Cops and Robbers on Geometric Graphs

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joint work with
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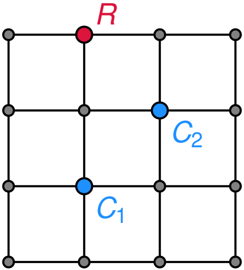
Outline

The Game of Cops and Robbers

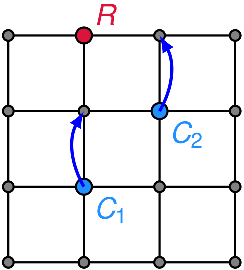
Introduced independently by Quilliot (1978), and Nowakowski & Winkler (1983).

An Example of 2 Cop Pursuit



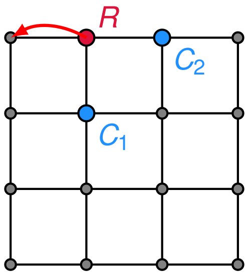
- First the cops C_1, C_2 choose their starting locations.
- Next, the robber R chooses his starting location.

2 Cop Pursuit: Turn One, Cop Move



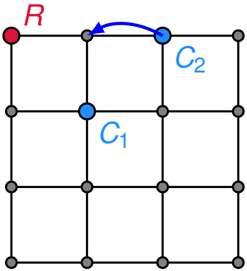
First turn: Cops move first. Each can move to an adjacent vertex (or stay where he is).

2 Cop Pursuit: Turn One, Robber Move



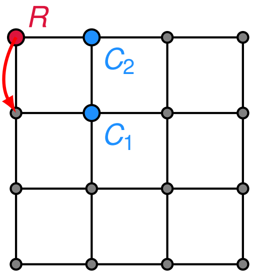
First turn: Robber moves second. He moves similarly.

2 Cop Pursuit: Turn Two, Cop Move



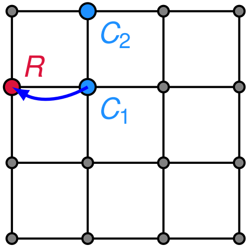
Second turn: Cops move. Here C_1 chooses to remain stationary.

2 Cop Pursuit: Turn Two, Robber Move



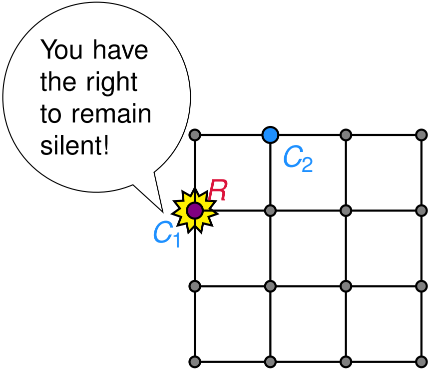
Second turn: Robber move.

2 Cop Pursuit: Turn Three, Cop Move



Third turn: Cop move.

2 Cop Pursuit: The Arrest



Third turn: Cop move, giving a Cop Win!

How to Play Cops and Robbers

Set Up:

- Chose a graph G and a positive integer k.
- Cops $C_1, C_2, \ldots, C_k$ are placed on vertices of G
- Next, the robber R is placed on a vertex of G.

A Game Turn:

- Each cop moves to an adjacent vertex, or remains in place
- Next, the robber moves similarly.

Victory Conditions:

- Cops: one cop becomes co-located with the robber
- Robber: evades capture forever

How to Study Cops and Robbers

- Pick a graph G and a number of cops k.
- Who has a winning strategy: the k cops or the robber?

Definition

The **cop number** $c(G) =$ *fewest number of cops with a winning strategy on G.*

The cop number is always defined, since $|V|$ cops trivially win.

Objective: Given a graph G, determine its cop number $c(G)$.

Cops and Robbers on Planar Graphs

Planar G has $c(G) \leq 3$

Theorem (Aigner and Fromme (1984))

If G is a planar graph then $c(G) \leq 3$.

Proof Idea:

- Cops work together to reduce the robber's free territory (vertices where R can move without being caught on the very next cop move).
- Eventually the robber is caught.

Path Guarding Lemma

Lemma (Path Guarding Lemma (Aigner and Fromme, 1984))

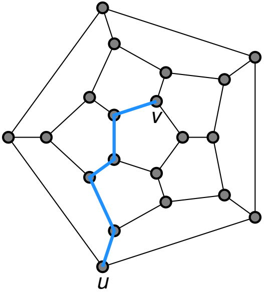
Let P be a shortest (u, v)-path on any graph G. Eventually, a cop C can position himself on P so that if R steps onto P, then he is caught on the very next cop move.

Proof Sketch:

- Cop moves onto P.
- Cop moves along P until he reaches the vertex on P closest to R.
- The path is now guarded
  - If R can beat C to a vertex on P, then there must be an even shorter (u, v)-path.

Planar Graphs have $c(G) \leq 3$

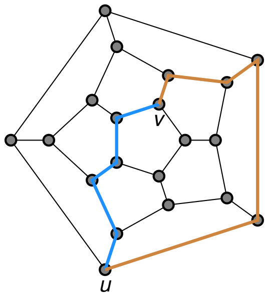
The winning 3-cop strategy for any planar graph.



First, cop C_1 takes control of a shortest (u, v)-path P_1.

Planar Graphs have $c(G) \leq 3$

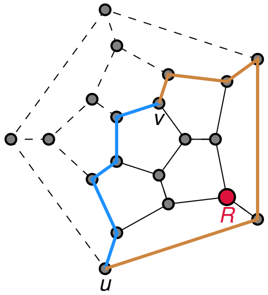
The winning 3-cop strategy for any planar graph.



Next, cop C_2 takes control of a shortest (u, v)-path P_2 in $G - P_1$.

Planar Graphs have $c(G) \leq 3$

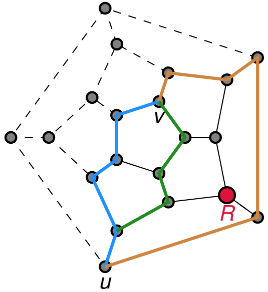
The winning 3-cop strategy for any planar graph.



Robber R is trapped inside or outside the cycle $P_1 + P_2$.

Planar Graphs have $c(G) \leq 3$

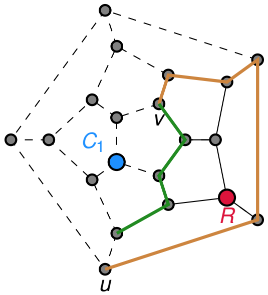
The winning 3-cop strategy for any planar graph.



Cop C_3 controls a shortest path P_3 in the robber territory of $G - P_1 - P_2$.

Planar Graphs have $c(G) \leq 3$

The winning 3-cop strategy for any planar graph.



The choice of P_3 actually frees up cop C_1, who takes control of a shortest path in the smaller robber territory inside $P_2 + P_3$.

Cops and Robbers on Geometric Graphs

Geometric Graphs

- Let $x_1, x_2, \ldots, x_n$ be points in $\mathbb{R}^2$.
- Let $r \in \mathbb{R}^+$

The **geometric graph**

$$G(x_1, x_2, \ldots, x_n; r)$$

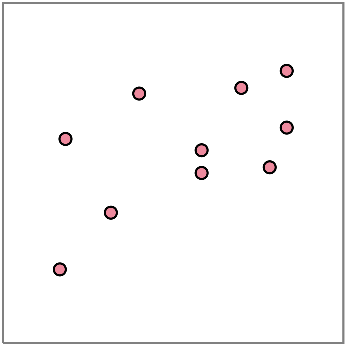
is the graph with vertices

$$V = \{x_1, x_2, \ldots, x_n\}$$

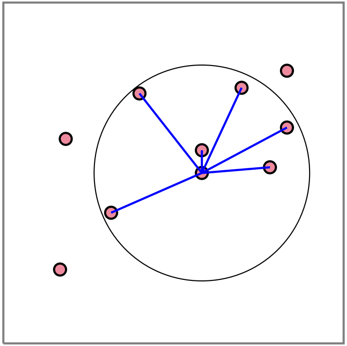
with

$$(x_i, x_j) \in E \iff \|x_i - x_j\| \leq r.$$

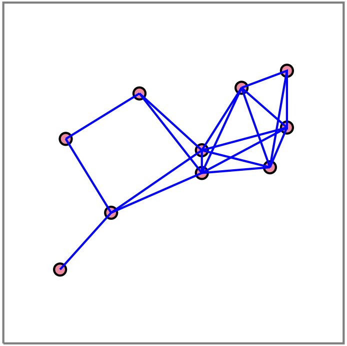
Geometric Graph Example



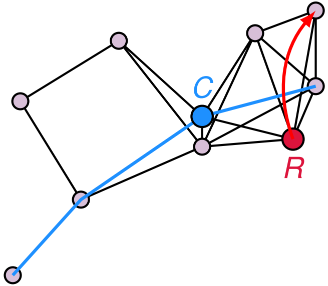
Geometric Graph Example



Geometric Graph Example



Controlling Shortest Paths on Geometric Graphs

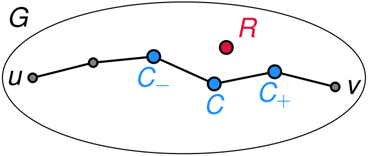


One cop controlling a shortest path of a geometric graph does not necessarily trap the robber!

Geometric Graphs have $c(G) \leq 9$

Lemma (B, Dudek, Frieze, Müller, (2012+))

Three cops moving in tandem on a shortest path can prevent the robber from crossing.

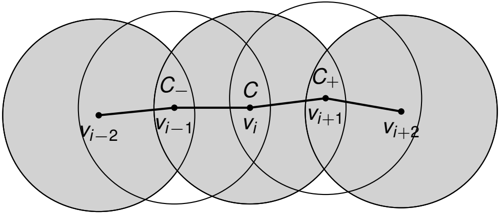


Theorem (B, Dudek, Frieze, Müller (2012+))

If G is a geometric graph then $c(G) \leq 9$.

3 Cops Patrolling a Geometric Shortest Path

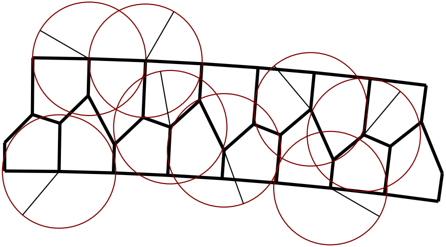
Three cops moving in tandem on a shortest path can prevent the robber from crossing.



- R must cross between v_{i-2} and v_{i+2}
- R cannot lie in the gray region $B(v_{i-2}, r) \cup B(v_i, r) \cup B(v_{i+2}, r)$.

A Geometric Graph on 1440 vertices with $c(G) = 3$

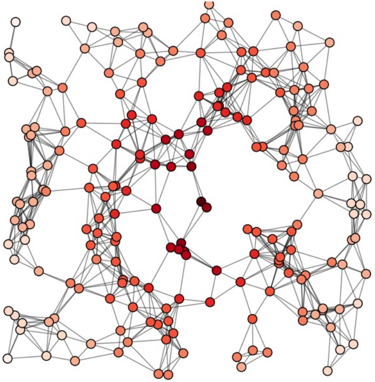
- Aigner and Fromme (1984) : A graph with minimum degree k and girth ≥ 5 has $c(G) \geq k$
- Create an annular graph with repeating interior 5-cycles. Minimum degree is 3 and the girth is 5.



$$3 \leq \max_{\text{geometric}} c(G) \leq 9$$

Random Geometric Graphs

Random Geometric Graph (RGG for short)



- Pick n random points in $[0, 1]^2$
- Consider the radius $r(n)$ as a function of n
- Study expected behavior as $n \to \infty$

RGGs with $c(G) = 1$

Theorem (BDFM (2012+))

A RGG on $[0, 1]^2$ *with*

$$r^5(n) > K_1 \frac{\log n}{n}$$

satisfies $c(G) = 1$ *whp.*

whp $\iff$ with high probability $\iff$ $\lim_{n\to\infty} \Pr[A] = 1$

Note: This result was proven independently by Alon and Prałat using a “lion’s move” strategy

Pitfall Vertex

- **neighborhood** of v

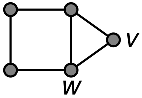
$$N(v) = \{u \in V \mid (u, v) \in E)\}$$

- **closed neighborhood** of v

$$\overline{N}(v) = N(v) \cup \{v\}$$

$v \in V$ is a **pitfall** (or **corner**) if $\exists w \in V$ such that

$$\overline{N}(v) \subseteq \overline{N}(w).$$

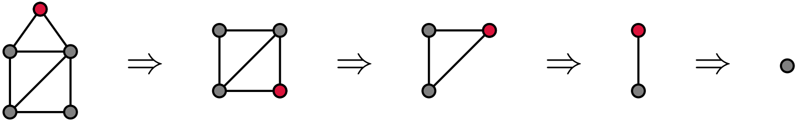


Cop Win = Dismantlable

Lemma (Quilliot, and Nowakowski & Winkler)

If v is a pitfall then $c(G) = c(G - v)$.

G is **dismantlable** if G can be reduced to a single vertex by successively removing pitfalls



Theorem (Quilliot, and Nowakowski & Winkler)

$c(G) = 1$ if and only if G is dismantlable.

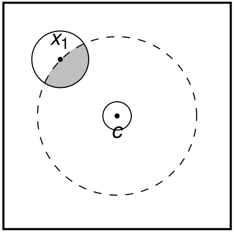
RGGs with $r^5(n) > K_1 \frac{\log n}{n}$ have $c(G) = 1$

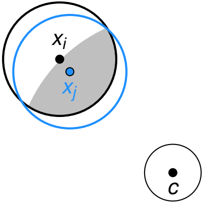
Claim: the RGG is dismantlable whp.

Let $N_c(x_i)$ = all neighbors of x_i closer to $c = (1/2, 1/2)$.

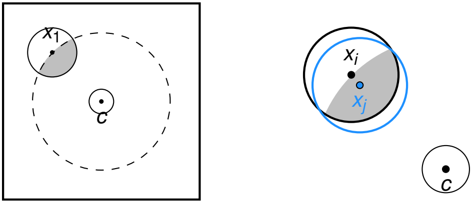
Lemma

Suppose that $r^5(n) > K_1 \frac{\log n}{n}$. Then whp, for every x_i such that $\|x_i - c\| \geq r/2$, there exists $x_j \in N_c(x_i)$ such that $N_c(x_i) \subset N(x_j)$.





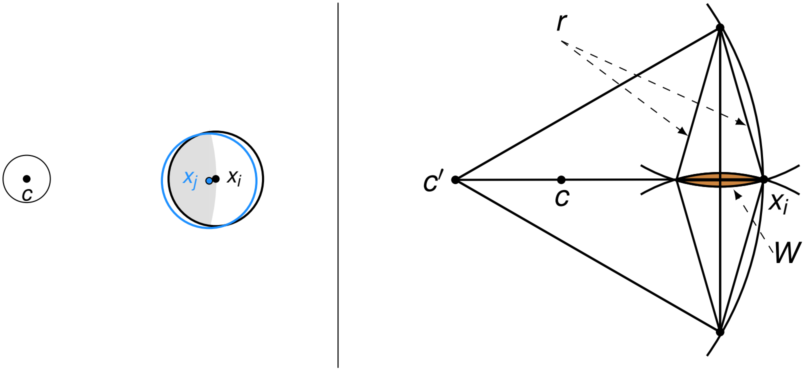
RGGs with $r^5(n) > K_1 \frac{\log n}{n}$ have $c(G) = 1$



Using this lemma, dismantle the RGG like peeling an onion.

- Order vertices by distance to c
- Furthest vertex is a pitfall by the lemma
- Repeat until only points left are within $r/2$ of c.
- These vertices form a clique, which is dismantlable. □

Why $r^5 = K_1 \log n/n$?



- If $x_j \in W$ then $N_c(x_i) \in N(x_j)$
- $\text{Area}(W) = \Omega(r^5)$

RGGs with $c(G) \leq 2$

Theorem (BDFM (2012+))

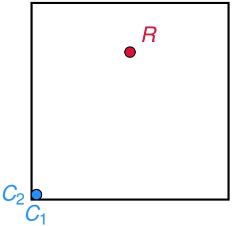
A RGG with

$$r^4(n) > K_2 \frac{\log n}{n}$$

satisfies $c(G) \leq 2$ whp.

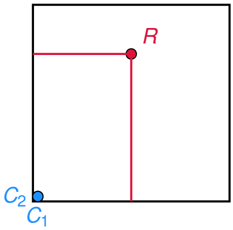
- Proof idea: two cops can eventually chase the robber into a corner of $[0,1]^2$.
- Why $r^4 > K_2 \log n / n$?
  - Enables the cop to get within $s \approx r^2$ of a target point
  - Ensures that the pursuit does not go on too long (so accumulated error is not too bad)

RGGs with $r^4(n) > K_2 \frac{\log n}{n}$ have $c(G) \leq 2$ whp



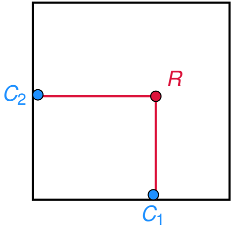
- Cops C_1, C_2 start near $(0, 0)$
- The **robber lines** L_1, L_2 are the vertical and horizontal lines through R

RGGs with $r^4(n) > K_2 \frac{\log n}{n}$ have $c(G) \leq 2$ whp



- Cops C_1, C_2 start near $(0, 0)$
- The **robber lines** L_1, L_2 are the vertical and horizontal lines through R
- **Phase 1:** The cops get within $s := 5(\log n/n)^{1/2}$ of L_1, L_2 near the axes.

RGGs with $r^4(n) > K_2 \frac{\log n}{n}$ have $c(G) \leq 2$ whp



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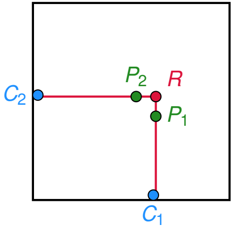
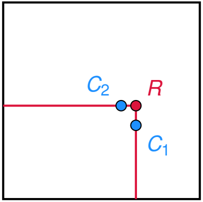


- **Phase 1:** The cops get within $s := 5(\log n/n)^{1/2}$ of L_1, L_2 near the axes.
- Define P_1 (P_2) to be the points $r/3$ below (left of) R.
- **Phase 2:** The cops get within $s := 5(\log n/n)^{1/2}$ of P_1, P_2 while staying near L_1, L_2.

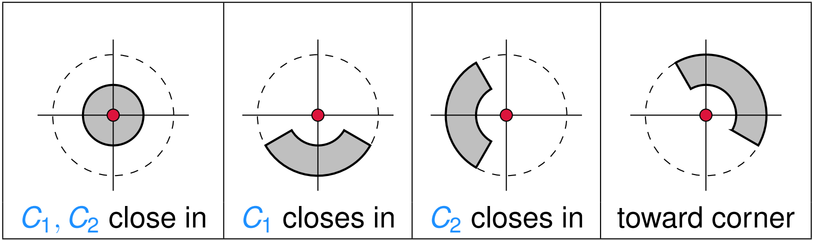
RGGs with $r^4(n) > K_2 \frac{\log n}{n}$ have $c(G) \leq 2$ whp



- **Phase 1:** The cops get within $s := 5(\log n/n)^{1/2}$ of L_1, L_2 near the axes.
- **Phase 2:** The cops get within $s := 5(\log n/n)^{1/2}$ of P_1, P_2 while staying near L_1, L_2.
- **Phase 3:** R is forced into the upper right corner by C_1, C_2 and eventually caught.

RGGs with $r^4(n) > K_2 \frac{\log n}{n}$ have $c(G) \leq 2$ whp

Why can the cops gain on R in Stage 2 and Stage 3?



This bounds the number of moves required in Stage 2 and Stage 3. $\square$

Open Problems

Summary of Results

Arbitrary geometric graphs in $[0,1]^2$

$$3 \leq \max_{\text{geometric}} c(G) \leq 9$$

Random geometric graphs in $[0,1]^2$

$$r^5 \geq K_1 \frac{logn}{n} \quad \rightarrow \quad c(G) = 1$$
$$r^4 \geq K_2 \frac{logn}{n} \quad \rightarrow \quad c(G) \leq 2$$

We have also shown:

$$r^2 \leq K_3 \frac{\log^2 n}{n} \quad \rightarrow \quad c(G) \geq 2$$

Summary and Open Problems

Arbitrary geometric graphs in $[0,1]^2$

- What is the true upper bound for $c(G)$? 9 feels too big; 6 seems optimistic.

Random geometric graphs in $[0,1]^2$

- Additional upper and lower bounds on the cop number
- Is our 2-cop result tight?