Chasing Cops on Random Graphs

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(Joint work with Nick Wormald and Noga Alon)

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Outline

1. Introduction and Definitions
2. Meyniel's Conjecture holds for random graphs
3. Radom Geometric Graphs

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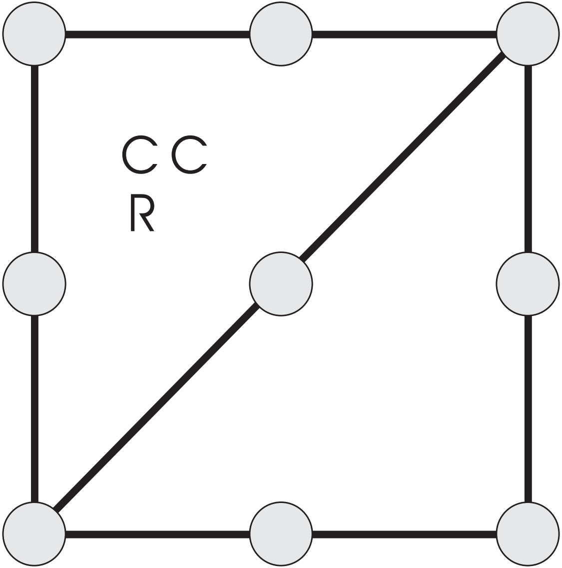
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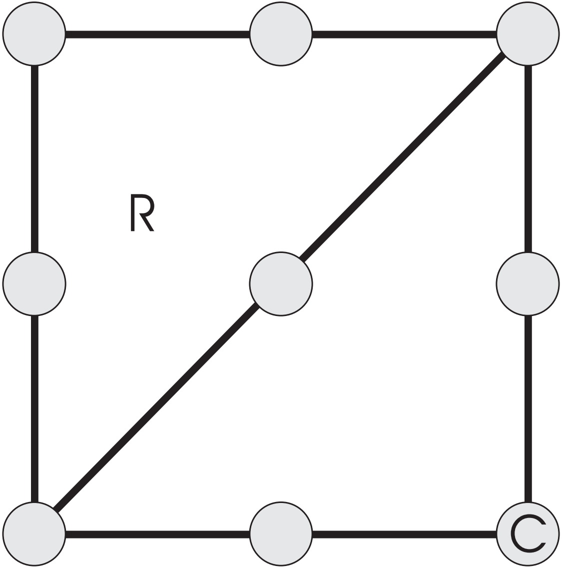
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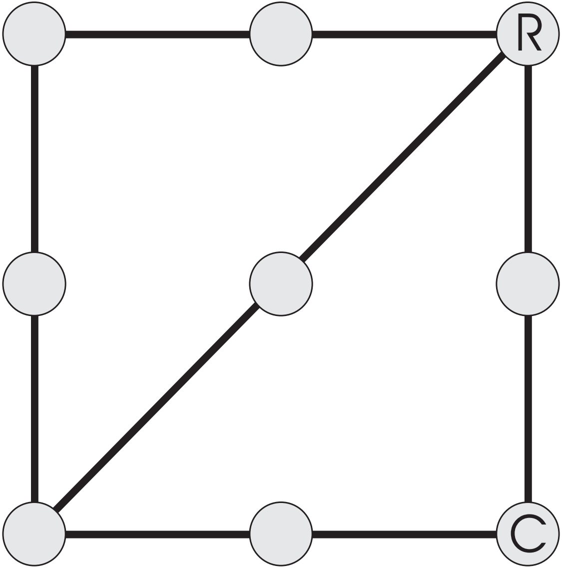
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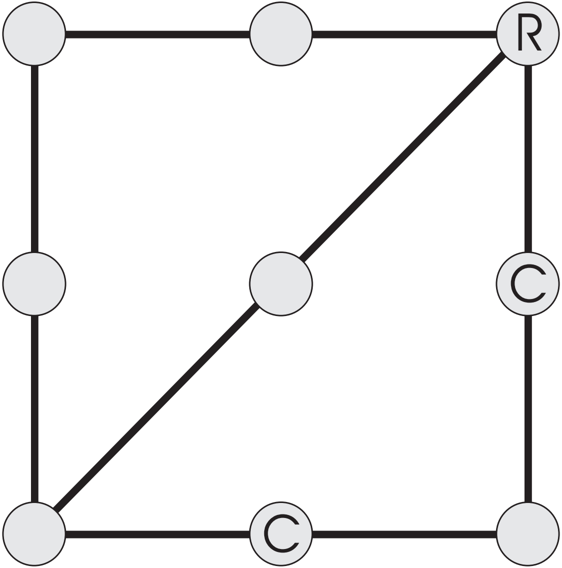
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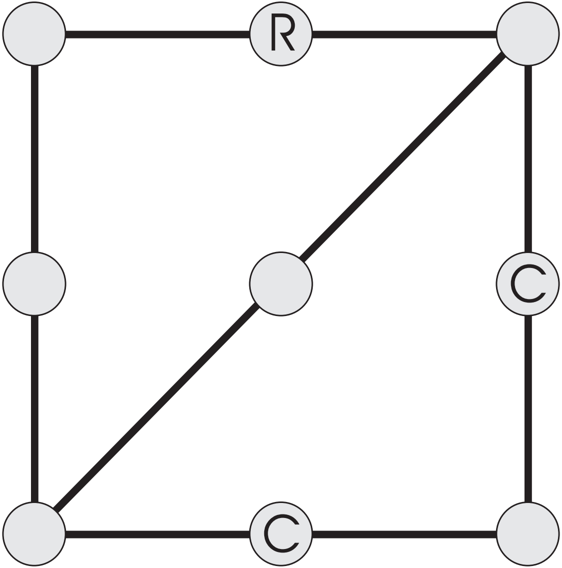
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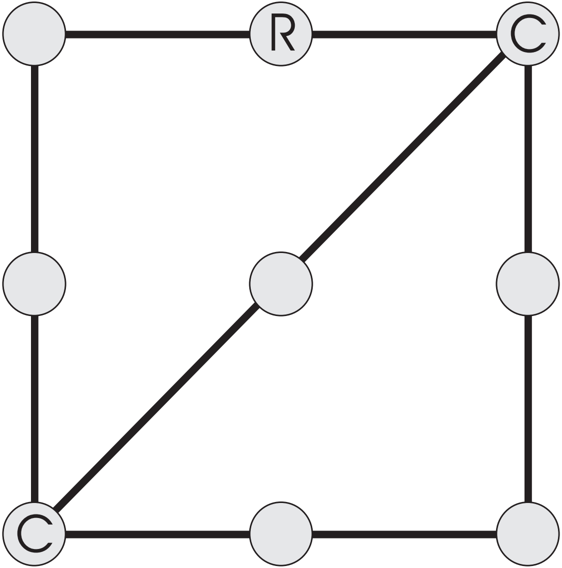
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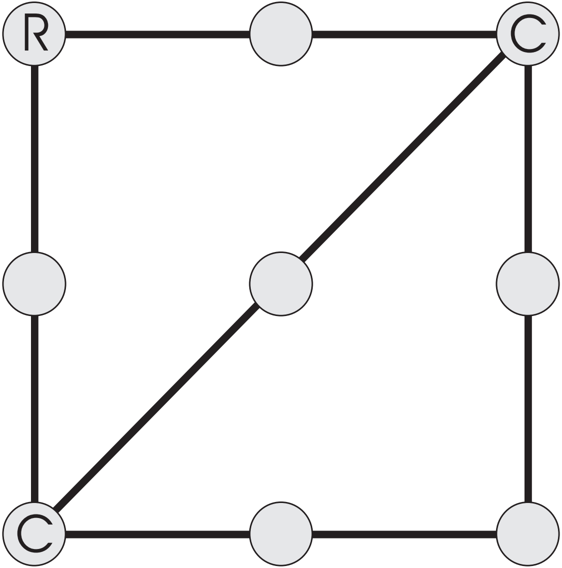
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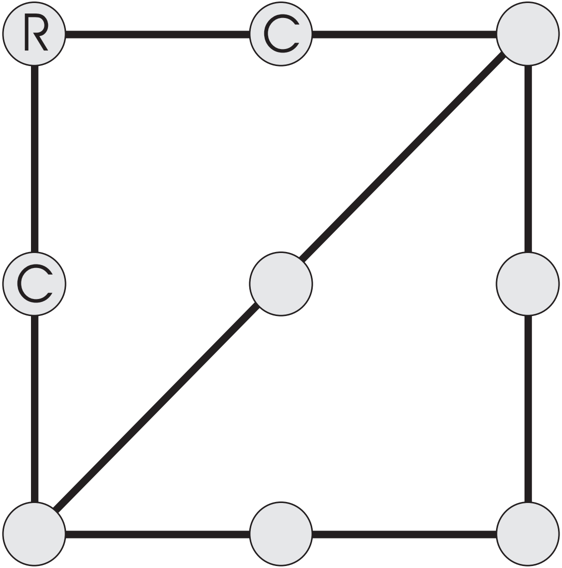
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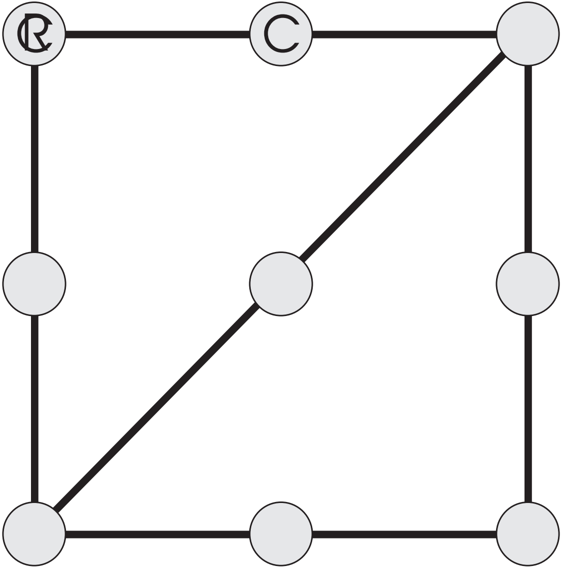
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Definition

As placing a cop on each vertex guarantees that the cops win, we may define the *cop number*, written $c_0(G)$, which is the minimum number of cops needed to win on G.

Example

- $c_0(T) = 1$ for any tree T,
- $c_0(K_n) = 1$ for $n \geq 3$,
- $c_0(C_n) = 2$ for $n \geq 4$,
- $c_0(G) = 1$ for any chordal graph G,
- $c_0(G) \leq 3$ for any planar graph G (Aigner, Fromme, 1984),
- $c_0(G) \leq 3 \cdot 3 = 9$ for any connected geometric graph G (previous talk).

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Outline

Our main results refer to the probability space $\mathcal{G}(n,p) = (\Omega, \mathcal{F}, \mathbb{P})$ of random graphs, where Ω is the set of all graphs with vertex set $[n] = \{1, 2, \dots, n\}$, $\mathcal{F}$ is the family of all subsets of Ω, and for every $G \in \Omega$

$$\mathbb{P}(G) = p^{|E(G)|}(1-p)^{\binom{n}{2}-|E(G)|}.$$

It can be viewed as a result of $\binom{n}{2}$ independent coin flipping, one for each pair of vertices, with the probability of success (that is, drawing an edge) equal to p ($p = p(n)$ can tend to zero with n).

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We consider also the probability space of random d-regular graphs on n vertices with uniform probability distribution. This space is denoted $\mathcal{G}_{n,d}$, with $d \geq 2$ fixed, and n even if d is odd.

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Conjecture (Meyniel's Conjecture, communicated by Frankl)

$$c_0(n) = O(\sqrt{n}),$$

where $c_0(n)$ is the maximum of $c_0(G)$ over all n-vertex connected graphs.

Theorem (Frankl, 1987)

$$c_0(n) = O\left(\frac{n \log \log n}{\log n}\right)$$

Theorem (Lu, Peng, 2012+)

$$c_0(n) \leq n2^{-(1-o(1))\sqrt{\log_2 n}} = n^{1-o(1)}$$

The result proved independently by
– Scott and Sudakov (2011)
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Theorem (Bonato, Prałat, Wang, 2009)

If $d = np = n^{\alpha+o(1)}$*, where* $1/2 < \alpha < 1$*, then a.a.s.*

$$c_0(G(n,p)) = \Theta(\log n/p) = n^{1-\alpha+o(1)}$$

and $c_0(G(n, n^{-1/2+o(1)})) = n^{1/2+o(1)}$ *a.a.s.*

Let us define the function $f : (0,1) \to \mathbb{R}$ as

$$f(\alpha) = \log_n \bar{c}_0(G(n, n^{\alpha-1})) = \frac{\log \bar{c}_0(G(n, n^{\alpha-1}))}{\log n},$$

where $\bar{c}_0(G(n,p))$ denotes the median of the cop number for $G(n,p)$.

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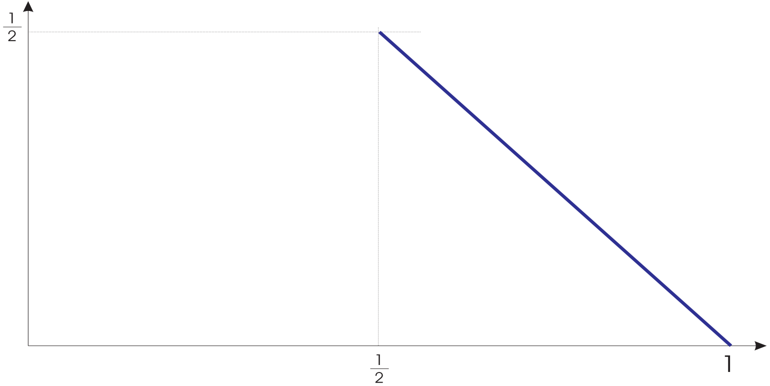
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Theorem (Bollobás, Kun, Leader, 2012+)

If $p(n) \geq 2.1 \log n/n$, then a.a.s.

$$\frac{1}{(np)^2} n^{\frac{1}{2} \frac{\log\log(np) - 9}{\log\log(np)}} \leq c_0(G(n,p)) \leq 160000\sqrt{n}\log n\,.$$

Since if either $np = n^{o(1)}$ or $np = n^{1/2+o(1)}$ then a.a.s. $c_0(G(n,p)) = n^{1/2+o(1)}$, it would be natural to assume that the cops number of $G(n,p)$ is close to $\sqrt{n}$ also for $np = n^{\alpha+o(1)}$, where $0 < \alpha < 1/2$.

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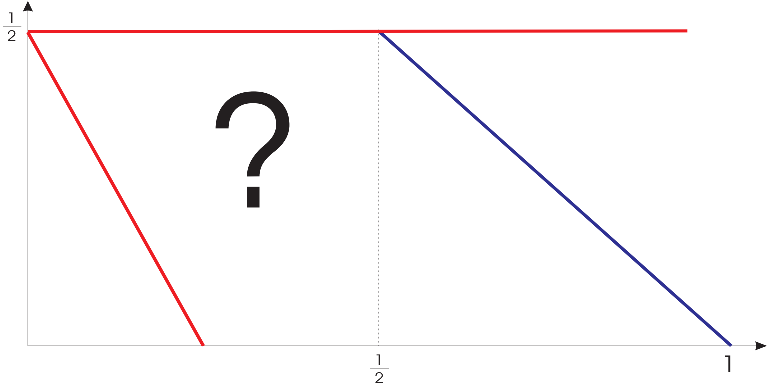

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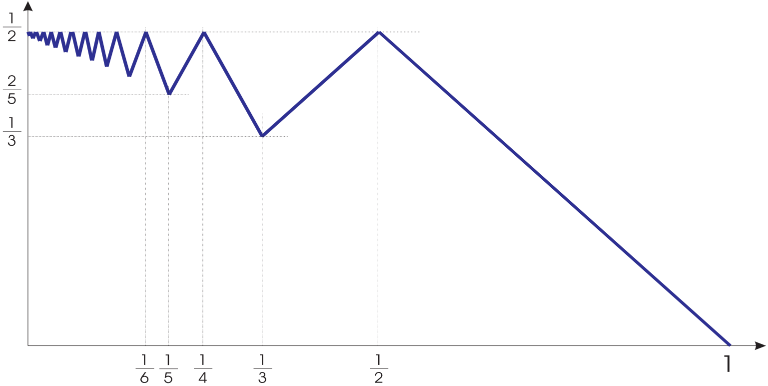
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Theorem (Łuczak, Prałat, 2010)

Let $0 < \alpha < 1$ *and* $d = d(n) = np = n^{\alpha+o(1)}$.

1. *If* $\frac{1}{2j+1} < \alpha < \frac{1}{2j}$ *for some* $j \geq 1$*, then a.a.s.*
$$c_0(G(n,p)) = \Theta(d^j)\,.$$
2. *If* $\frac{1}{2j} < \alpha < \frac{1}{2j-1}$ *for some* $j \geq 1$*, then a.a.s.*
$$\Omega\left(\frac{n}{d^j}\right) = c_0(G(n,p)) = O\left(\frac{n}{d^j}\log n\right)\,.$$

We get a good upper estimate for $c_0(G(n,p))$ also for $d = n^{1/k+o(1)}$ ($k = 2, 3, \dots$), and our argument for lower bound can be repeated in this case to determine $c_0(G(n,p))$ up to $\log^{O(1)} n$ factor in the whole range of p, provided $n^{\varepsilon-1} \leq p \leq n^{-\varepsilon}$ for some $\varepsilon > 0$.

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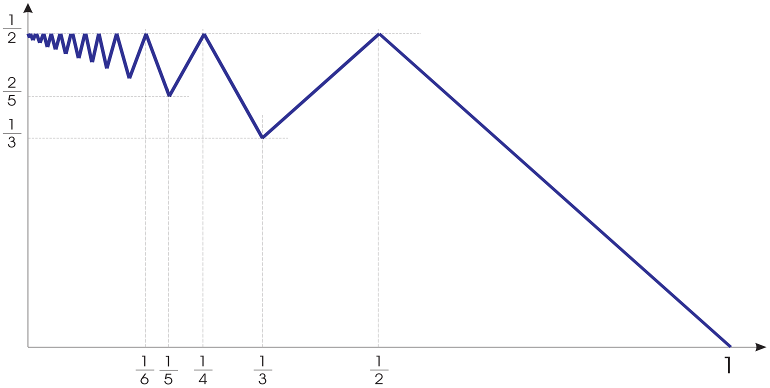
Meyniel's conjecture holds a.a.s. for random graphs except perhaps when $np = n^{1/(2k)+o(1)}$ for some $k \in \mathbb{N}$, or $np = n^{o(1)}$.



Theorem (Prałat, Wormald, 2012+)

Let $\varepsilon > 0$ *and suppose that* $d = d(n) \geq (1/2 + \varepsilon) \log n$. *Let* $G \in G(n, p)$ *with* $p = d/(n-1)$. *Then a.a.s.*

$$c_0(G) = O(\sqrt{n})\,.$$

Theorem (Prałat, Wormald, 2012+)

Fix $d = d(n) \geq 3$. *Then, a.a.s.*

$$c_0(\mathcal{G}_{n,d}) = O(\sqrt{n})\,.$$

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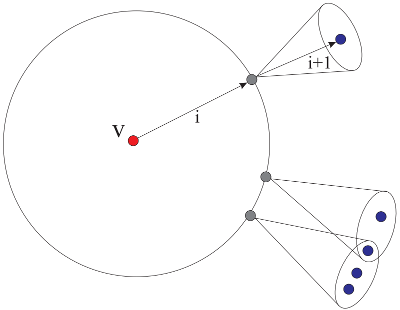
$c_0(G) = O(\sqrt{n})$ — sketch for $G(n,p)$ with $p(n-1) > \log^3 n$

$i = \max\{j : d^j \leq \sqrt{n}\}$, $C\sqrt{n}$ cops in the first team.

Case 1: $d^{i+1} \geq \sqrt{n} \log n$ — easy

Case 2: $d^{i+1} = \sqrt{n}\omega$ with $1 \leq \omega < \log n$ — we need one more (independent) team of $C\sqrt{n}$ cops.



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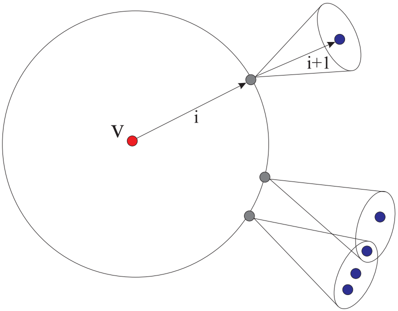

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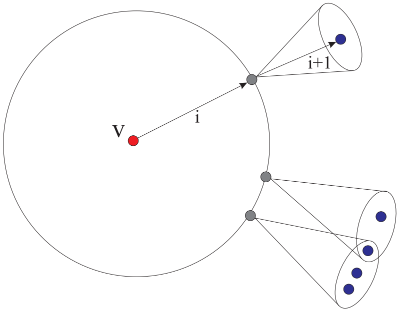

The first team 'densely covers' the sphere $S(v,i)$: $u \in S(v,i)$ is covered with probability at most

$$\left(1-\frac{C}{\sqrt{n}}\right)^{|W(u)|} \le \exp\left(-\frac{C}{\sqrt{n}}\cdot\frac{1}{2}\sqrt{n}\cdot\omega\right) = \exp\left(-\frac{C}{2}\cdot\omega\right) < \frac{1}{10\omega},$$

for C sufficiently large.

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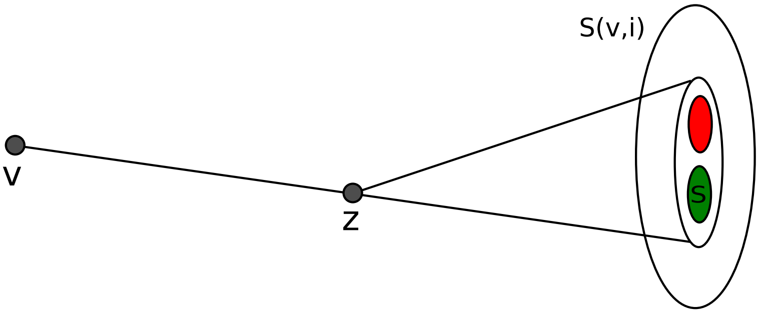
The second team of cops is released when the robber is at $z \in S(v, \lceil i/2 \rceil)$.

We may assume that she is heading to the set $S \subseteq S(z, \lfloor i/2 \rfloor) \cap S(v, i)$ that is not covered by cops from the first team.



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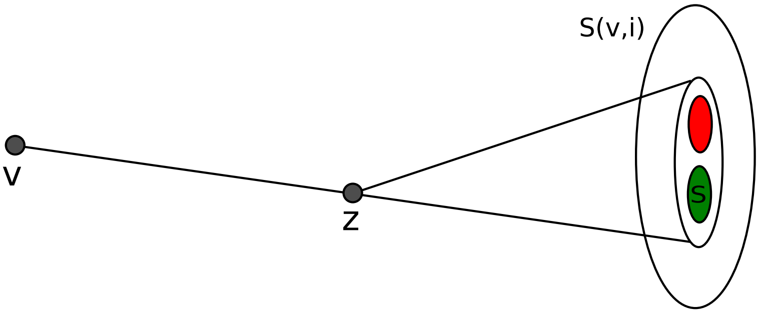
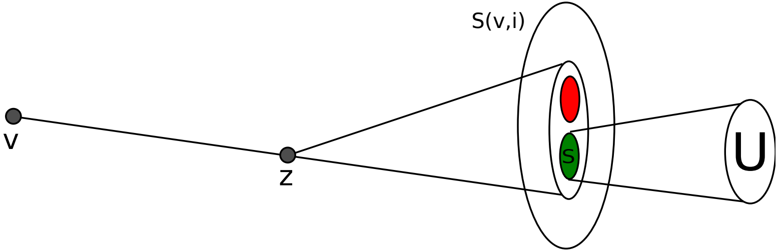


The second team has to cover the set of vertices

$$U = \bigcup_{s \in S} S(s, \lceil i/2 \rceil + 1),$$

of size at most

$$2d^{\lceil r/2 \rceil + 1}|S| \leq \frac{d^{r+1}}{2\omega} < \sqrt{n}.$$



For $u \in U$, we need to search for cops within distance $r + 2$.

sparse case is more complicated:

- $F \log \log n$ independent teams of cops
- the ith team consists of $c_i = Ce^{-i}\sqrt{n}$ cops ($\Theta(\sqrt{n})$ in total)
- $F \log \log n$ rounds
- the duration of the round is until the robber reaches a vertex v_{i+1} of $S(v_i, r_i)$ where $r_1 = \log_d(\varepsilon_0 n)/4$, and r_i (for $i \geq 2$) is defined recursively

$$\frac{\sqrt{\varepsilon_0 n}}{d} < \frac{d^{r_{i-1}+r_i}}{e^{2(i-1)}} \leq \sqrt{\varepsilon_0 n},$$

- at the start of round i ($i \geq 2$), the cops from team $(i-1)$ are already heading towards some of the vertices in the sphere $S(v_i, r_i)$; team i is released

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- at the start of round i ($i \geq 2$), the cops from team $(i-1)$ are already heading towards some of the vertices in the sphere $S(v_i, r_i)$; team i is released

sparse case is more complicated:

- $F \log\log n$ independent teams of cops
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Outline

(Random subgraph of) random geometric graph $\mathcal{G}_d(n,r,p)$
- vertex set $[n] = \{1, 2, \ldots, n\}$: n vertices are chosen uniformly at random and independently from $[0,1]^d$,
- a pair of vertices within Euclidean distance $r = r(n)$ appears as an edge with probability $p = p(n)$, independently for each such a pair.

Today, we will focus on $\mathcal{G}_d(n,r) = \mathcal{G}_d(n,r,1)$ – *(classic) random geometric graph*

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Theorem (Alon, Prałat, 2012+)

There exists an absolute constant $c > 0$ so that if $r^5 > c\frac{\log n}{n}$ then a.a.s. $c(\mathcal{G}_2(n,r)) = 1$.

Independently proved by Beveridge, Dudek, Frieze, and Müller (previous talk).

Proof is quite different and also gives the following.

Theorem (Alon, Prałat, 2012+)

For each fixed $d > 1$ there exists a constant $c_d > 0$ so that if $r^{3d-1} > c_d \frac{\log n}{n}$ then a.a.s. $c(\mathcal{G}_d(n,r)) = 1$.

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$\mathcal{G}_2(r)$: continuous (infinite) graph whose vertices are all of the points of $[0,1]^2$, where two of them are adjacent if and only if their distance is at most r.

Theorem (Known? Similar to *the Lion and the Christian*)

$c(\mathcal{G}_2(r)) = 1$ for any $r > 0$.

- the cop places himself at the center O of $[0,1]^2$
- catch the bad guy if you can; otherwise:

+ move to a point C that lies on the segment OR, making sure his distance from the robber is at least, say, $r^2/100$,

+ in each step the square of the distance between the location of the cop and O increases by at least $r^2/5$.

The game ends in at most $O(1/r^2)$ steps.

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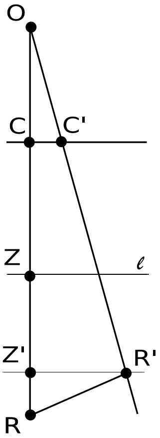


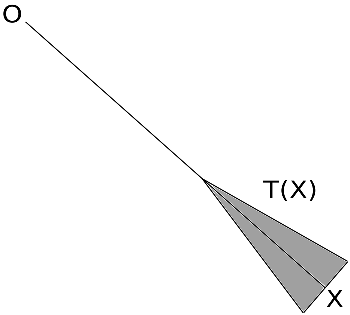
- R' is below ℓ (otherwise the game ends)
- $CC' \leq Z'R' \leq RR' \leq r$ so the cop may move to C'

Case 1: $CC' > r/2$. Go to C', and move towards O if too close to R' to make sure the distance between players is at least $r^2/100$.
The square distance increases by at least $r^2/4 - 2r^2/100 > r^2/5$.

Case 2: $CC' \leq r/2$. Go to C', and move towards R'.
Since $OC' \geq OC$, the cop increases his distance from O by more than $r/2$.
Move back, if too close to R'.

Adopting strategy for $\mathcal{G}_2(n, r)$.

Let $X \in [0,1]^d$ so that $OX \geq r/2$ and the distance from the boundary is at least $r^2/10^3$.
$T(X)$ is an isosceles triangle of height $r^2/100$ and the base of length $r^3/10^5$.



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Lemma

There exists an absolute constant $c > 0$ so that a.a.s. every triangle $T(X)$ contains a vertex of $\mathcal{G}_2(n,r)$, provided that $r^5 > c\frac{\log n}{n}$.

F - fixed collection of $O((1/r)^6)$ rectangles, each of area $\Omega(r^5)$, so that every triangle $T(X)$ fully contains at least one of these rectangles. (For example, take $10^6 r^3$ by $10^6 r^3$ grid and for each point we take the rectangle of width $r^3/10^6$ and height $r^2/10^6$ in which Y is the midpoint of the edge of length $10^6 r^3$ and the other edge is in direction YO.)

A.a.s. each rectangle in F contains at least one vertex of $\mathcal{G}_2(n,r)$.

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Adopting strategy for $\mathcal{G}_2(n, r)$.

The cop will follow essentially the continuos strategy, but will always place himself at a vertex of the graph which is sufficiently close to where he wants to be in the continuous variant of the game.