Cops and Scared Robber

Shannon Fitzpatrick
University of Prince Edward Island

Stephen Finbow
St Francis Xavier University

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Cops and Robber

Nowakowski & Winkler and Quilliot (1983) (1 cop)

Aigner & Fromme (1984) (k cops)

Played on a finite, simple graph

Cops have perfect information

Two players alternate moves

On Cops' move, some subset of cops (possibly empty) can each move to an adjacent vertex

On Robber's move, the Robber can move to an adjacent vertex or pass

How many Cops are required to catch a Robber?

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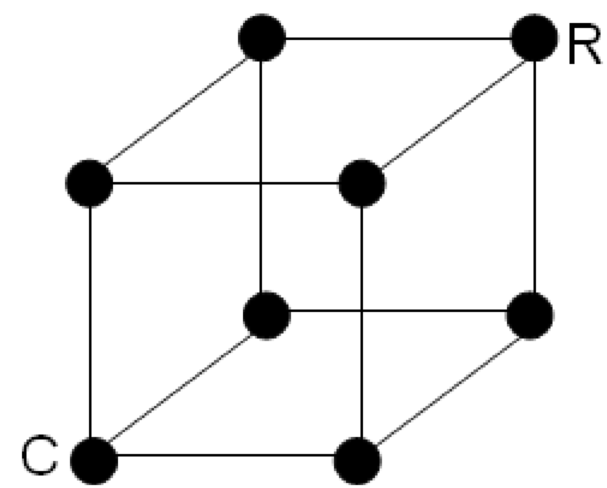
C&M	Characterization of graphs in which k cops can win

Cop vs Scared Robber

The Robber always moves so that his distance from the Cop never decreases

Cop free to move to any adjacent vertex or pass



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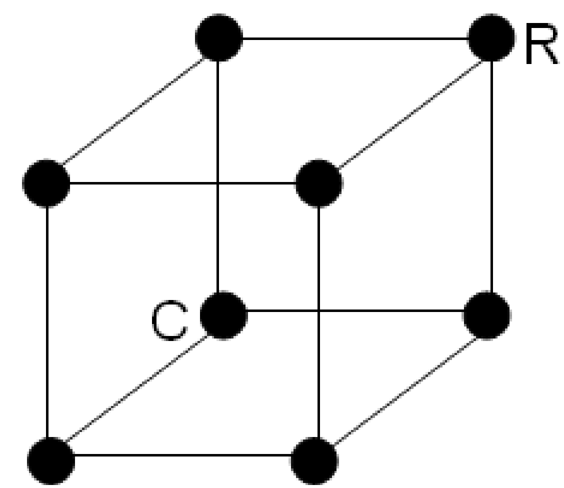

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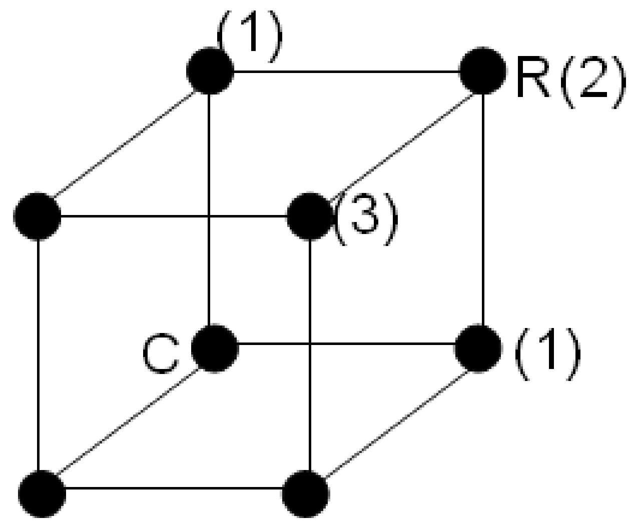


Scared Robber

Single Robber (R) vs k Cops (C_1, C_2, ..., C_k)

Say it's R's move and R is on v

Label each vertex in $N[v]$ with a k–tuple whose entries are the distances from that vertex to the vertices occupied by C_1,C_2,..., C_k, respectively.

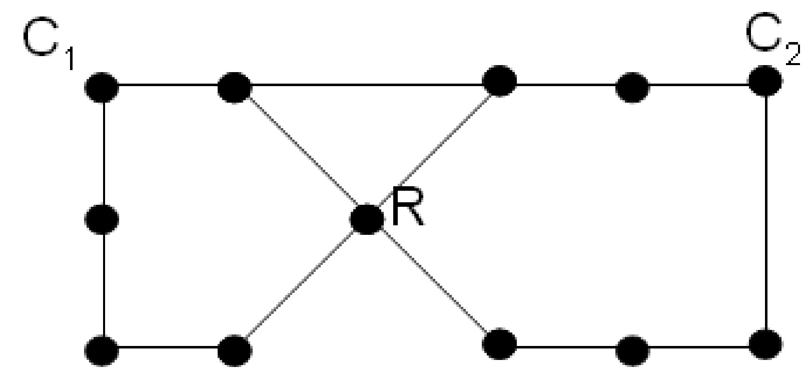
Say v has label (v_1, v_2,...,v_k)

R may move to vertex w with label (w_1, w_2,... w_k) if $w_i \geq v_i$ for **all** i.

2 Cops vs Scared Robber

The Robber always moves so that his distance from each of the two Cops never decreases

Cops free to move to any adjacent vertex or pass



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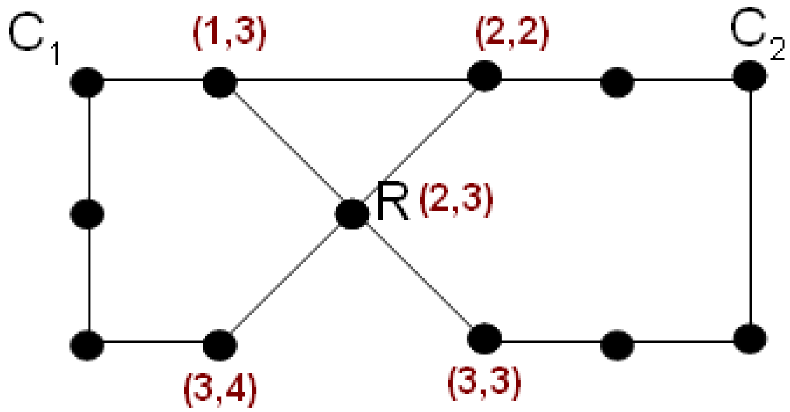

If we change the rules from the original game to the “scared robber” game, the copnumber will either stay the same or go down.

$cs(G) \leq c(G)$

cs(G) is the minimum number of cops required to catch the robber in the scared game.

c(G) is the copnumber of the graph

Theorem (N&W, '83):

c(G) = 1 if and only if c(G-v)where v is a corner (pitfall)

A vertex v is a corner in G if there is some vertex d such that $N[v] \subseteq N[d]$

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Theorem:

cs(G) = 1 if and only if cs(G-v) where v is a corner (pitfall)

Lemma: cs(G) = 1 if and only if c(G) = 1

Corollary: If c(G) ≥ 2 then cs(G) ≥ 2.

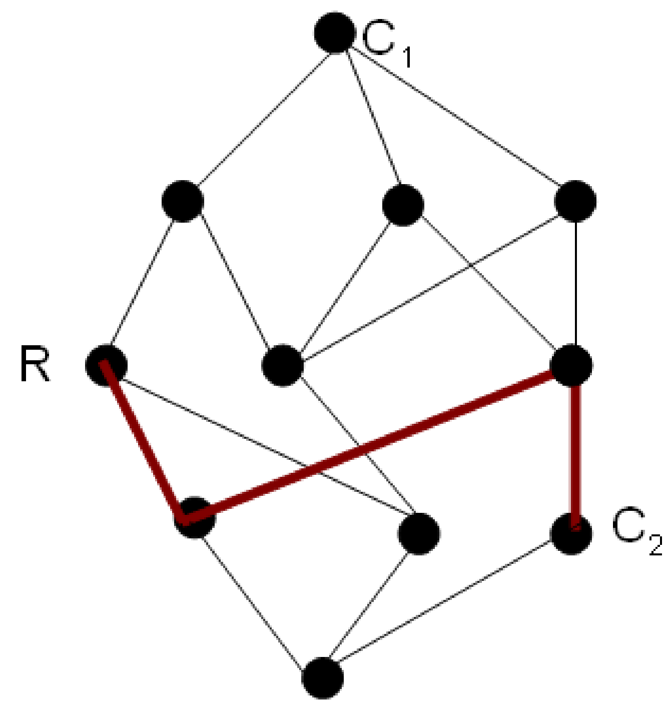
Conclusion: If you play the original game with a single cop, it is never to the robber's advantage to move toward the cop.

Lemma: If G is a graph with diameter 2
cs(G) = c(G) .

"*Proof* ": If a Robber moves closer to one of the cops, he is either moving adjacent to that cop's current position or onto the cop.

Bipartite Graphs

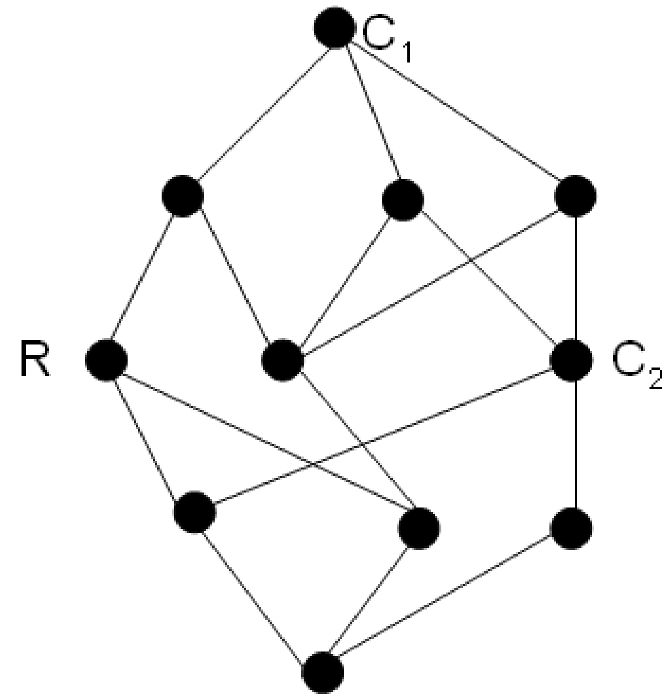
- C_1 and C_2 choose vertices
- C_1 stays on the same vertex for the entire game
- C_2 moves along a shortest path between his current position and the robber



Bipartite Graphs

R can now pass or move further away from C_1

Every time R passes C_2 gets closer.



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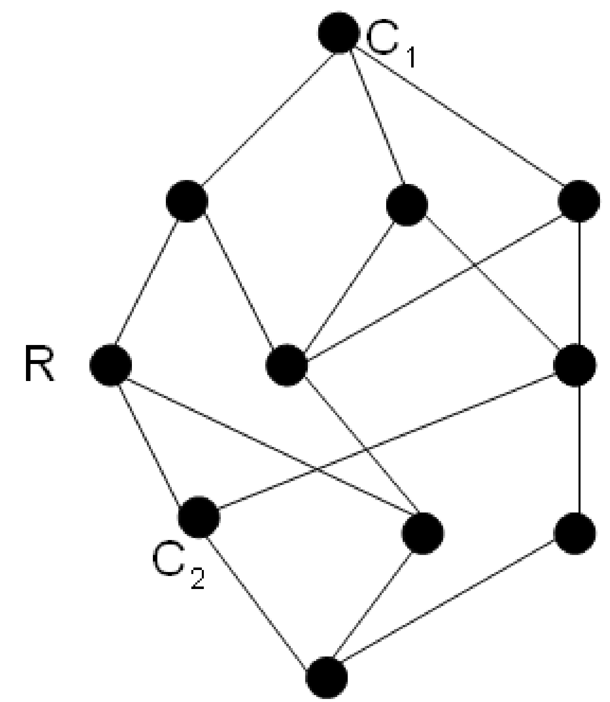

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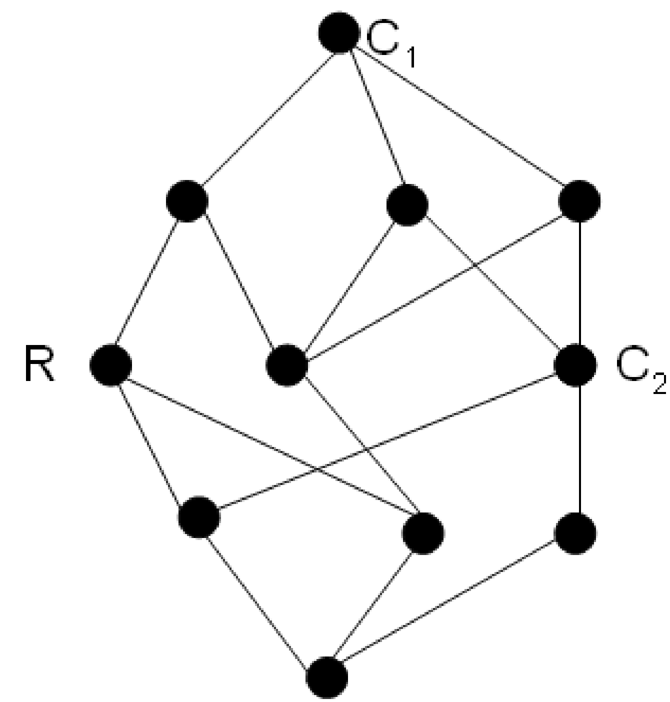

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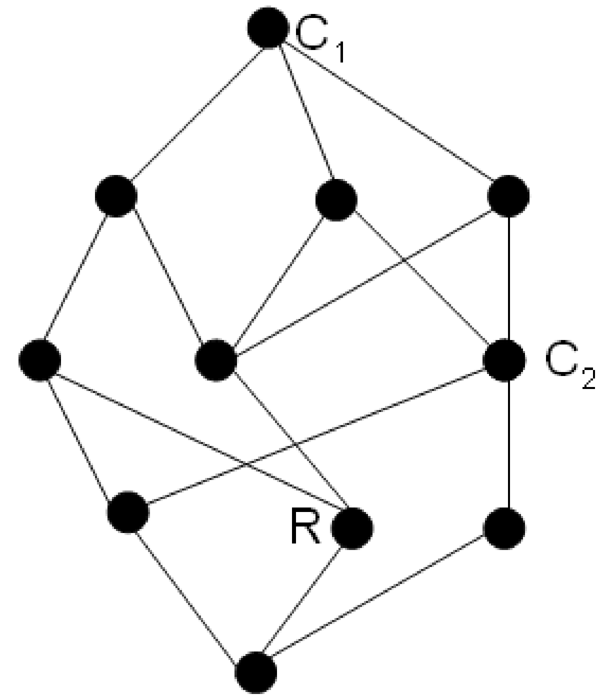
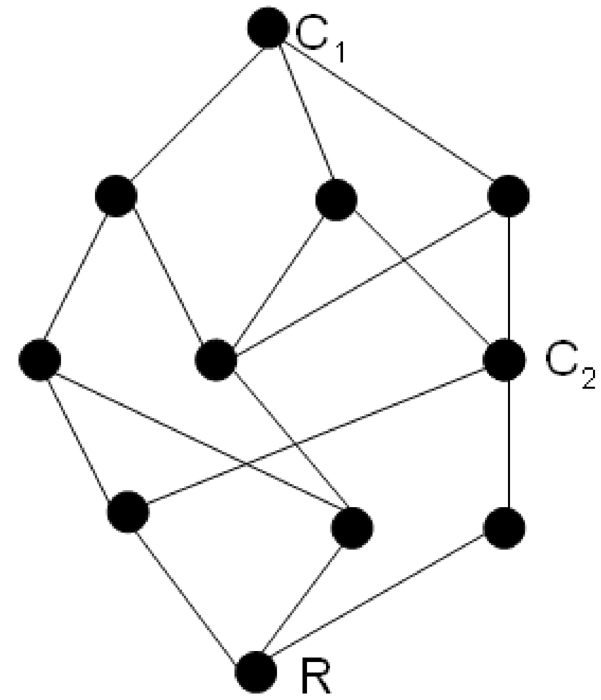


Bipartite Graphs

If R is forced onto a vertex at maximal distance from C_1 then he can only pass from that point on



Bipartite Graphs

Lemma: If G is bipartite then $cs(G) \leq 2$

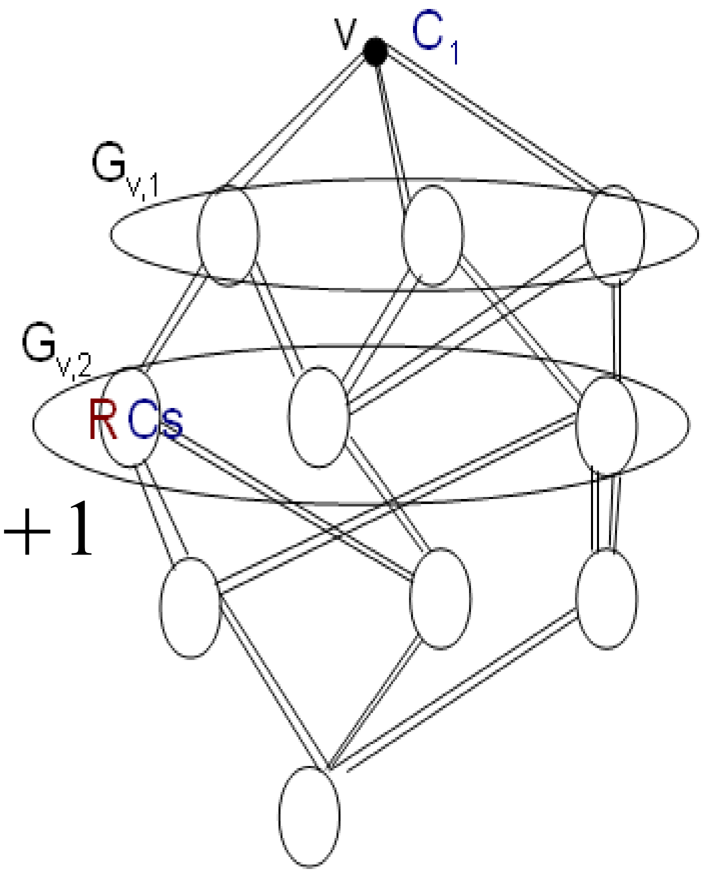
Corollary: If G is bipartite and $cs(G) = 1$ then G is a tree.

Corollary: For bipartite graphs, $c(G)-cs(G)$ can be made arbitrarily large.

General Result

Let $G_{v,d}$ denote the subgraph induced on the vertices of G at distance *d* from vertex *v* . Then

$$cs(G) \leq min_{v \in V(G)} max_{d \geq 1} max_{H \in G_{v,d}} cs(H') + 1$$



where H is a subgraph of H' such

$d_{H'}(u,v) = d_G(u,v)$ for all u,v in H.

1-Stay Strategy

We say that the cs(*G*) cops have a *1-stay strategy* on *G* if they can win with one cop staying on the same vertex for the duration of the game.

Examples of *G* such that cs(*G*) = 2 with 1-stay strategy

G bipartite

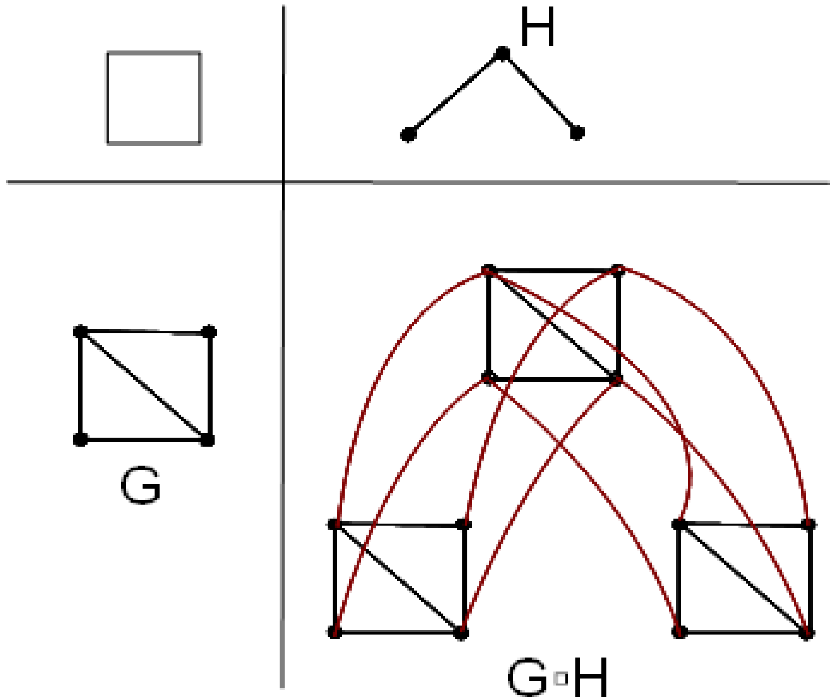
G-v copwin for some vertex *v* (*cycles*, $\gamma(G) = 2$)

Cartesian Product

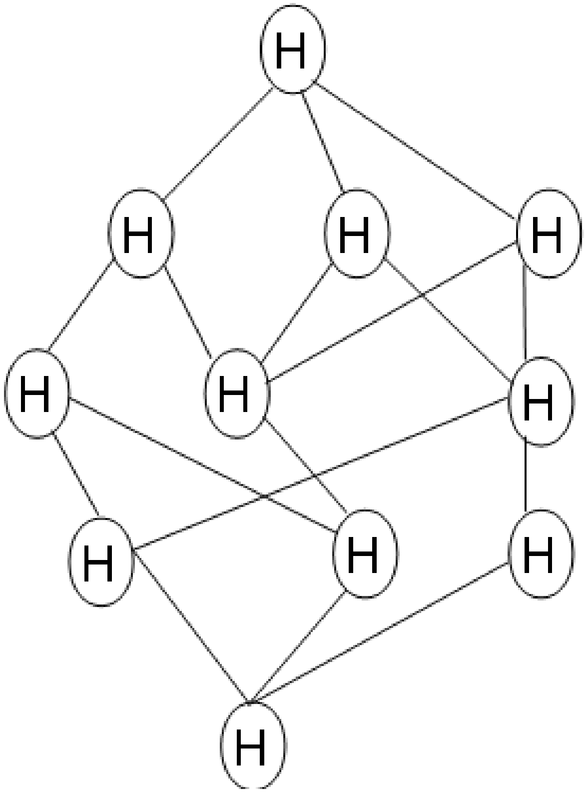
Denoted G□H

Replace each vertex of H with a copy of G

If two vertices are adjacent in H, join the “same” vertices in each associated copy of G with an edge.



Lemma: If G is bipartite and cs(*H*)=*k* with a 1-stay strategy, then cs(G□H) =*k* with a 1-stay strategy.



V(G□H) ={$(u, v): u \in V(G), v \in V(H)$}

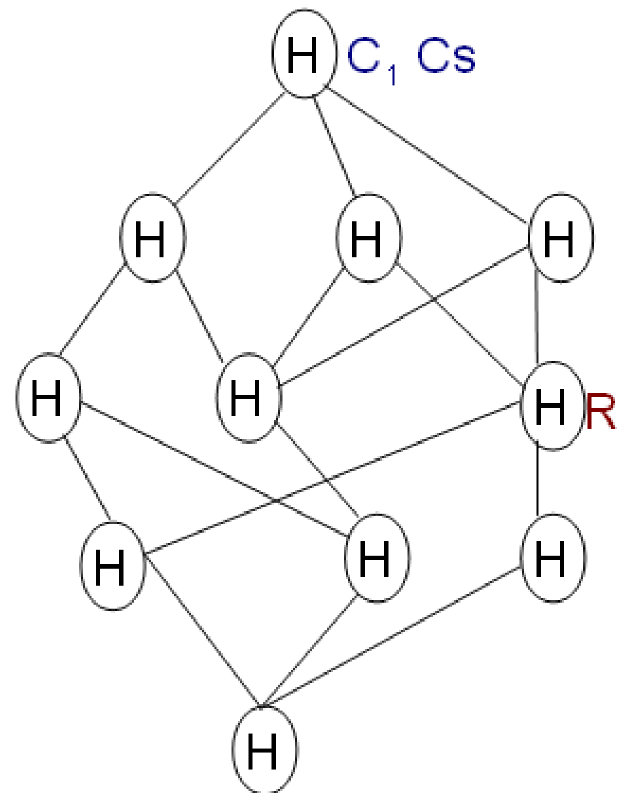
d((u_1,v_1), (u_2,v_2)) = $d_G(u_1,u_2)+d_H(v_1,v_2)$

(u_1,v_1) ~(u_2,v_2) if $u_1 \sim u_2$ and $v_1 = v_2$, *or*
$u_1 = u_2$ and $v_1 = v_2$

Place cs(*H*) cops on vertex (*u*,*v*) such that *v* is the “stay vertex” in *H*.

If C_1 stays put, R's moves projected onto G are “scared” moves in G.

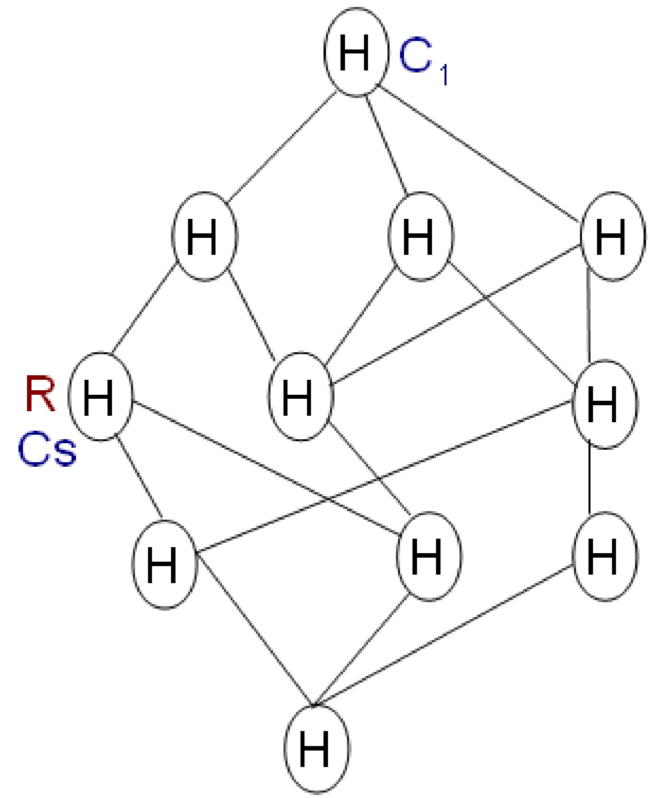
cs(H)-1 Cops move together between copies of H according to bipartite strategy on *G*.



Eventually R and the other Cops are in the same copy of H.

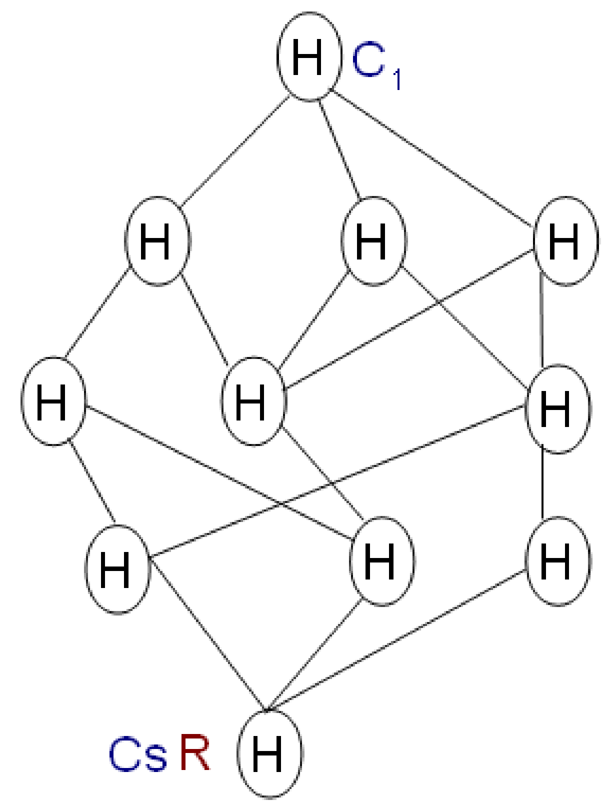
If R stays in this copy of H the cs(H)-1 cops will capture him.

C_1 on (*u*,*v*) has the same effect as having a cop “staying” on the equivalent vertex in any copy of *H.*



Eventually R will be forced onto a copy of *H* at maximal distance from C_1

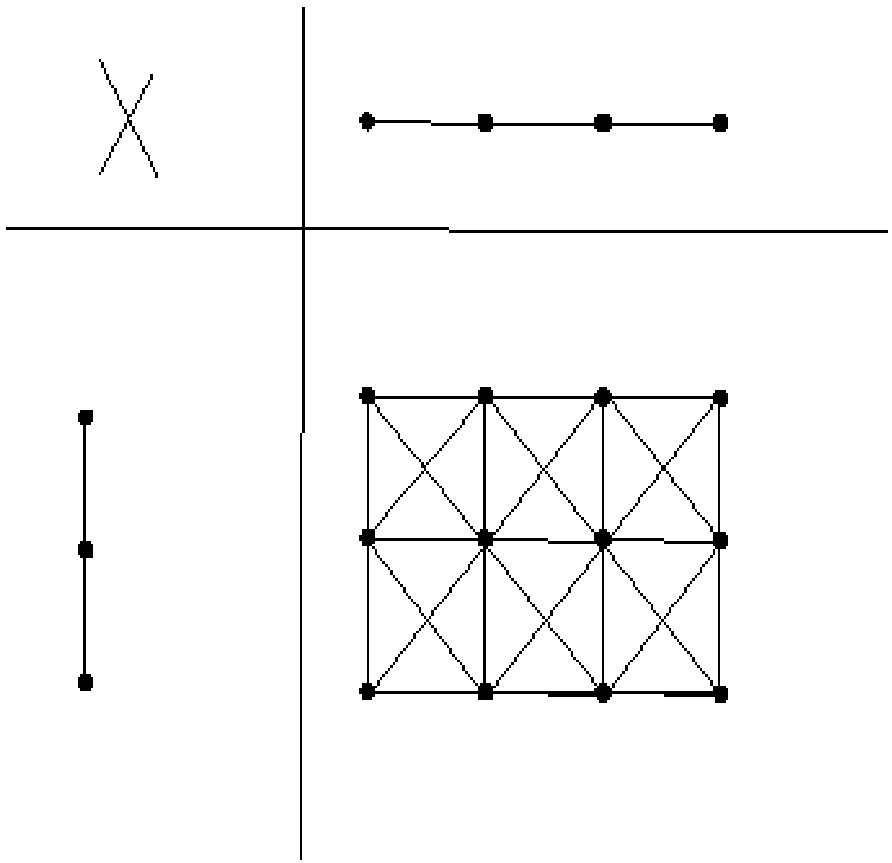
The cs(*H*)-1 other cops move into this copy and capture R



Lemma: If G is bipartite and cs(H)=k with a 1- stay strategy, then cs($G□H$) =k with a 1-stay strategy.

Corollary: If G is bipartite, then for any graph H, cs($G□H$) $\leq$ *cs*(H) + 1.

Strong Product of Paths



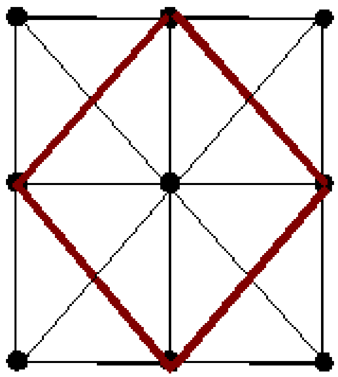
$P_3 \times P_4$

Strong Isometric Dimension of G

idim(G) = k if k is the minimum integer such that G is an isometric subgraph of the strong product of some *k* paths.

Isometric subgraph is a distance preserving subgraph

idim(C_4) = 2



Strong Isometric Dimension

F&N (2001)

If idim(G) = 2 then c(G) ≤ 2

If idim(G) = 3 then c(G) ≤ diam(G)+3

Strong Isometric Dimension

F&N (2001)

If idim(G) = 2 then c(G) ≤ 2

If idim(G) = 3 then c(G) ≤ diam(G)+3

Theorem: If idim(G)=3 then cs(G)≤4.

Future Investigations

Meyniel's Conjecture:

For any connected graph G on n vertices $c(n) = O(\sqrt{n})$.

c(n) denotes the maximum copnumber of a connected graph of order n.

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Question: .... $cs(n) = O(\sqrt{n})$