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All countable monoids embed into the monoid of the infinite random graph

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ABSTRACT

We prove that the full transformation monoid on a countably infinite set is isomorphic to a submonoid of End(R), the endomorphism monoid of the infinite random graph R. Consequently, End(R) embeds each countable monoid, satisfies no nontrivial monoid identity, and has an undecidable universal theory.

The infinite random graph, written R, has many remarkable properties which have attracted the attention of several researchers, including graph theorists, logicians, and algebraists. The graph R is the unique (up to isomorphism) countable graph that satisfies the *existentially closed* adjacency property: for all finite disjoint sets of vertices A and B, there is a vertex joined to each vertex of A and not joined nor equal to a vertex of B. An interesting property of R, known to Fraïssé [6] in 1953, is that it is a *universal* graph: each countable graph is isomorphic to an induced subgraph of R. For other properties of R, the reader is directed to the surveys of Cameron [3,4]. All graphs we consider are undirected, countable, and simple. Let $\aleph_0$ denote the cardinality of $\mathbb{N}$. For additional background on graphs and graph homomorphisms, the reader is directed to the excellent text of Hell and Nešetřil [7].

While the automorphism group of R has been thoroughly investigated (see the references in [3,4]), properties of the endomorphism monoid of R have been largely overlooked. In [2], the first two authors studied the monoid End(R), and characterized the properties of its retracts. This monoid was further studied in [1,5].

We prove in this short note that End(R) is universal as a monoid; that is, it contains as a submonoid an isomorphic copy of each countable monoid. To this end, we use a well-known fact that each countable monoid embeds in the *full transformation monoid* $T(X)$, the monoid of all mappings from X to itself under composition, where X is a countably infinite set. Our main result is now as follows.

Theorem 1. *If X is a countable set, then $T(X)$ embeds in* End(R).

The principal idea is to use the universality property of R. We start with a graph G, and then inductively construct a graph R_G containing it, so that $R_G \cong R$. More formally, let G be a fixed countable graph. First, we define G^* by adding a new vertex

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v_S for each finite subset $S \subseteq V$, so that v_S is joined to the vertices belonging to S and no other vertex from G^*. Define a chain of graphs by setting $G_0 = G$ and $G_{n+1} = G_n^*$ for all $n \geqslant 0$. The union of this chain is denoted by R_G; that is,

$$R_G = \bigcup_{n\in\mathbb{N}} G_n.$$

The above is one of the canonical ways of constructing R (see for example [3]), as it is quickly verified that R_G satisfies the existentially closed property. We record these facts in the form of the following lemma.

Lemma 1. *For any countable graph G, the graph R_G is isomorphic to R.*

The key ingredient in our main proof is the fact that any endomorphism of G can be extended to a endomorphism of G^*.

Lemma 2. *If f is an endomorphism of a graph G, then there is an endomorphism f^* of G^* that extends f.*

Proof. If $G = (V, E)$ and $G^* = (V^*, E^*)$, then define $f : V^* \to V^*$ in the following way:

$$f^*(x) = \begin{cases} f(x) & \text{if } x \in V, \\ v_{f(S)} & \text{if } x = v_S \text{ for a finite } S \subseteq V. \end{cases}$$

Clearly, $f^* \restriction V = f$. To see that f^* is a graph homomorphism, since any two vertices $x, y \in V^* \setminus V$ are not joined, we need only to consider the case when $x \in V$ and $y \in V^* \setminus V$. Then $y = v_S$ for a finite $S \subseteq V$ and $x \in S$. However, these assumptions imply $f(x) \in f(S)$ and $f(y) = v_{f(S)}$, showing that $f(x)$ and $f(y)$ are joined, as required. □

Now we exploit the feature of the infinite co-clique $\overline{K_{\aleph_0}}$ that each transformation of its vertex set is a graph endomorphism.

Proof of Theorem 1. Start with $\overline{K_{\aleph_0}}$ as G_0. As already noted, $\mathrm{End}(G_0) \cong T(X)$ (in fact, we may take the vertex set of G_0 as X). By Lemma 1, $R_{G_0} \cong R$, so we identify R with R_{G_0} in what follows.

By Lemma 2 and a straightforward induction, it follows that for any $f \in T(X)$ and for each $n > 0$ there is an endomorphism f_n of the graph G_n such that we have $f_n \restriction V(G_0) = f$ and, moreover, $f_n \restriction V(G_m) = f_m$ for all $0 < m < n$. Define a mapping $\phi : T(X) \to \mathrm{End}(R_{G_0})$ by

$$\phi(f) = \bigcup_{n\in\mathbb{N}} f_n,$$

where $f_0 = f$. This mapping is well defined, since for any $x, y \in V(R_{G_0})$ there is an $n \geqslant 0$ such that $x, y \in V(G_n)$. Thus, if x and y are joined by an edge, so are $(\phi(f))(x) = f_n(x)$ and $(\phi(f))(y) = f_n(y)$, as f_n is an endomorphism of G_n.

It is straightforward to see that ϕ is injective, as $\phi(f) \restriction V(G_0) = f$. Hence, it remains to show that ϕ is a homomorphism of monoids, that is, that

$$\phi(fg) = \phi(f)\phi(g)$$

holds for all $f, g \in T(X)$. We immediately have that ϕ preserves the identity transformation. It is sufficient to prove that $(fg)_n = f_n g_n$ for all $n \geqslant 0$, provided that we have $f_{n+1} = f_n^*$ and $g_{n+1} = g_n^*$ as in the proof of Lemma 2. The required equality is clear for $n = 0$. For the case of $n + 1$, we must show that

$$(fg)_{n+1}(x) = f_{n+1} g_{n+1}(x)$$

for all $x \in V(G_{n+1})$. Bearing in mind the induction hypothesis, we may assume that $x \in V(G_{n+1}) \setminus V(G_n)$. Therefore, $x = v_S$ for a unique finite $S \subseteq V(G_n)$, implying

$$(fg)_{n+1}(x) = v_{(fg)_n(S)} = v_{f_n(g_n(S))} = f_{n+1}(v_{g_n(S)}) = f_{n+1}(g_{n+1}(x)),$$

where the second equality follows by the induction hypothesis. □

Theorem 1 has the following consequences. We refer the reader to Hodges [8] for any terms not explicitly defined.

Corollary 2. *The monoid* $\mathrm{End}(R)$ *does not satisfy any nontrivial monoid identity. In particular,* $\mathrm{End}(R)$ *generates the variety of all monoids.*

Proof. Since every countable monoid embeds into $\mathrm{End}(R)$ by Theorem 1, so does the free monoid on a countable set of generators, written $F(X)$. If there were an equation $s = t$ in the language of monoids that is not a consequence of the associative law, and satisfied by $\mathrm{End}(R)$, then $s = t$ would be satisfied by $F(X)$, which is a contradiction. □

Corollary 3. *The universal theory of* $\mathrm{End}(R)$ *is undecidable.*

Proof. We first observe that the universal theory of End(R) equals the universal theory of all monoids. To see this, note that since every countable monoid embeds into End(R) by Theorem 1, every universal sentence true in End(R) will be true in all countable monoids and, by the Löwenheim–Skolem Theorem (see [8]), in all monoids.

It is well known that the universal theory of monoids (semigroups) is undecidable. This fact follows this by the existence of a semigroup with an undecidable word problem [10,11]. Hence, the universal theory of End(R) is undecidable. □

We note that the monoid End(R) (which has cardinality $2^{\aleph_0}$) does not embed all monoids of cardinality at most $2^{\aleph_0}$. The reason for this is that $T(X)$, where X is countable, does not embed all monoids of cardinality at most $2^{\aleph_0}$, and by Theorem 1, $T(X)$ and End(R) are mutually embeddable. For example, an uncountable direct sum of countable simple groups does not embed into $T(X)$ [9]. We do not know, however, exactly which uncountable monoids embed in End(R).

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