Chasing Cops on Random Graphs

Paweł Prałat

Department of Mathematics, Ryerson University, Toronto, ON, Canada
(Joint work with Nick Wormald and Noga Alon)

SIAM DM, June 2012
Outline

1. Introduction and Definitions
2. Meyniel’s Conjecture holds for random graphs
3. Random Geometric Graphs
Outline

1. Introduction and Definitions
2. Meyniel’s Conjecture holds for random graphs
3. Random Geometric Graphs
Chasing Cops on Random Graphs
Chasing Cops on Random Graphs

Pawel Pralat
Introduction Random Graphs Radom Geometric Graphs

Chasing Cops on Random Graphs
Chasing Cops on Random Graphs
Introduction Random Graphs Random Geometric Graphs

Paweł Prałat

Chasing Cops on Random Graphs
Introduction

Chasing Cops on Random Graphs
Definition

As placing a cop on each vertex guarantees that the cops win, we may define the *cop number*, written $c_0(G)$, which is the minimum number of cops needed to win on $G$.

Example

- $c_0(T) = 1$ for any tree $T$,
- $c_0(K_n) = 1$ for $n \geq 3$,
- $c_0(C_n) = 2$ for $n \geq 4$,
- $c_0(G) = 1$ for any chordal graph $G$,
- $c_0(G) \leq 3$ for any planar graph $G$ (Aigner, Fromme, 1984),
- $c_0(G) \leq 3 \cdot 3 = 9$ for any connected geometric graph $G$ (previous talk).
### Definition

As placing a cop on each vertex guarantees that the cops win, we may define the *cop number*, written $c_0(G)$, which is the minimum number of cops needed to win on $G$.

### Example

- $c_0(T) = 1$ for any tree $T$,
- $c_0(K_n) = 1$ for $n \geq 3$,
- $c_0(C_n) = 2$ for $n \geq 4$,
- $c_0(G) = 1$ for any chordal graph $G$,
- $c_0(G) \leq 3$ for any planar graph $G$ (Aigner, Fromme, 1984),
- $c_0(G) \leq 3 \cdot 3 = 9$ for any connected geometric graph $G$ (previous talk).
**Definition**

As placing a cop on each vertex guarantees that the cops win, we may define the *cop number*, written $c_0(G)$, which is the minimum number of cops needed to win on $G$.

**Example**

- $c_0(T) = 1$ for any tree $T$,
- $c_0(K_n) = 1$ for $n \geq 3$,
- $c_0(C_n) = 2$ for $n \geq 4$,
- $c_0(G) = 1$ for any chordal graph $G$,
- $c_0(G) \leq 3$ for any planar graph $G$ (Aigner, Fromme, 1984),
- $c_0(G) \leq 3 \cdot 3 = 9$ for any connected geometric graph $G$ (previous talk).
Definition

As placing a cop on each vertex guarantees that the cops win, we may define the cop number, written \( c_0(G) \), which is the minimum number of cops needed to win on \( G \).

Example

- \( c_0(T) = 1 \) for any tree \( T \),
- \( c_0(K_n) = 1 \) for \( n \geq 3 \),
- \( c_0(C_n) = 2 \) for \( n \geq 4 \),
- \( c_0(G) = 1 \) for any chordal graph \( G \),
- \( c_0(G) \leq 3 \) for any planar graph \( G \) (Aigner, Fromme, 1984),
- \( c_0(G) \leq 3 \cdot 3 = 9 \) for any connected geometric graph \( G \) (previous talk).
Definition
As placing a cop on each vertex guarantees that the cops win, we may define the cop number, written $c_0(G)$, which is the minimum number of cops needed to win on $G$.

Example
- $c_0(T) = 1$ for any tree $T$,
- $c_0(K_n) = 1$ for $n \geq 3$,
- $c_0(C_n) = 2$ for $n \geq 4$,
- $c_0(G) = 1$ for any chordal graph $G$,
- $c_0(G) \leq 3$ for any planar graph $G$ (Aigner, Fromme, 1984),
- $c_0(G) \leq 3 \cdot 3 = 9$ for any connected geometric graph $G$ (previous talk).
Definition

As placing a cop on each vertex guarantees that the cops win, we may define the cop number, written \( c_0(G) \), which is the minimum number of cops needed to win on \( G \).

Example

- \( c_0(T) = 1 \) for any tree \( T \),
- \( c_0(K_n) = 1 \) for \( n \geq 3 \),
- \( c_0(C_n) = 2 \) for \( n \geq 4 \),
- \( c_0(G) = 1 \) for any chordal graph \( G \),
- \( c_0(G) \leq 3 \) for any planar graph \( G \) (Aigner, Fromme, 1984),
- \( c_0(G) \leq 3 \cdot 3 = 9 \) for any connected geometric graph \( G \) (previous talk).
**Definition**

As placing a cop on each vertex guarantees that the cops win, we may define the *cop number*, written $c_0(G)$, which is the minimum number of cops needed to win on $G$.

**Example**

- $c_0(T) = 1$ for any tree $T$,
- $c_0(K_n) = 1$ for $n \geq 3$,
- $c_0(C_n) = 2$ for $n \geq 4$,
- $c_0(G) = 1$ for any chordal graph $G$,
- $c_0(G) \leq 3$ for any planar graph $G$ (Aigner, Fromme, 1984),
- $c_0(G) \leq 3 \cdot 3 = 9$ for any connected geometric graph $G$ (previous talk).
Outline

1. Introduction and Definitions
2. Meyniel’s Conjecture holds for random graphs
3. Random Geometric Graphs
Our main results refer to the probability space \( G(n, p) = (\Omega, \mathcal{F}, \mathbb{P}) \) of random graphs, where \( \Omega \) is the set of all graphs with vertex set \([n] = \{1, 2, \ldots, n\}\), \( \mathcal{F} \) is the family of all subsets of \( \Omega \), and for every \( G \in \Omega \) 

\[
\mathbb{P}(G) = p^{|E(G)|} (1 - p)^{\binom{n}{2} - |E(G)|}.
\]

It can be viewed as a result of \( \binom{n}{2} \) independent coin flipping, one for each pair of vertices, with the probability of success (that is, drawing an edge) equal to \( p \) (\( p = p(n) \) can tend to zero with \( n \)).

We say that an event holds *asymptotically almost surely* (a.a.s.) if it holds with probability tending to 1 as \( n \to \infty \).
Our main results refer to the probability space $\mathcal{G}(n, p) = (\Omega, \mathcal{F}, \mathbb{P})$ of random graphs, where $\Omega$ is the set of all graphs with vertex set $[n] = \{1, 2, \ldots, n\}$, $\mathcal{F}$ is the family of all subsets of $\Omega$, and for every $G \in \Omega$

$$\mathbb{P}(G) = p^{|E(G)|} (1 - p)^{{n\choose 2} - |E(G)|}.$$  

It can be viewed as a result of $\binom{n}{2}$ independent coin flipping, one for each pair of vertices, with the probability of success (that is, drawing an edge) equal to $p$ ($\rho = p(n)$ can tend to zero with $n$).

We say that an event holds *asymptotically almost surely* (a.a.s.) if it holds with probability tending to 1 as $n \to \infty$. 
We consider also the probability space of random $d$-regular graphs on $n$ vertices with uniform probability distribution. This space is denoted $\mathcal{G}_{n,d}$, with $d \geq 2$ fixed, and $n$ even if $d$ is odd.

We say that an event holds \textit{asymptotically almost surely} (a.a.s.) if it holds with probability tending to 1 as $n \to \infty$. 
Conjecture (Meyniel’s Conjecture, communicated by Frankl)

\[ c_0(n) = O(\sqrt{n}), \]

where \( c_0(n) \) is the maximum of \( c_0(G) \) over all \( n \)-vertex connected graphs.

Theorem (Frankl, 1987)

\[ c_0(n) = O\left(\frac{n \log \log n}{\log n}\right) \]

Theorem (Lu, Peng, 2012+)

\[ c_0(n) \leq n2^{-(1-o(1))\sqrt{\log_2 n}} = n^{1-o(1)} \]

The result proved independently by
– Scott and Sudakov (2011)
– Frieze, Krivelevich, and Loh (2012)
Conjecture (Meyniel’s Conjecture, communicated by Frankl)

\[ c_0(n) = O(\sqrt{n}), \]

where \( c_0(n) \) is the maximum of \( c_0(G) \) over all \( n \)-vertex connected graphs.

Theorem (Frankl, 1987)

\[ c_0(n) = O\left(\frac{n \log \log n}{\log n}\right) \]

Theorem (Lu, Peng, 2012+)

\[ c_0(n) \leq n2^{-\left(1-o(1)\right)}\sqrt{\log n} = n^{1-o(1)} \]

The result proved independently by
– Scott and Sudakov (2011)
– Frieze, Krivelevich, and Loh (2012)
Conjecture (Meyniel’s Conjecture, communicated by Frankl)

\[ c_0(n) = O(\sqrt{n}), \]

where \( c_0(n) \) is the maximum of \( c_0(G) \) over all \( n \)-vertex connected graphs.

Theorem (Frankl, 1987)

\[ c_0(n) = O\left( \frac{n \log \log n}{\log n} \right) \]

Theorem (Lu, Peng, 2012+)

\[ c_0(n) \leq n2^{-(1-o(1))}\sqrt{\log_2 n} = n^{1-o(1)} \]

The result proved independently by
– Scott and Sudakov (2011)
– Frieze, Krivelevich, and Loh (2012)
Conjecture (Meyniel’s Conjecture, communicated by Frankl)

\[ c_0(n) = O(\sqrt{n}), \]

where \( c_0(n) \) is the maximum of \( c_0(G) \) over all \( n \)-vertex connected graphs.

Theorem (Frankl, 1987)

\[ c_0(n) = O\left(\frac{n \log \log n}{\log n}\right) \]

Theorem (Lu, Peng, 2012+)

\[ c_0(n) \leq n2^{-(1-o(1))\sqrt{\log_2 n}} = n^{1-o(1)} \]

The result proved independently by
– Scott and Sudakov (2011)
– Frieze, Krivelevich, and Loh (2012)
Theorem (Bonato, Prałat, Wang, 2009)

If \( d = np = n^{\alpha + o(1)} \), where \( 1/2 < \alpha < 1 \), then a.a.s.

\[
c_0(G(n, p)) = \Theta(\log n/p) = n^{1-\alpha + o(1)}
\]

and \( c_0(G(n, n^{-1/2+o(1)})) = n^{1/2+o(1)} \) a.a.s.

Let us define the function \( f : (0, 1) \rightarrow \mathbb{R} \) as

\[
f(\alpha) = \log_n \bar{c}_0(G(n, n^{\alpha-1})) = \frac{\log \bar{c}_0(G(n, n^{\alpha-1}))}{\log n},
\]

where \( \bar{c}_0(G(n, p)) \) denotes the median of the cop number for \( G(n, p) \).
Theorem (Bonato, Prałat, Wang, 2009)

If \( d = np = n^{\alpha + o(1)} \), where \( 1/2 < \alpha < 1 \), then a.a.s.

\[
c_0(G(n, p)) = \Theta(\log n/p) = n^{1-\alpha+o(1)}
\]

and \( c_0(G(n, n^{-1/2+o(1)})) = n^{1/2+o(1)} \) a.a.s.

Let us define the function \( f : (0, 1) \to \mathbb{R} \) as

\[
f(\alpha) = \log_n \bar{c}_0(G(n, n^{\alpha-1})) = \frac{\log \bar{c}_0(G(n, n^{\alpha-1}))}{\log n},
\]

where \( \bar{c}_0(G(n, p)) \) denotes the median of the cop number for \( G(n, p) \).
Introduction Random Graphs Radom Geometric Graphs

Chasing Cops on Random Graphs

\[ \frac{1}{2} \]

[1/2]

[1]
Theorem (Bollobás, Kun, Leader, 2012+)

If \( p(n) \geq 2.1 \log n/n \), then a.a.s.

\[
\frac{1}{(np)^2} n^2 \frac{1 \log \log(np) - 9}{\log \log(np)} \leq c_0(G(n, p)) \leq 160000 \sqrt{n} \log n.
\]

Since if either \( np = n^{o(1)} \) or \( np = n^{1/2+o(1)} \) then a.a.s. \( c_0(G(n, p)) = n^{1/2+o(1)} \), it would be natural to assume that the cops number of \( G(n, p) \) is close to \( \sqrt{n} \) also for \( np = n^{\alpha+o(1)} \), where \( 0 < \alpha < 1/2 \).
Theorem (Bollobás, Kun, Leader, 2012+)

If \( p(n) \geq 2.1 \log n/n \), then a.a.s.

\[
\frac{1}{(np)^2} n^2 \frac{1}{\log \log(np) - 9} \leq c_0(G(n, p)) \leq 160000 \sqrt{n \log n}.
\]

Since if either \( np = n^{o(1)} \) or \( np = n^{1/2 + o(1)} \) then a.a.s. \( c_0(G(n, p)) = n^{1/2 + o(1)} \), it would be natural to assume that the cops number of \( G(n, p) \) is close to \( \sqrt{n} \) also for \( np = n^{\alpha + o(1)} \), where \( 0 < \alpha < 1/2 \).
Chasing Cops on Random Graphs
Theorem (Łuczak, Prałat, 2010)

Let $0 < \alpha < 1$ and $d = d(n) = np = n^{\alpha + o(1)}$.

1. If $\frac{1}{2^{j+1}} < \alpha < \frac{1}{2^j}$ for some $j \geq 1$, then a.a.s.

$$c_0(G(n, p)) = \Theta(d^j).$$

2. If $\frac{1}{2^j} < \alpha < \frac{1}{2^{j-1}}$ for some $j \geq 1$, then a.a.s.

$$\Omega \left( \frac{n}{d^j} \right) = c_0(G(n, p)) = O \left( \frac{n}{d^j \log n} \right).$$

We get a good upper estimate for $c_0(G(n, p))$ also for $d = n^{1/k + o(1)}$ ($k = 2, 3, \ldots$), and our argument for lower bound

can be repeated in this case to determine $c_0(G(n, p))$ up to

$\log^{O(1)} n$ factor in the whole range of $p$, provided

$n^{\varepsilon-1} \leq p \leq n^{-\varepsilon}$ for some $\varepsilon > 0$. 

Paweł Prałat

Chasing Cops on Random Graphs
Theorem (Łuczak, Prałat, 2010)

Let $0 < \alpha < 1$ and $d = d(n) = np = n^{\alpha + o(1)}$.

1. If $\frac{1}{2^{j+1}} < \alpha < \frac{1}{2^j}$ for some $j \geq 1$, then a.a.s.
   
   $$c_0(G(n, p)) = \Theta(d^j).$$

2. If $\frac{1}{2^j} < \alpha < \frac{1}{2^{j-1}}$ for some $j \geq 1$, then a.a.s.

   $$\Omega\left(\frac{n}{d^j}\right) = c_0(G(n, p)) = O\left(\frac{n}{d^j \log n}\right).$$

We get a good upper estimate for $c_0(G(n, p))$ also for $d = n^{1/k + o(1)}$ ($k = 2, 3, \ldots$), and our argument for lower bound can be repeated in this case to determine $c_0(G(n, p))$ up to $\log^{O(1)} n$ factor in the whole range of $p$, provided $n^{\varepsilon - 1} \leq p \leq n^{-\varepsilon}$ for some $\varepsilon > 0$. 
Meyniel’s conjecture holds a.a.s. for random graphs except perhaps when \( np = n^{1/(2k)+o(1)} \) for some \( k \in \mathbb{N} \), or \( np = n^{o(1)} \).
Theorem (Prałat, Wormald, 2012+)

Let $\varepsilon > 0$ and suppose that $d = d(n) \geq (1/2 + \varepsilon) \log n$. Let $G \in G(n, p)$ with $p = d/(n - 1)$. Then a.a.s.

$$c_0(G) = O(\sqrt{n}).$$

Theorem (Prałat, Wormald, 2012+)

Fix $d = d(n) \geq 3$. Then, a.a.s.

$$c_0(G_{n,d}) = O(\sqrt{n}).$$
**Theorem (Prałat, Wormald, 2012+)**

Let $\varepsilon > 0$ and suppose that $d = d(n) \geq (1/2 + \varepsilon) \log n$. Let $G \in G(n, p)$ with $p = d/(n - 1)$. Then a.a.s.

$$c_0(G) = O(\sqrt{n}).$$

**Theorem (Prałat, Wormald, 2012+)**

Fix $d = d(n) \geq 3$. Then, a.a.s.

$$c_0(G_{n,d}) = O(\sqrt{n}).$$
$c_0(G) = O(\sqrt{n})$ — sketch for $G(n, p)$ with $p(n - 1) > \log^3 n$

$i = \max\{j : d^i \leq \sqrt{n}\}$, $C\sqrt{n}$ cops in the first team.

Case 1: $d^{i+1} \geq \sqrt{n}\log n$ — easy

Case 2: $d^{i+1} = \sqrt{n}\omega$ with $1 \leq \omega < \log n$ — we need one more (independent) team of $C\sqrt{n}$ cops.
\[ c_0(G) = O(\sqrt{n}) \quad \text{— sketch for } G(n, p) \text{ with } p(n - 1) > \log^3 n \]

\( i = \max\{j : d^j \leq \sqrt{n}\}, C\sqrt{n} \text{ cops in the first team.} \)

**Case 1:** \( d^{i+1} \geq \sqrt{n} \log n \quad \text{— easy} \)

**Case 2:** \( d^{i+1} = \sqrt{n} \omega \text{ with } 1 \leq \omega < \log n \quad \text{— we need one more (independent) team of } C\sqrt{n} \text{ cops.} \)
$c_0(G) = O(\sqrt{n})$ — sketch for $G(n, p)$ with $p(n - 1) > \log^3 n$

\[ i = \max\{j : d^j \leq \sqrt{n}\}, \text{ } C\sqrt{n} \text{ cops in the first team.} \]

Case 1: $d^{i+1} \geq \sqrt{n} \log n$ — easy

Case 2: $d^{i+1} = \sqrt{n}\omega$ with $1 \leq \omega < \log n$ — we need one more (independent) team of $C\sqrt{n}$ cops.
The first team ‘densely covers’ the sphere $S(v, i)$: $u \in S(v, i)$ is covered with probability at most

$$
\left(1 - \frac{C}{\sqrt{n}}\right)^{|W(u)|} \leq \exp\left(- \frac{C}{\sqrt{n}} \cdot \frac{1}{2} \sqrt{n} \cdot \omega\right) = \exp\left(- \frac{C}{2} \cdot \omega\right) < \frac{1}{10\omega},
$$

for $C$ sufficiently large.

We may assume that the robber heads directly to some vertex in $S(v, i)$ and reaches it in $i$ steps.
The first team ‘densely covers’ the sphere $S(v, i)$: $u \in S(v, i)$ is covered with probability at most

$$\left(1 - \frac{C}{\sqrt{n}}\right)^{|W(u)|} \leq \exp\left(-\frac{C}{\sqrt{n}} \cdot \frac{1}{2}\sqrt{n} \cdot \omega\right) = \exp\left(-\frac{C}{2} \cdot \omega\right) < \frac{1}{10^\omega},$$

for $C$ sufficiently large.

We may assume that the robber heads directly to some vertex in $S(v, i)$ and reaches it in $i$ steps.
The second team of cops is released when the robber is at $z \in S(v, \lceil i/2 \rceil)$.

We may assume that she is heading to the set $S \subseteq S(z, \lceil i/2 \rceil) \cap S(v, i)$ that is not covered by cops from the first team.
The second team of cops is released when the robber is at \( z \in S(v, \lceil i/2 \rceil) \).

We may assume that she is heading to the set \( S \subseteq S(z, \lceil i/2 \rceil) \cap S(v, i) \) that is not covered by cops from the first team.
The second team has to cover the set of vertices

\[ U = \bigcup_{s \in S} S(s, \lceil i/2 \rceil + 1), \]

of size at most

\[ 2d^{\lceil r/2 \rceil + 1} |S| \leq \frac{d^{r+1}}{2\omega} < \sqrt{n}. \]

For \( u \in U \), we need to search for cops within distance \( r + 2 \).
sparse case is more complicated:

- $F \log \log n$ independent teams of cops
- the $i$th team consists of $c_i = Ce^{-i}\sqrt{n}$ cops ($\Theta(\sqrt{n})$ in total)
- $F \log \log n$ rounds
- the duration of the round is until the robber reaches a vertex $v_{i+1}$ of $S(v_i, r_i)$ where $r_1 = \log_d(\varepsilon_0 n)/4$, and $r_i$ (for $i \geq 2$) is defined recursively

$$\frac{\sqrt{\varepsilon_0 n}}{d} < \frac{d^{r_i-1+r_i}}{e^{2(i-1)}} \leq \sqrt{\varepsilon_0 n},$$

- at the start of round $i$ ($i \geq 2$), the cops from team $(i - 1)$ are already heading towards some of the vertices in the sphere $S(v_i, r_i)$; team $i$ is released
sparse case is more complicated:

- $F \log \log n$ independent teams of cops
- the $i$th team consists of $c_i = Ce^{-i}i\sqrt{n}$ cops ($\Theta(\sqrt{n})$ in total)
- $F \log \log n$ rounds
- the duration of the round is until the robber reaches a vertex $v_{i+1}$ of $S(v_i, r_i)$ where $r_1 = \log_d(\varepsilon_0 n)/4$, and $r_i$ (for $i \geq 2$) is defined recursively

$$\frac{\sqrt{\varepsilon_0 n}}{d} < \frac{d^{r_{i-1}+r_i}}{e^{2(i-1)}} \leq \sqrt{\varepsilon_0 n},$$

- at the start of round $i$ ($i \geq 2$), the cops from team $(i - 1)$ are already heading towards some of the vertices in the sphere $S(v_i, r_i)$; team $i$ is released
sparse case is more complicated:

- $F \log \log n$ independent teams of cops
- the $i$th team consists of $c_i = C e^{-i} \sqrt{n}$ cops ($\Theta(\sqrt{n})$ in total)
- $F \log \log n$ rounds
- the duration of the round is until the robber reaches a vertex $v_{i+1}$ of $S(v_i, r_i)$ where $r_1 = \log_d(\varepsilon_0 n) / 4$, and $r_i$ (for $i \geq 2$) is defined recursively

$$\frac{\sqrt{\varepsilon_0 n}}{d} < \frac{d^{r_{i-1}+r_i}}{e^{2(i-1)}} \leq \sqrt{\varepsilon_0 n},$$

- at the start of round $i$ ($i \geq 2$), the cops from team $(i - 1)$ are already heading towards some of the vertices in the sphere $S(v_i, r_i)$; team $i$ is released
sparse case is more complicated:

- $F \log \log n$ independent teams of cops
- the $i$th team consists of $c_i = C e^{-i} \sqrt{n}$ cops ($\Theta(\sqrt{n})$ in total)
- $F \log \log n$ rounds
- the duration of the round is until the robber reaches a vertex $v_{i+1}$ of $S(v_i, r_i)$ where $r_1 = \log_d(\varepsilon_0 n)/4$, and $r_i$ (for $i \geq 2$) is defined recursively

$$\frac{\sqrt{\varepsilon_0 n}}{d} < \frac{d^{r_{i-1}+r_i}}{e^{2(i-1)}} \leq \sqrt{\varepsilon_0 n},$$

- at the start of round $i$ ($i \geq 2$), the cops from team $(i - 1)$ are already heading towards some of the vertices in the sphere $S(v_i, r_i)$; team $i$ is released
sparse case is more complicated:

- $F \log \log n$ independent teams of cops
- the $i$th team consists of $c_i = C e^{-i} \sqrt{n}$ cops ($\Theta(\sqrt{n})$ in total)
- $F \log \log n$ rounds
- the duration of the round is until the robber reaches a vertex $v_{i+1}$ of $S(v_i, r_i)$ where $r_1 = \log_d(\varepsilon_0 n)/4$, and $r_i$ (for $i \geq 2$) is defined recursively

$$\frac{\sqrt{\varepsilon_0 n}}{d} < \frac{d^{r_i-1+r_i}}{e^{2(i-1)}} \leq \sqrt{\varepsilon_0 n},$$

- at the start of round $i$ ($i \geq 2$), the cops from team $(i - 1)$ are already heading towards some of the vertices in the sphere $S(v_i, r_i)$; team $i$ is released
sparse case is more complicated:

- $F \log \log n$ independent teams of cops
- the $i$th team consists of $c_i = Ce^{-i} \sqrt{n}$ cops ($\Theta(\sqrt{n})$ in total)
- $F \log \log n$ rounds
- the duration of the round is until the robber reaches a vertex $v_{i+1}$ of $S(v_i, r_i)$ where $r_1 = \log_d(\varepsilon_0 n)/4$, and $r_i$ (for $i \geq 2$) is defined recursively

$$\frac{\sqrt{\varepsilon_0 n}}{d} < \frac{d^{r_{i-1} + r_i}}{e^{2(i-1)}} \leq \sqrt{\varepsilon_0 n},$$

- at the start of round $i$ ($i \geq 2$), the cops from team $(i - 1)$ are already heading towards some of the vertices in the sphere $S(v_i, r_i)$; team $i$ is released
Outline

1. Introduction and Definitions
2. Meyniel’s Conjecture holds for random graphs
3. Random Geometric Graphs
(Random subgraph of) random geometric graph $G_d(n, r, p)$
- vertex set $[n] = \{1, 2, \ldots, n\}$: $n$ vertices are chosen uniformly at random and independently from $[0, 1]^d$,
- a pair of vertices within Euclidean distance $r = r(n)$ appears as an edge with probability $p = p(n)$, independently for each such a pair.

Today, we will focus on $G_d(n, r) = G_d(n, r, 1) – (\text{classic})$ random geometric graph
(Random subgraph of) random geometric graph $\mathcal{G}_d(n, r, p)$
- vertex set $[n] = \{1, 2, \ldots, n\}$: $n$ vertices are chosen uniformly at random and independently from $[0, 1]^d$,
- a pair of vertices within Euclidean distance $r = r(n)$ appears as an edge with probability $p = p(n)$, independently for each such a pair.

Today, we will focus on $\mathcal{G}_d(n, r) = \mathcal{G}_d(n, r, 1) – (\text{classic}) \text{ random geometric graph}$
Theorem (Alon, Prałat, 2012+)

There exists an absolute constant $c > 0$ so that if $r^5 > c \frac{\log n}{n}$ then a.a.s. $c(G_2(n, r)) = 1$.

Independently proved by Beveridge, Dudek, Frieze, and Müller (previous talk).

Proof is quite different and also gives the following.

Theorem (Alon, Prałat, 2012+)

For each fixed $d > 1$ there exists a constant $c_d > 0$ so that if $r^{3d-1} > c_d \frac{\log n}{n}$ then a.a.s. $c(G_d(n, r)) = 1$. 
Theorem (Alon, Prałat, 2012+)

There exists an absolute constant $c > 0$ so that if $r^5 > c \frac{\log n}{n}$ then a.a.s. $c(G_2(n, r)) = 1$.

Independently proved by Beveridge, Dudek, Frieze, and Müller (previous talk).

Proof is quite different and also gives the following.

Theorem (Alon, Prałat, 2012+)

For each fixed $d > 1$ there exists a constant $c_d > 0$ so that if $r^{3d-1} > c_d \frac{\log n}{n}$ then a.a.s. $c(G_d(n, r)) = 1$. 
$G_2(r)$: continuous (infinite) graph whose vertices are all of the points of $[0, 1]^2$, where two of them are adjacent if and only if their distance is at most $r$.

**Theorem (Known? Similar to the Lion and the Christian)**

$c(G_2(r)) = 1$ for any $r > 0$.

- the cop places himself at the center $O$ of $[0, 1]^2$
- catch the bad guy if you can; otherwise:

+ move to a point $C$ that lies on the segment $OR$, making sure his distance from the robber is at least, say, $r^2/100$,

+ in each step the square of the distance between the location of the cop and $O$ increases by at least $r^2/5$.

The game ends in at most $O(1/r^2)$ steps.
$G_2(r)$: continuous (infinite) graph whose vertices are all of the points of $[0, 1]^2$, where two of them are adjacent if and only if their distance is at most $r$.

**Theorem (Known? Similar to the Lion and the Christian)**

$c(G_2(r)) = 1$ for any $r > 0$.

- the cop places himself at the center $O$ of $[0, 1]^2$
- catch the bad guy if you can; otherwise:
  + move to a point $C$ that lies on the segment $OR$, making sure his distance from the robber is at least, say, $r^2/100$,
  + in each step the square of the distance between the location of the cop and $O$ increases by at least $r^2/5$.

The game ends in at most $O(1/r^2)$ steps.
$G_2(r)$: continuous (infinite) graph whose vertices are all of the points of $[0, 1]^2$, where two of them are adjacent if and only if their distance is at most $r$.

**Theorem (Known? Similar to the Lion and the Christian)**

$c(G_2(r)) = 1 \text{ for any } r > 0.$

- the cop places himself at the center $O$ of $[0, 1]^2$
- catch the bad guy if you can; otherwise:
  + move to a point $C$ that lies on the segment $OR$, making sure his distance from the robber is at least, say, $r^2/100$,
  + in each step the square of the distance between the location of the cop and $O$ increases by at least $r^2/5$.

The game ends in at most $O(1/r^2)$ steps.
$G_2(r)$: continuous (infinite) graph whose vertices are all of the points of $[0, 1]^2$, where two of them are adjacent if and only if their distance is at most $r$.

**Theorem (Known? Similar to *the Lion and the Christian*):**

$c(G_2(r)) = 1$ for any $r > 0$.

- the cop places himself at the center $O$ of $[0, 1]^2$
- catch the bad guy if you can; otherwise:

  + move to a point $C$ that lies on the segment $OR$, making sure his distance from the robber is at least, say, $r^2/100$,

  + in each step the square of the distance between the location of the cop and $O$ increases by at least $r^2/5$.

The game ends in at most $O(1/r^2)$ steps.
- $R'$ is below $\ell$ (otherwise the game ends)
- $CC' \leq Z'R' \leq RR' \leq r$ so the cop may move to $C'$

Case 1: $CC' > r/2$. Go to $C'$, and move towards $O$ if too close to $R'$ to make sure the distance between players is at least $r^2/100$. The square distance increases by at least $r^2/4 - 2r^2/100 > r^2/5$.

Case 2: $CC' \leq r/2$. Go to $C'$, and move towards $R'$. Since $OC' \geq OC$, the cop increases his distance from $O$ by more than $r/2$. Move back, if too close to $R'$. 

Paweł Prałat

Chasing Cops on Random Graphs
Adopting strategy for $G_2(n, r)$.

Let $X \in [0, 1]^d$ so that $OX \geq r/2$ and the distance from the boundary is at least $r^2/10^3$.

$T(X)$ is an isosceles triangle of height $r^2/100$ and the base of length $r^3/10^5$. 
Adopting strategy for $G_2(n, r)$.

**Lemma**

There exists an absolute constant $c > 0$ so that a.a.s. every triangle $T(X)$ contains a vertex of $G_2(n, r)$, provided that $r^5 > c \frac{\log n}{n}$.

$F$ - fixed collection of $O((1/r)^6)$ rectangles, each of area $\Omega(r^5)$, so that every triangle $T(X)$ fully contains at least one of these rectangles. (For example, take $10^6 r^3$ by $10^6 r^3$ grid and for each point we take the rectangle of width $r^3/10^6$ and height $r^2/10^6$ in which $Y$ is the midpoint of the edge of length $10^6 r^3$ and the other edge is in direction $YO$.)

A.a.s. each rectangle in $F$ contains at least one vertex of $G_2(n, r)$. 
Adopting strategy for $G_2(n, r)$.

**Lemma**

*There exists an absolute constant $c > 0$ so that a.a.s. every triangle $T(X)$ contains a vertex of $G_2(n, r)$, provided that $r^5 > c \frac{\log n}{n}$.***

$F$ - fixed collection of $O((1/r)^6)$ rectangles, each of area $\Omega(r^5)$, so that every triangle $T(X)$ fully contains at least one of these rectangles. (For example, take $10^6 r^3$ by $10^6 r^3$ grid and for each point we take the rectangle of width $r^3/10^6$ and height $r^2/10^6$ in which $Y$ is the midpoint of the edge of length $10^6 r^3$ and the other edge is in direction $YO$.)

A.a.s. each rectangle in $F$ contains at least one vertex of $G_2(n, r)$. 
Adopting strategy for $G_2(n, r)$.

**Lemma**

There exists an absolute constant $c > 0$ so that a.a.s. every triangle $T(X)$ contains a vertex of $G_2(n, r)$, provided that $r^5 > c \frac{\log n}{n}$.

$F$ - fixed collection of $O((1/r)^6)$ rectangles, each of area $\Omega(r^5)$, so that every triangle $T(X)$ fully contains at least one of these rectangles. (For example, take $10^6r^3$ by $10^6r^3$ grid and for each point we take the rectangle of width $r^3/10^6$ and height $r^2/10^6$ in which $Y$ is the midpoint of the edge of length $10^6r^3$ and the other edge is in direction $YO$.)

A.a.s. each rectangle in $F$ contains at least one vertex of $G_2(n, r)$.
Adopting strategy for $\mathcal{G}_2(n, r)$.

The cop will follow essentially the continuous strategy, but will always place himself at a vertex of the graph which is sufficiently close to where he wants to be in the continuous variant of the game.