On 2-e.c. graphs, tournaments, and hypergraphs

Anthony Bonato, [a] Kathie Cameron [b]

[a] *Department of Mathematics*
Wilfrid Laurier University
Waterloo, ON
Canada, N2L 3C5
abonato@wlu.ca

[b] *Department of Mathematics*
Wilfrid Laurier University
Waterloo, ON
Canada, N2L 3C5
kcameron@wlu.ca

Abstract

We study adjacency properties in the class of k-colourable graphs, the class of K_m-free graphs, and the class of tournaments. In particular, we investigate the 2-existentially closed (or 2-e.c.) adjacency property in these classes which was studied in the class of graphs in [6], [7] and [8]. For a fixed integer $n \geq 1$, a graph G is called n-existentially closed or n-e.c. if for for every n-element subset S of the vertices, and for every subset T of S, there is a vertex not in S which is joined to every vertex in T and to no vertex in $S \setminus T$. In each of the classes described above, we determine the possible orders of the 2-e.c. members of the class, and give explicit examples for all such orders. Some results on adjacency properties in the class of k-uniform hypergraphs are discussed.

Key words: adjacency property, n-existentially closed, graph, tournament, hypergraph

1 Introduction

A graph is called *n-existentially closed* or *n-e.c.* if it satisfies the following adjacency property: for every n-element subset S of the vertices, and for every subset T of S, there is a vertex not in S which is joined to every vertex of T and to no vertex of $S \setminus T$. Fagin [10] and Blass, Harary [4] proved that almost all finite graphs are n-e.c.. The cases $n = 1, 2$ were studied in [6], [7] and [8]. For $n > 2$ very few explicit examples of n-e.c. graphs are known other than large Paley graphs (see [2]).

We now investigate similar adjacency properties in other classes, such as in various subclasses of graphs, in tournaments, and in hypergraphs. We consider the following general adjacency property of relational structures, first given in [5]. Before we can state it we introduce some terminology.

Let L consist of a single symbol R of arity $r \geq 2$. An *L-structure* A consists of a nonempty set $V(A)$ called the *vertex set*, and a set $R(A)$ which is a set of ordered r-tuples from $V(A)$. The *order* of A is $|V(A)|$. An *embedding* between L-structures $A, B \in \mathcal{K}$ is an injective vertex-mapping such that $(a_1, \ldots, a_r) \in R(A)$ if and only if $(f(a_1), \ldots, f(a_r)) \in R(B)$. An *isomorphism* is a bijective embedding. A is an *induced substructure* of B if $V(A) \subseteq V(B)$ and $R(A)$ is the restriction of $R(B)$ to $V(A)$; we write $A \leq B$.

Definition 1 *Fix an integer $n \geq 1$, and let $\mathcal{K}$ be a class of L-structures that is closed under isomorphisms and taking induced substructures.*

*(1) Let A, B, C be in $\mathcal{K}$ with $B \leq A$ and $B \leq C$. C is **realized in** A **over** B if there is an embedding f of C into A so that f is the identity mapping on B.*
*(2) $A \in \mathcal{K}$ is n-$\mathcal{K}$-**e.c.** if the following conditions hold.*
  (a) A embeds each one element structure in $\mathcal{K}$;
  (b) Let $B \leq A$ with $|V(B)| = n$. If $A \leq D \in \mathcal{K}$ with D finite, then if $B \leq C \leq D$ with $|V(C)| = n + 1$, then C is realized in A over B.

Informally, an n-$\mathcal{K}$-e.c. structure realizes all possible ways of extending a set of n vertices. It is not hard to see that if $\mathcal{K}$ is the class of all graphs, then the above Definition is equivalent to the one given at the beginning of the Introduction.

As studied by model theorists, it often happens that almost all finite structures in a class $\mathcal{K}$ satisfy an adjacency property since some countable universal structure in $\mathcal{K}$ has the finite model property. The best known example of such a structure is the countable random graph which is the unique countable graph that is n-e.c. for all $n \geq 1$. (Similar structures exist in the classes of tournaments and k-uniform hypergraphs; see [9].) For many classes however, even though a countable universal structure S exists, whether S has the finite model property is open. Examples of this sort are the classes of K_m-free graphs; see [5].

In this paper we study 2-e.c. properties in various well-known subclasses of graphs and in the class of tournaments. We note that adjacency properties of tournaments were studied in [3] and in [11], and of K_3-free graphs in [1].

We focus our study on the *spectrum problem* for 2-e.c. structures: determine the orders for which 2-e.c. structures in a class exist, and if they do exist, find explicit examples. Variants of the operation of replication, which proved valuable in [6] and [7], are an important tool in our approach to solving the spectrum problem for 2-e.c. structures in the classes we consider.

2 Main result

The main results of the paper are summarized as follows.

Theorem 2 *For each of the classes of $\mathcal{C}(k)$ of k-colourable graphs, $k \geq 2$, and the classes $\mathcal{K}(m)$ of K_m-free graphs, $m \geq 3$, the spectrum of 2-e.c. structures has been classified, and examples have been found for all orders in the spectrum. In particular,*

(1) There are 2-$\mathcal{C}(2)$-e.c. graphs for all orders ≥ 10, and a unique one of order 10.
(2) For each $k \geq 3$, there are 2-$\mathcal{C}(k)$-e.c. graphs for all orders ≥ 9, and a unique one of order 9.
(3) There are 2-$\mathcal{K}(3)$-e.c. graphs for all orders ≥ 8.
(4) For all $m \geq 4$, there are 2-$\mathcal{K}(m)$-e.c. graphs for all orders ≥ 9, and a unique one of order 9.

Theorem 3 *If $\mathcal{T}$ is the class of tournaments, then 2-$\mathcal{T}$-e.c. tournaments exist for orders ≥ 7 except 8, and there is a unique one of order 7 (the Paley tournament on 7 vertices).*

We also consider n-e.c. properties in the classes $\mathcal{H}(k)$ of k-uniform hypergraphs. In particular, the spectrum of n-$\mathcal{H}(k)$-e.c. k-uniform hypergraphs is determined, for $1 \leq n \leq k-1$.

References

[1] B. Alspach, C.C. Chen, and K. Heinrich, Characterization of a class of triangle-free graphs with a certain adjacency property, *J. Graph Theory* **15** (1991) 375–388.

[2] W. Ananchuen and L. Caccetta, On the adjacency properties of Paley graphs, *Networks* **23** (1993) 227-236.

[3] W. Ananchuen, L. Caccetta, On tournaments with a prescribed property, *Ars Combin.* **36** (1993) 89–96.

[4] A. Blass and F. Harary, Properties of almost all graphs and complexes, *J. Graph Theory* **3** (1979) 225-240.

[5] A. Bonato, Constrained classes closed under unions and n-e.c. structures, to appear in *Ars Combin.*

[6] A. Bonato, K. Cameron, On an adjacency property of almost all graphs, to appear in *Discrete Math.*

[7] A. Bonato, K. Cameron, On 2-e.c. line-critical graphs, to appear in *J. Combin. Math. Combin. Comput.*

[8] L. Caccetta, L., P. Erdős, and K. Vijayan, A property of random graphs, *Ars Combin.* **19** (1985) 287–294.

[9] P.J. Cameron, The random graph, in: *Algorithms and Combinatorics* **14** (eds. R.L. Graham and J. Nešetřil), Springer Verlag, New York (1997) 333–351.

[10] R. Fagin, Probabilities on finite models, *J. Symbolic Logic* **41** (1976) 50–58.

[11] R.L. Graham, J.H. Spencer, A constructive solution to a tournament problem, *Canad. Math. Bull.* **14** (1971) 45–48.