}, x, y)\) forms a optimal burning sequence for \( G_0 \), which is again a contradiction with the assumption. Therefore, in this case, \( b(G_1) = b(G_0) \) is impossible, and hence, \( b(G_1) = b(G_0) + 1 \).

Otherwise, suppose that \( b(G_1) = b(G_0) + 1 \), and \((x_1, \ldots, x_k)\) is a optimal burning sequence for \( G_0 \). If \( x_k \) has a neighbor that is burned at stage \( k - 1 \), then \( x'_k \) is also burned at stage \( k \). Therefore, \((x_1, \ldots, x_k)\) is a burning sequence for \( G_1 \), and we have that \( b(G_1) = b(G_0) \), which is a contradiction. Thus, \( b(G_1) = b(G_0) + 1 \), if and only if for every optimal burning sequence of \( G_0 \), say \((x_1, \ldots, x_k)\), all the neighbours of \( x_k \) are burned in stage \( k \). By induction, we can conclude that \( b(G_t) = b(G_0) + 1 \) if and only if for every optimal burning sequence of \( G_0 \), say \((x_1, \ldots, x_k)\), all the neighbours of \( x_k \) are burned in stage \( k \). Since starting from any graph \( G_0 \), for any \( t \geq 1 \), \( b(G_t) = b(G_0) \), or \( b(G_t) = b(G_0) + 1 \), we conclude that \( b(G_t) = b(G_0) \) if and only if for every optimal burning sequence of \( G_0 \), say \((x_1, \ldots, x_k)\) one of the neighbours of \( x_k \) is burned in stage \( k - 1 \).

\( \square \)

4 Cartesian Grids

The Cartesian product of graphs \( G \) and \( H \), written \( G \square H \), has nodes \( V(G) \times V(H) \) with \((u, v)\) adjacent to \((x, y)\) if \( u = x \) and \( vy \in E(H) \) or \( v = y \) and \( ux \in E(G) \). The Cartesian \( m \times n \) grid is \( P_m \square P_n \). We prove the following theorem.

**Theorem 9.** If \( G \) is a Cartesian \( m \times n \) grid \( G \), then we have that

\[
b(G) = \Theta((mn)^{1/3}).
\]
Proof. Assume that $S = (x_1, x_2, \ldots, x_k)$ is a burning sequence for $G$. We can assume that (in the worse possible case) there is no source of fire in the $(k-i)$-th neighborhood of each $x_i$, for $1 \leq i \leq k$. On the other hand, by direct checking, the number of nodes in the $r$-th closed neighborhood of $x$ in $G$ equals

$$|N_r[x]| = |\{y \in G : d(x, y) \leq r\}| = 1 + 4 + \cdots + 4r = 1 + 2r(r + 1).$$

Since $S$ is a burning sequence for $G$, every node in $G$ must be in the $(k-i)$-th neighborhood of a node $x_i$, for some $1 \leq i \leq k$. Thus, by double counting the nodes of $G$ and by (1), we have that

$$mn = |G| \leq |N_{k-1}[x_1]| + |N_{k-2}[x_2]| + \cdots + |N_0[x_k]|$$

$$= k + \sum_{i=1}^{k-1} 2i(i + 1)$$

$$= k + 2 \left[ \sum_{i=1}^{k-1} i^2 + \sum_{i=1}^{k-1} i \right]$$

$$= k + 2 \left[ \frac{k(k-1)(2k-1)}{6} + \frac{k(k-1)}{2} \right]$$

$$= \frac{2k^3 + k}{3}. $$

Since the above inequality holds for all burning sequences, we conclude that $b(G) = \Omega((mn)^{1/3})$.

To show the upper bound, using a layout as shown in Figure 3 we may provide a covering of $G$ by a collection of $t$ closed neighborhoods of radius $k$. Note that the $k$-th neighbourhood of a vertex in a grid is a subset of a “diamond” with diameter $2k + 1$ in the Cartesian grid plane. We compute $t$ as follows:

$$t = \left\lceil \frac{m}{2k+1} \right\rceil \left\lceil \frac{n}{2k+1} \right\rceil + \left( \left\lceil \frac{m}{2k+1} \right\rceil + 1 \right) \left( \left\lceil \frac{n}{2k+1} \right\rceil + 1 \right)$$

$$\leq 2 \left( \left\lceil \frac{m}{2k+1} \right\rceil + 1 \right) \left( \left\lceil \frac{n}{2k+1} \right\rceil + 1 \right)$$
By Theorem 4, we conclude that $b(G) \leq t+k$. By taking $k = (mn)^{1/3}$, we derive that $b(G) = O((mn)^{1/3})$. \hfill \square

5 Conclusions and future work

We introduced a new graph parameter, the burning number of a graph, written $b(G)$. The burning number measures how rapidly social contagion spreads in a given graph. We gave a characterization of the burning number in terms of decompositions into trees, and gave bounds on the burning number which allow us to compute it for a variety of graphs. We determined the asymptotic order of the burning number of grids, and determined the burning number in the Iterated Local Transitive model for social networks.

Several problems remain on the burning number. We conjecture that for a connected graph $G$ of order $n$, $b(G) \leq \lceil n^{1/2} \rceil$. Determining the burning number remains open for many classes of graphs, including trees and disconnected graphs. It remains open to consider the burning number in real-world social networks such as Facebook or LinkedIn. As Theorem 8 suggests, the burning number of on-line social networks is likely of constant order as the network grows over time. We remark that burning number generalizes naturally to directed graphs; one interesting direction is to determine the burning number on Kleinberg’s small world model [15], which adds random directed edges to the Cartesian grid.

A simple variation which leads to complex dynamics is to change the rules for nodes to burn. As in graph bootstrap percolation [2], the rules could be varied so nodes burn only if they are adjacent to at least $r$ burned neighbors, where $r > 1$. We plan on studying this variation in future work.
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