Cops and Scared Robber

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SIAM Conference on Discrete Math
Dalhousie University
June 20, 2012
Cops and Robber

Nowakowski & Winkler and Quilliot (1983) (1 cop)

Aigner & Fromme (1984) (k cops)

Played on a finite, simple graph

Cops have perfect information

Two players alternate moves

On Cops’ move, some subset of cops (possibly empty) can each move to an adjacent vertex

On Robber’s move, the Robber can move to an adjacent vertex or pass
How many Cops are required to catch a Robber?

N&W (1983) Graphs in which one cop can catch a robber are characterized.
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C&M Characterization of graphs in which k cops can win
Cop vs Scared Robber

The Robber always moves so that his distance from the Cop never decreases.

Cop free to move to any adjacent vertex or pass.
Cop vs Scared Robber

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Scared Robber

Single Robber ($R$) vs $k$ Cops ($C_1, C_2, \ldots, C_k$)

Say it's $R$'s move and $R$ is on $v$

Label each vertex in $N[v]$ with a $k$–tuple whose entries are the distances from that vertex to the vertices occupied by $C_1, C_2, \ldots, C_k$, respectively.

Say $v$ has label $(v_1, v_2, \ldots, v_k)$

$R$ may move to vertex $w$ with label $(w_1, w_2, \ldots, w_k)$ if $w_i \geq v_i$ for all $i$. 
2 Cops vs Scared Robber

The Robber always moves so that his distance from each of the two Cops never decreases

Cops free to move to any adjacent vertex or pass
2 Cops vs Scared Robber

The Robber always moves so that his distance from each of the two Cops never decreases.

Cops free to move to any adjacent vertex or pass.
If we change the rules from the original game to the “scared robber” game, the copnumber will either stay the same or go down.

\[ cs(G) \leq c(G) \]

\( cs(G) \) is the minimum number of cops required to catch the robber in the scared game.

\( c(G) \) is the copnumber of the graph
Theorem (N&W, '83):

c(G) = 1 if and only if c(G-v) where v is a corner (pitfall)

A vertex v is a corner in G if there is some vertex d such that $N[v] \subseteq N[d]$
Theorem (N&W, '83):

\[ c(G) = 1 \text{ if and only if } c(G-v) \] where \( v \) is a corner (pitfall)

A vertex \( v \) is a corner in \( G \) if there is some vertex \( d \) such that \( N[v] \subseteq N[d] \)

Theorem:

\[ cs(G) = 1 \text{ if and only if } cs(G-v) \] where \( v \) is a corner (pitfall)
Lemma: \( cs(G) = 1 \) if and only if \( c(G) = 1 \)

Corollary: If \( c(G) \geq 2 \) then \( cs(G) \geq 2 \).

Conclusion: If you play the original game with a single cop, it is never to the robber's advantage to move toward the cop.
Lemma: If $G$ is a graph with diameter 2
\[\text{cs}(G) = c(G) .\]

“Proof”: If a Robber moves closer to one of the cops, he is either moving adjacent to that cop's current position or onto the cop.
Bipartite Graphs

$C_1$ and $C_2$ choose vertices

$C_1$ stays on the same vertex for the entire game

$C_2$ moves along a shortest path between his current position and the robber
Bipartite Graphs

R can now pass or move further away from $C_1$

Every time R passes $C_2$ gets closer.
Bipartite Graphs

R can now pass or move
Every time R passes $C_2$
gets closer.
Bipartite Graphs

R can now pass or move
Every time R moves he gets further from $C_1$
Bipartite Graphs

R can now pass or move.
Every time R moves he gets further from $C_1$. 
Bipartite Graphs

If R is forced onto a vertex at maximal distance from $C_1$ then he can only pass from that point on...
Bipartite Graphs

Lemma: If G is bipartite then $cs(G) \leq 2$

Corollary: If G is bipartite and $cs(G) = 1$ then G is a tree.

Corollary: For bipartite graphs, $c(G) - cs(G)$ can be made arbitrarily large.
General Result

Let $G_{v,d}$ denote the subgraph induced on the vertices of $G$ at distance $d$ from vertex $v$. Then

$$cs(G) \leq \min_{v \in V(G)} \max_{d \geq 1} \max_{H \in G_{v,d}} cs(H') + 1$$

where $H$ is a subgraph of $H'$ such $d_{H'}(u,v) = d_{G}(u,v)$ for all $u,v$ in $H$. 
1-Stay Strategy

We say that the $\text{cs}(G)$ cops have a 1-\textit{stay} strategy on $G$ if they can win with one cop staying on the same vertex for the duration of the game.

Examples of $G$ such that $\text{cs}(G) = 2$ with 1-\textit{stay} strategy

- $G$ bipartite
- $G-v$ copwin for some vertex $v$ \textit{(cycles, $\gamma(G) = 2$)}
Cartesian Product

Denoted $G \square H$

Replace each vertex of $H$ with a copy of $G$

If two vertices are adjacent in $H$, join the “same” vertices in each associated copy of $G$ with an edge.
Lemma: If $G$ is bipartite and $\text{cs}(H)=k$ with a 1-stay strategy, then $\text{cs}(G \Box H) = k$ with a 1-stay strategy.

\[ V(G \Box H) = \{(u, v): u \in V(G), v \in V(H)\} \]

\[ d((u_1, v_1), (u_2, v_2)) = d_G(u_1, u_2) + d_H(v_1, v_2) \]

$(u_1, v_1) \sim (u_2, v_2)$ if $u_1 \sim u_2$ and $v_1 = v_2$, or $u_1 = u_2$ and $v_1 = v_2$
Place $\text{cs}(H)$ cops on vertex $(u,v)$ such that $v$ is the “stay vertex” in $H$.

If $C_1$ stays put, R's moves projected onto $G$ are “scared” moves in $G$.

$\text{cs}(H) - 1$ Cops move together between copies of $H$ according to bipartite strategy on $G$. 
Eventually R and the other Cops are in the same copy of H.

If R stays in this copy of H the cs(H)-1 cops will capture him.

$C_1$ on $(u,v)$ has the same effect as having a cop “staying” on the equivalent vertex in any copy of $H$. 
Eventually R will be forced onto a copy of $H$ at maximal distance from $C_1$

The $\text{cs}(H)-1$ other cops move into this copy and capture R
**Lemma:** If $G$ is bipartite and $\text{cs}(H) = k$ with a 1-stay strategy, then $\text{cs}(G \Box H) = k$ with a 1-stay strategy.

**Corollary:** If $G$ is bipartite, then for any graph $H$, $\text{cs}(G \Box H) \leq \text{cs}(H) + 1$. 
Strong Product of Paths

\[ P_3 \times P_4 \]
Strong Isometric Dimension of $G$

$idim(G) = k$ if $k$ is the minimum integer such that $G$ is an isometric subgraph of the strong product of some $k$ paths.

Isometric subgraph is a distance preserving subgraph

$idim(C_4) = 2$
Strong Isometric Dimension

F&N (2001)

If \( \text{idim}(G) = 2 \) then \( c(G) \leq 2 \)

If \( \text{idim}(G) = 3 \) then \( c(G) \leq \text{diam}(G) + 3 \)
Strong Isometric Dimension

F&N (2001)

If \(\text{idim}(G) = 2\) then \(c(G) \leq 2\)
If \(\text{idim}(G) = 3\) then \(c(G) \leq \text{diam}(G)+3\)

Theorem: If \(\text{idim}(G)=3\) then \(c_s(G)\leq 4\).
Future Investigations

Meyniel's Conjecture:
For any connected graph $G$ on $n$ vertices
$c(n) = O(\sqrt{n})$.

$c(n)$ denotes the maximum copnumber of a connected graph of order $n$. 
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Question: ....... $cs(n) = O(\sqrt{n})$