Worked Exercises for Linear Diophantine Equations

Exercise 1. Solve the linear Diophantine equation: $7x - 9y = 3$.

Exercise 2. Find all integers $x$ and $y$ such that: $2173x + 2491y = 53$.

Exercise 3. Find all integers $x$ and $y$ such that: $2173x + 2491y = 159$.

Exercise 4. Show there is no integers solution to: $2173x + 2491y = 210$.

Exercise 5. Solve the linear Diophantine equation: $858x + 253y = 33$.

Exercise 6. Find all integer solutions to: $258x + 147y = 369$.

Exercise 7. Show there are no integers solution to: $155x + 45y = 7$.

Exercise 8. Solve the linear Diophantine equation: $60x + 33y = 9$.

Solutions

Exercise 1. Solve the linear Diophantine equation: $7x - 9y = 3$.

Solution. We find a particular solution of the given equation. Such a solution exists because \( \gcd(7,9) = 1 \) and 3 is divisible by 1. One solution, found by inspection, of the given equation is

\[
    x = 3, \quad y = 2.
\]

We obtain all integer solutions of the given equation: $x = 3 + 9k$, $y = 2 + 7k$, for $k$ an integer.

NOTE: we can use any variable name for $k$ here, such as $t$.

Exercise 2: Find integers $x$ and $y$ such that $2173x + 2491y = 53$.

Solution: From the Euclidean Algorithm:

\[
\begin{align*}
2491 &= 1 \cdot 2173 + 318 \\
2173 &= 6 \cdot 318 + 265 \\
318 &= 1 \cdot 265 + 53 \\
265 &= 5 \cdot 53 + 0
\end{align*}
\]

Thus, \( \gcd(2173, 2491) = 53 \), which divides $c = 53$, so there is a solution to this linear Diophantine equation.
Working the Euclidean Algorithm backwards we find:

\[ 53 = 318 - 1 \cdot 265 \]
\[ = 318 - (2173 - 6 \cdot 318) \]
\[ = 7 \cdot 318 - 1 \cdot 2173 \]
\[ = 7(2491 - 1 \cdot 2173) - 1 \cdot 2173 \]
\[ = 7(2491) - 8(2173) \]

Therefore, a particular solution is \( x = -8 \) and \( y = 7 \). The general solution is

\[ x = -8 + (2491/53)k = -8 + 47k \]
\[ y = 7 - (2173/53)k = 7 - 41k, \text{ where } k \text{ is any integer} \]

**Exercise 3:** Find all integers \( x \) and \( y \) such that \( 2173x + 2491y = 159 \).

Solution: Since \( \gcd(2173, 2491) = 53 \) and 53 divides 159, we know that there is a solution to this linear Diophantine equation.

We note that \( \gcd(2173, 2491) = 53 \) from the previous exercise. Multiplying both sides by 3 we obtain 159. We know from Exercise 2 that

\[ 53 = 7(2491) - 8(2173) \]

So one solution of \( 2173x + 2491y = 159 \) is \( x = -24, y = 21 \). The general solution is therefore

\[ x = -24 + (2491/53)k = -24 + 47k, \]
\[ y = 21 - (2173/53)k = 21 - 41k, \text{ where } k \text{ is any integer} \]

**Exercise 4:** Show there is no integer solution to: \( 2173x + 2491y = 210 \).

Solution: In Exercise 2, we found that \( \gcd(2173, 2491) = 53 \), and since 53 does not divide 210, there is no integer solution.

**Exercise 5.** Solve the linear Diophantine equation \( 858 x + 253 y = 33 \).

Solution. We apply the Euclidean algorithm to find \( \gcd(858, 253) \). We have that

\[ 858 = 253 \cdot 3 + 99 \]
\[ 253 = 99 \cdot 2 + 55 \]
\[ 99 = 55 \cdot 1 + 44 \]
\[ 55 = 44 \cdot 1 + 11 \]
\[ 44 = 11 \cdot 4 \]
So \( \gcd(858, 253) = 11 \). Since \( 11 \mid 33 \), the equation has solutions.

Working the Euclidean algorithm backwards, we find that

\[
11 = 55 \cdot 44 \cdot 1
= 55 - 1 \cdot (99 - 55 \cdot 1)
= 2 \cdot 55 - 99
= 2 \cdot (253 - 99 \cdot 2) - 99
= 2 \cdot 253 - 5 \cdot 99
= 2 \cdot 253 - 5 \cdot (858 - 253 \cdot 3)
= 17 \cdot 253 - 5 \cdot 858.
\]

In our equation, the right hand side is \( c = 33 \), so we have (after multiplying by 3):

\[
33 = 51 \cdot 253 - 15 \cdot 858.
\]

Thus, a particular solution is \( x = -15, y = 51 \).

The general solution of \( 858x + 253y = 33 \) is therefore: \( x = -15 - 253k, y = 51 + 858k \), where \( k \) is an integer.

**Exercise 6.** Find all integer solutions to: \( 258x + 147y = 369 \).

Solution. We use the Euclidean Algorithm to find \( \gcd(147,258) \):

\[
258 = 147 \cdot 1 + 111
147 = 111 \cdot 1 + 36
111 = 36 \cdot 3 + 3
36 = 3 \cdot 12.
\]

So \( \gcd(147,258)=3 \). Since \( 3 \mid 369 \), the equation has integer solutions. Working the Euclidean algorithm backwards, we have that:

\[
3 = 111 - 3 \cdot 36
= 111 - 3(147 - 111) = 4 \cdot 111 - 3 \cdot 147
= 4(258 - 147) - 3 \cdot 147
= 4 \cdot 258 - 7 \cdot 147.
\]

We take \( 258 \cdot 4 + 147 \cdot (-7) = 3 \), and multiply through by 123 as \( 3 \cdot 123 = 369 \).

So one solution is \( x = 492 \) and \( y = -861 \). All other solutions will have the form
\[ X = 492 - \left(\frac{147}{3}\right)k = 492 - 49k \]
\[ y = -861 + \left(\frac{258}{3}\right)k = 86k - 861, \text{ where } k \text{ is an integer.} \]

**Exercise 7.** Show there are no integers solution to: \(155x + 45y = 7\).

Solution. Check that \(\text{gcd}(155, 45) = 5\). As 5 does not divide 7, the equation has no integer solution.

**Exercise 8.** Solve the linear Diophantine equation: \(60x + 33y = 9\).

Solution. By the Euclidean algorithm we find:

\[
egin{align*}
60 &= 1 \cdot 33 + 27 \\
33 &= 1 \cdot 27 + 6 \\
27 &= 4 \cdot 6 + 3 \\
6 &= 2 \cdot 3 + 0.
\end{align*}
\]

We see the last nonzero remainder is 3 so \(\text{gcd}(60, 33) = 3\). As \(3|9\), the equation has integer solutions. Reversing the steps, we find:

\[
\begin{align*}
3 &= 27 - 4 \cdot 6 \\
&= 27 - 4(33 - 27) \\
&= 5 \cdot 27 - 4 \cdot 33 \\
&= 5 \cdot (60 - 33) - 4 \cdot 33 \\
&= 5 \cdot 60 - 9 \cdot 33.
\end{align*}
\]

One solution is then \(x = 15\) and \(y = 27\) (notice we multiplied by 3). All the solutions are given by

\[
\begin{align*}
x &= 15 + \left(\frac{33}{3}\right)k = 15 + 11k, \\
y &= -27 - \left(\frac{60}{3}\right)k = -27 - 20k, \text{ where } k \text{ is an integer.}
\end{align*}
\]