Cops and Robber on Circulant Graphs

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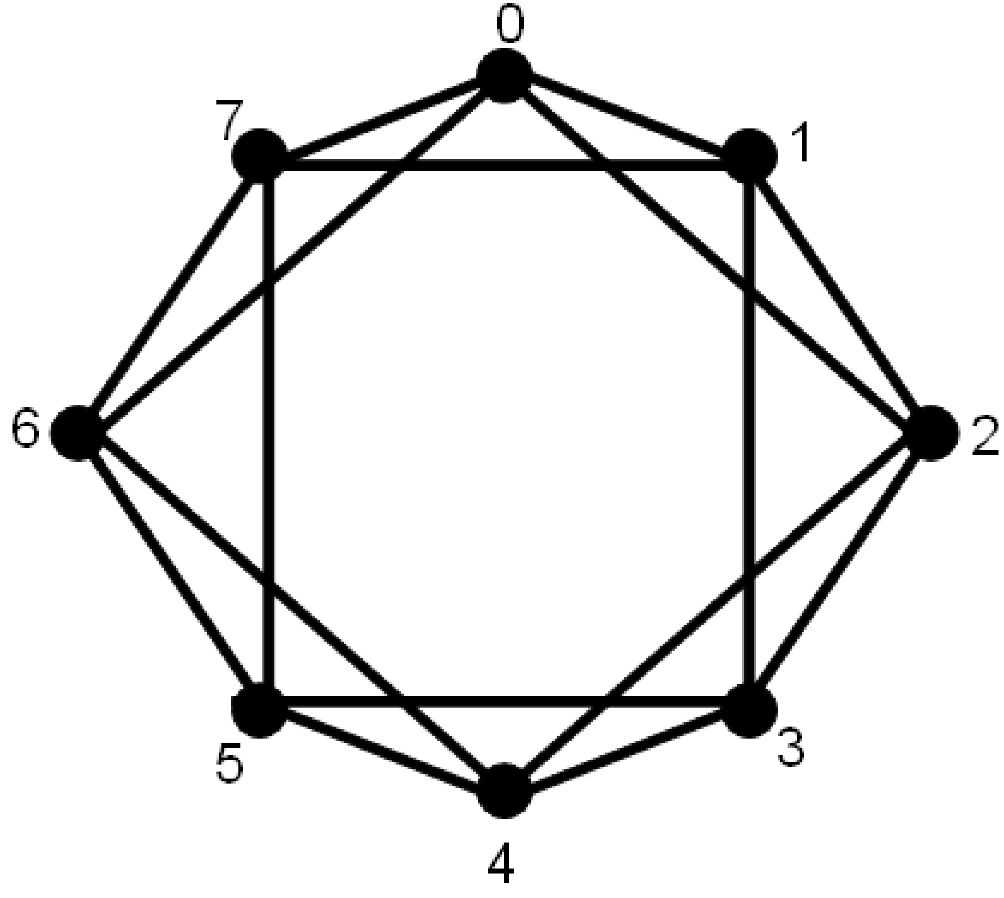
Cops and Robber

- Played on a finite, simple graph
- Cops have perfect information
- Two players alternate moves
- On Cops’ move, some subset of cops (possibly empty) can each move to an adjacent vertex
- On Robber’s move, the Robber can move to an adjacent vertex or pass

Circulant Graphs

- $C_{n;k_1,k_2,\ldots,k_m}$
- Vertex Set {0,1,2,…, *n*-1}
- Edge from vertex *i* to $i + k_j$ mod *n* for each *i* = 0,1,2,…, *n*-1 and *j* =1,2, …, m

$C_{8;1,2}$



$c(C_{n;m,k}) \leq 3$ whenever $\gcd(n,m,k)=1$

On a Pursuit Game on Cayley Graphs

P. Frankl (1987)

- H is an abelian group, S is a subset of H such that $S = S^{-1}$
- $C(H,S)$ is the graph with vertex set H and edge set $\{(h,hs) \mid h \text{ in } H, s \text{ in } S\}$
- At most $\left\lceil \frac{|S|+1}{2} \right\rceil$ cops are required to win on connected Cayley graph

$C_{n;l,k}$ with gcd(n,l+k)=1

Cops C_1 and C_2 on vertex 0

Robber on vertex r = -j(l+k) mod n

The cops respond to the robber's moves as follows

R	C_1	C_2
+m	-k	+m
+k	-m	+k
-m	-m	+k
-k	-k	+m
0	0	0

Suppose at some point in the game R has made j moves from {+m,+k}

R on -j(m+k) + im+(j-i)k + #(-m) + #(-k)

= -ki + (i-j)m + #(-m) + #(-k)

C_1 on i(-k) + (j-i)(-m) + #(-m) + #(-k)

C_1 has captured R

$c(C_{n;m,k}) \leq 3$ whenever $\gcd(n,m,k)=1$

C_1 wins if R makes enough moves from {+m,+k}

C_2 wins if R makes enough moves from {-m,-k}

C_3 forces one of the previous situations

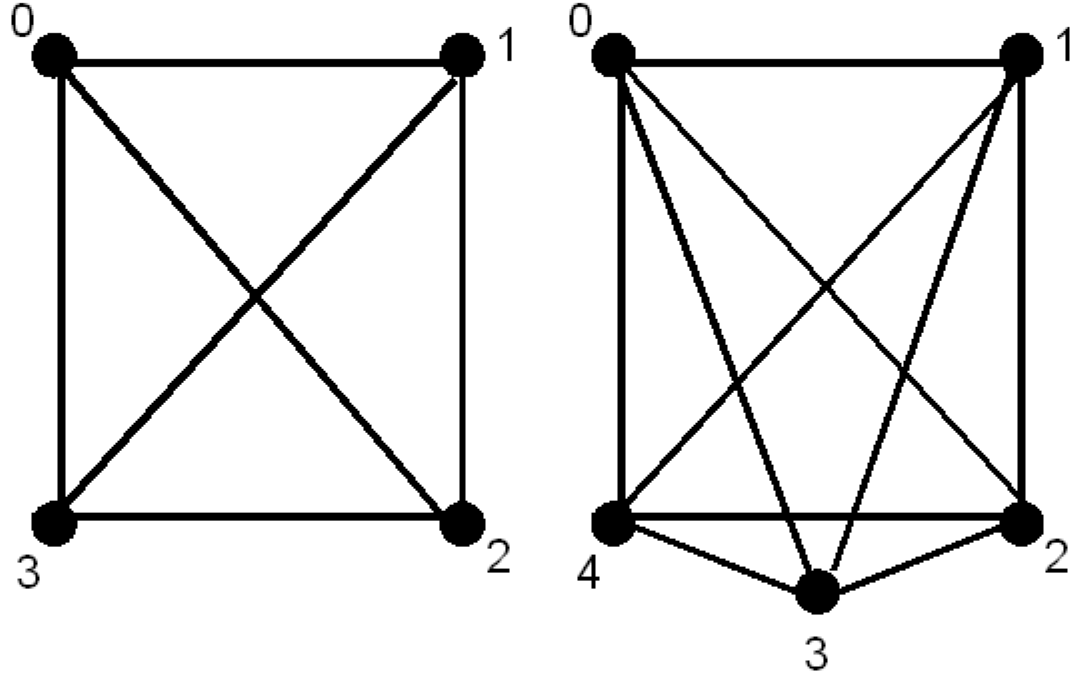
$c(C_{n;k_1,k_2,\ldots,k_m})=1$ if and only if $C_{n;k_1,k_2,\ldots,k_m}$ is complete

If $C_{n;k_1,k_2,\ldots,k_m}$ is not complete then for every vertex v, there is a vertex w such that v and w are not adjacent.

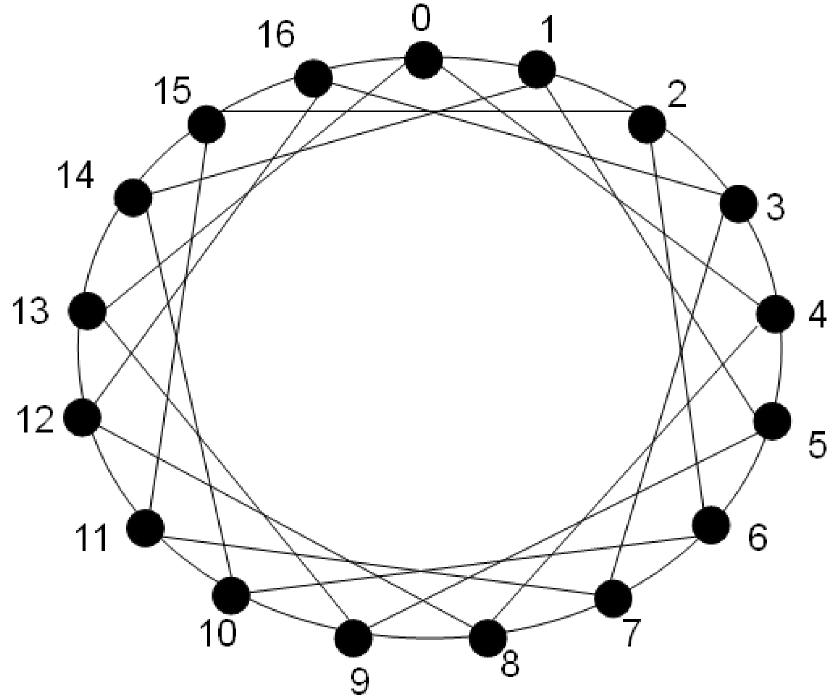
Cop chooses a vertex, v, Robber chooses a non adjacent vertex, w.

Robber's strategy is to now mirror the Cop's moves. If Cop moves from v to v+j, Robber moves from w to w+j.

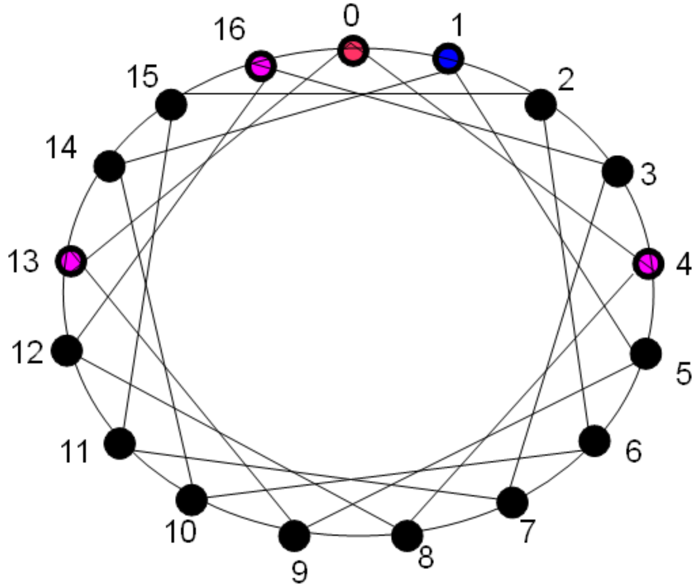
$c(C_{n;m,k})=1$ if and only if $C_{n;m,k}$ *is isomorphic to* $C_{4;1,2}$ *or* $C_{5;1,2}$



There exist connected $C_{n;m,k}$ such that $c(C_{n;m,k})=3$



$c(C_{17;1,4})=3$

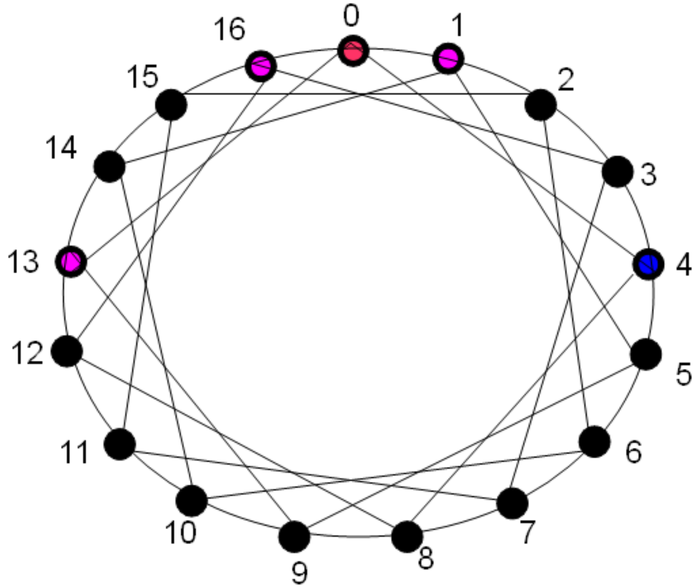


Suppose 2 cops could win.

Consider cops' final move

- R at 0
- C_1 at 1
- Impossible for C_2 to "cover" R's other neighbours

$c(C_{17;1,4})=3$



- Suppose 2 cops could win.
- Consider cops' final move
  - R at 0
  - C_1 at 4
  - Impossible for C_2 to “cover” R's other neighbours

Question: For what *m*, *k* and *n* with gcd(m,k,n)=1 is $c(C_{n;m,k}) = 2$?

Note: If gcd(n,m)=1 or gcd(n,k)=1, then $C_{n;m,k}$ *is isomorphic to* $C_{n;1,b}$ *for some b*.

$c(C_{n;1,2}) = 2$

$c(C_{n;1,3}) = 2$

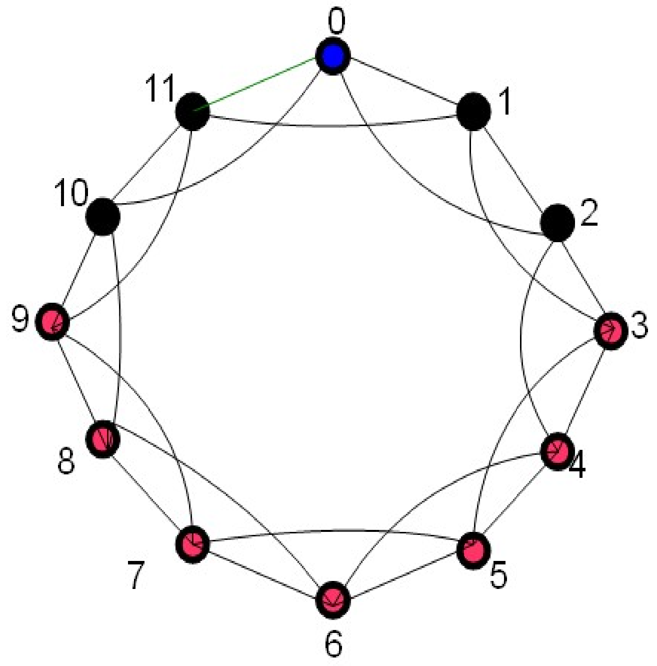
$c(C_{2k;1,k}) = 2$ (*Frankl*)

$c(C_{2k+2;1,k}) = 2$

$c(C_{n;1,2}) = 2$

Cops C_1 and C_2 on 0

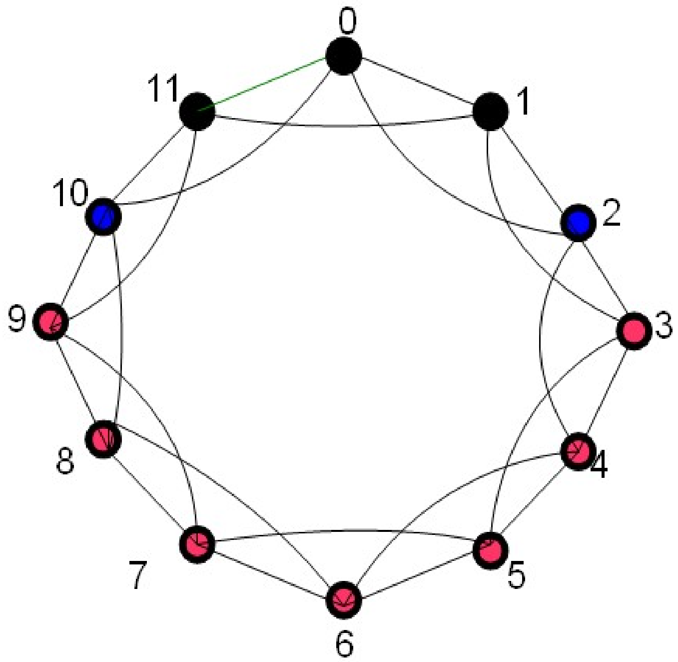
Assume R on vertex in {3, 4, ... , n-3}



$c(C_{n;1,2}) = 2$

C_1 moves to 2

C_2 move to n-2

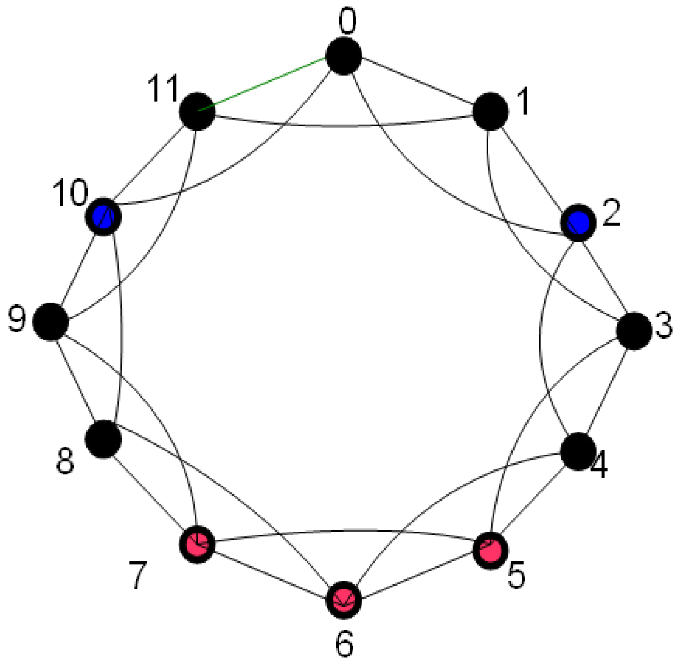


$c(C_{n;1,2}) = 2$

C_1 on *2*

C_2 on n-2

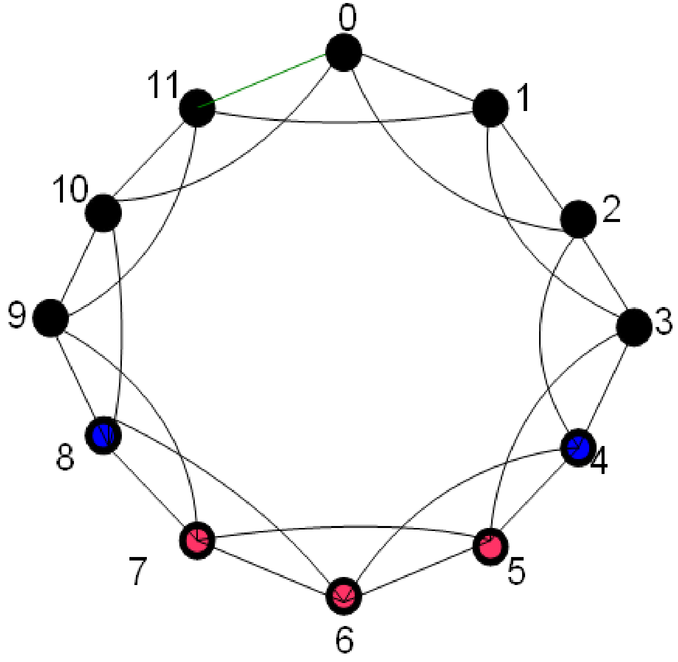
R moves to vertex in {5,6, ... n-5}



$c(C_{n;1,2}) = 2$

C_1 moves to 4

C_2 moves to n-4

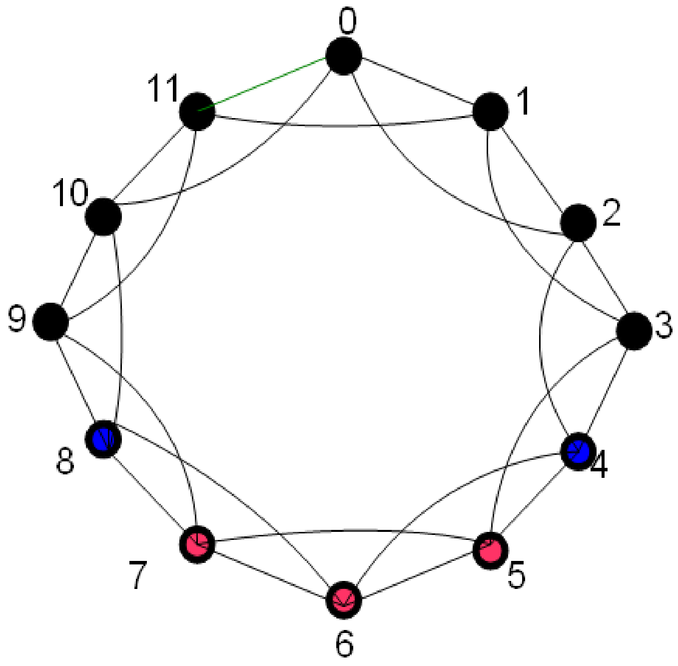


$c(C_{n;1,2}) = 2$

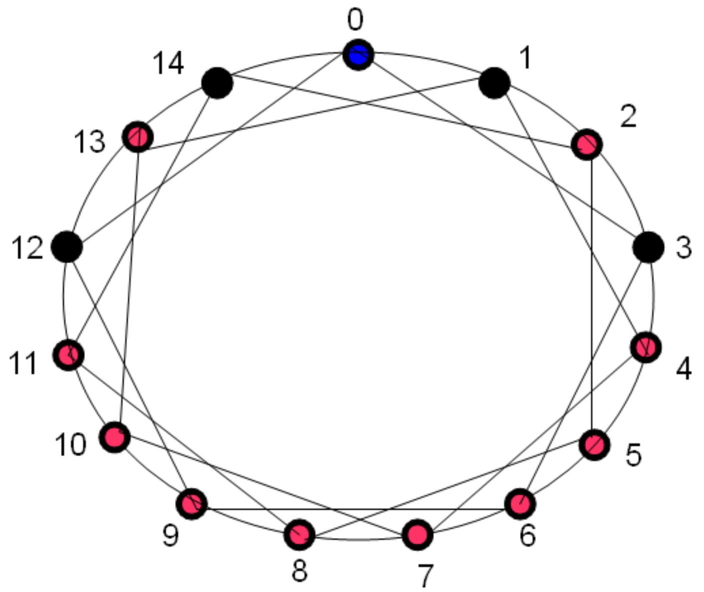
C_1 moves to 4

C_2 on vertex n-4

R on vertex in {7,8,...,n-7}



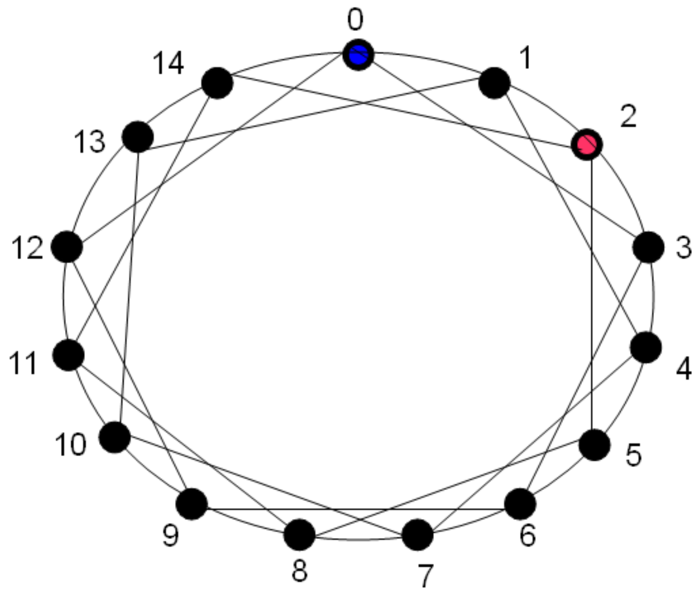
$c(C_{n;1,3}) = 2$



Cops start on 0

R on vertex in {2, 4, 5, 6, ...,n-5, n-4, n-2}

$c(C_{n;1,3}) = 2$

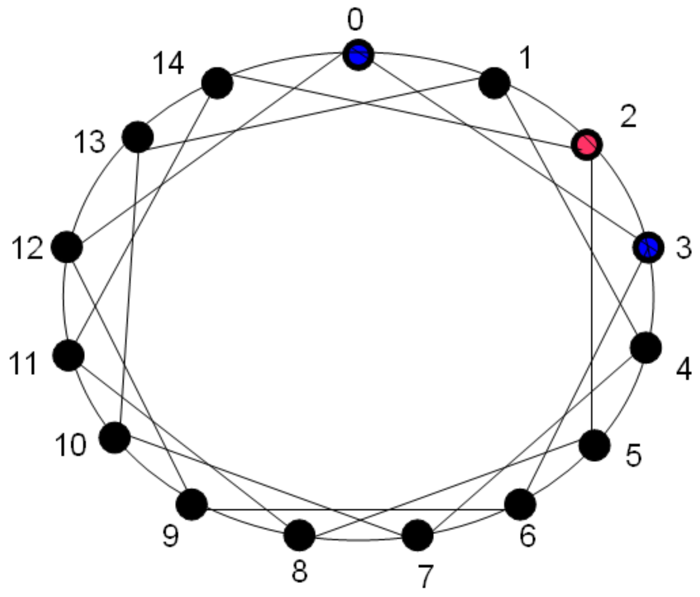


R on 2

C_1 moves to 3

C_2 passes

$c(C_{n;1,3}) = 2$



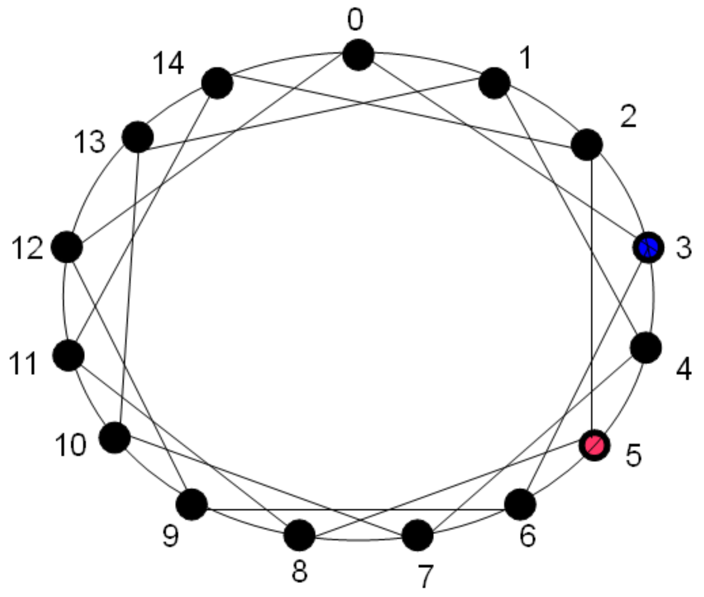
R on 2

C_1 passes

C_2 moves to 3

R forced to move to 5

$c(C_{n;1,3}) = 2$

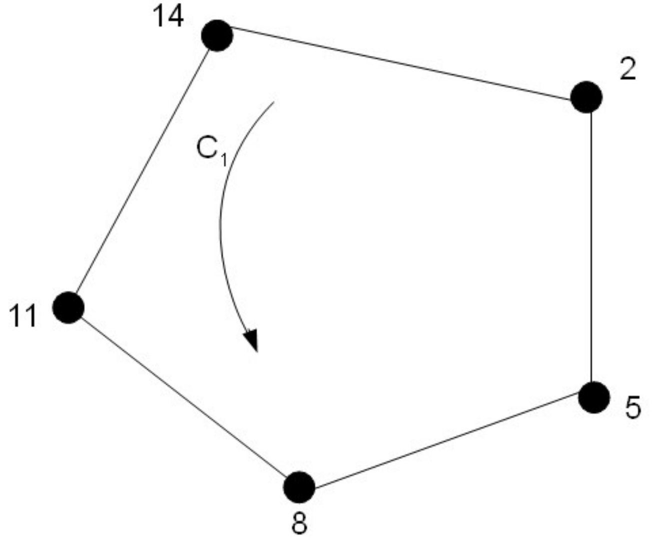


From this point on, R will be forced to either use a +3 edge or pass.

C_2 will imitate R

In this case, that will keep R restricted to the subgraph induced by {5+3h}, moving clockwise only.

$c(C_{n;1,3}) = 2$



C_1 moves onto subgraph (cycle) to which R is now confined.

C_1 moves counter-clockwise on every move.

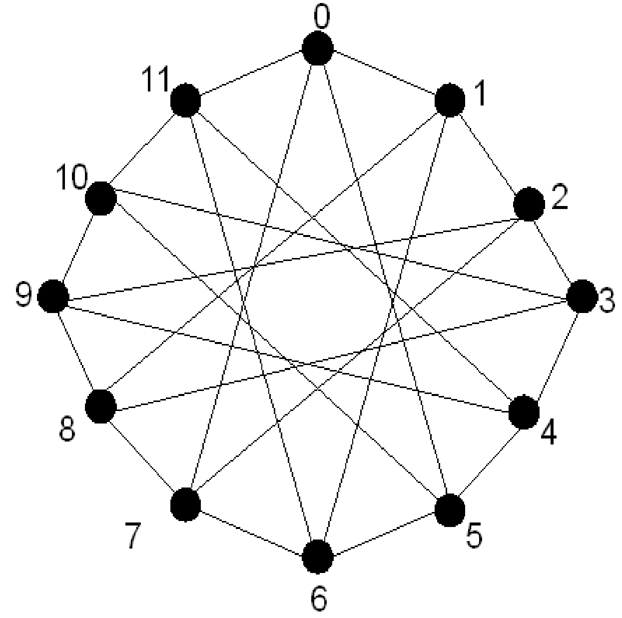
Eventually captures R

$c(C_{2k+2;1,k}) = 2$

A vertex *v* is an **open corner** if there is some vertex *y* such that N(v) is a subset of N[y].

A graph G is tandem-win if and only if G-*v* is tandem-win for open corner *v*.

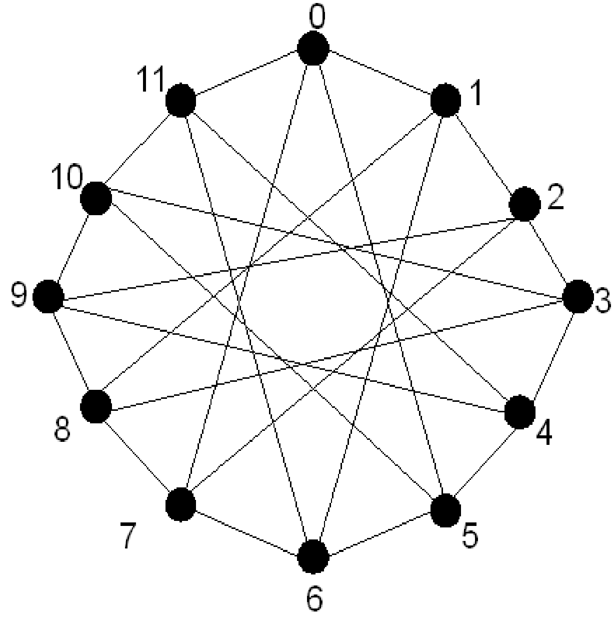
Clarke and Nowakowski (2005)



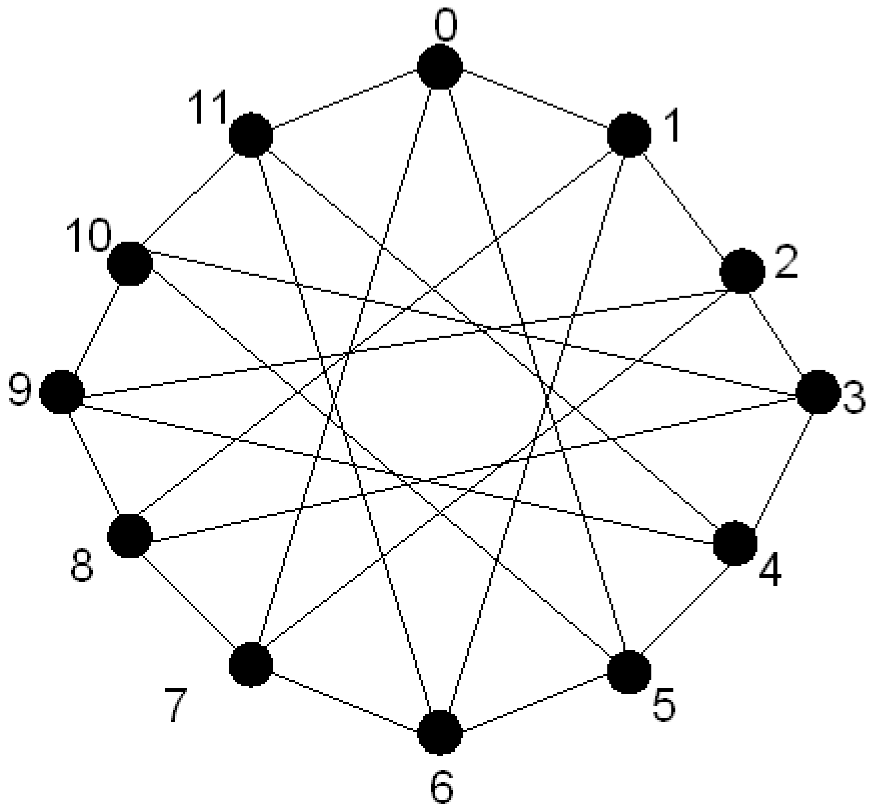
$c(C_{2k+2;1,k}) = 2$

Suppose G has an open corner v and c(G-v) = 2. Then c(G) = 2.

Suppose open corners are successively removed from G so that the resulting graph G' satisfies c(G')≤2, then c(G) ≤2.

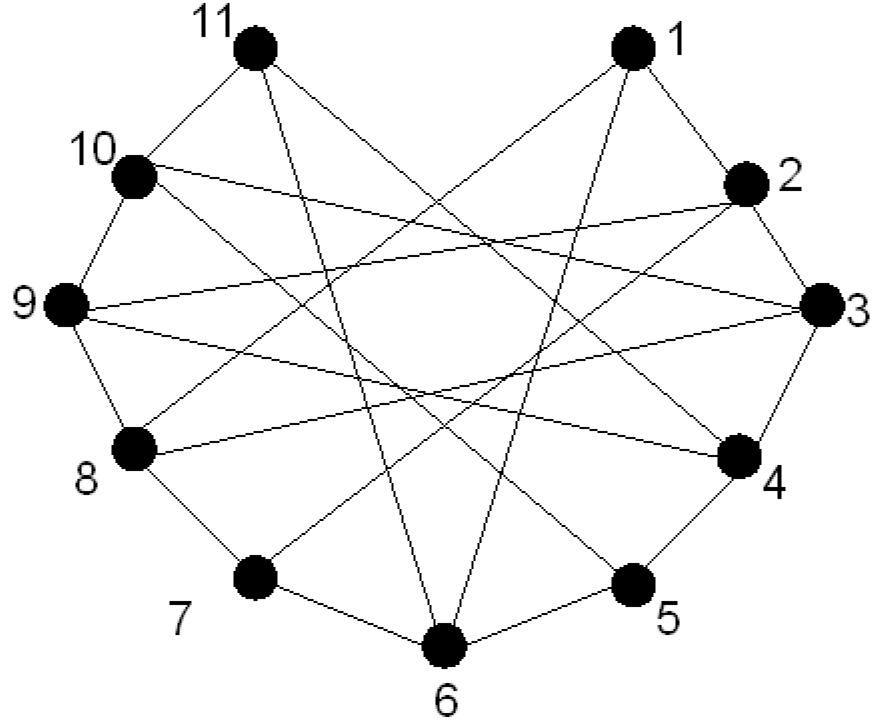


$c(C_{12;1,5}) = 2$



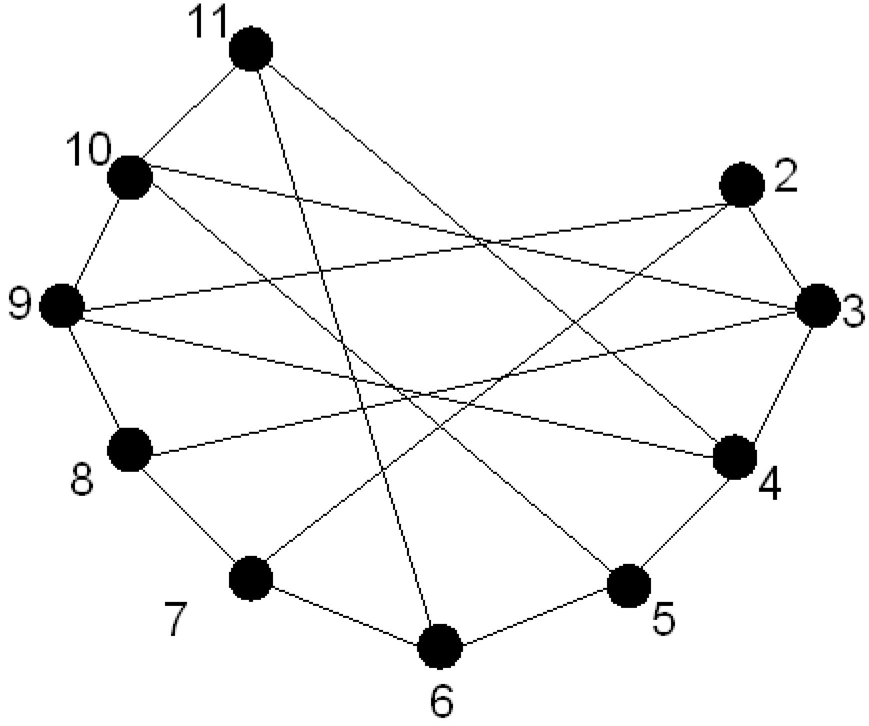
0 is an o-corner

$c(C_{12;1,5}) = 2$



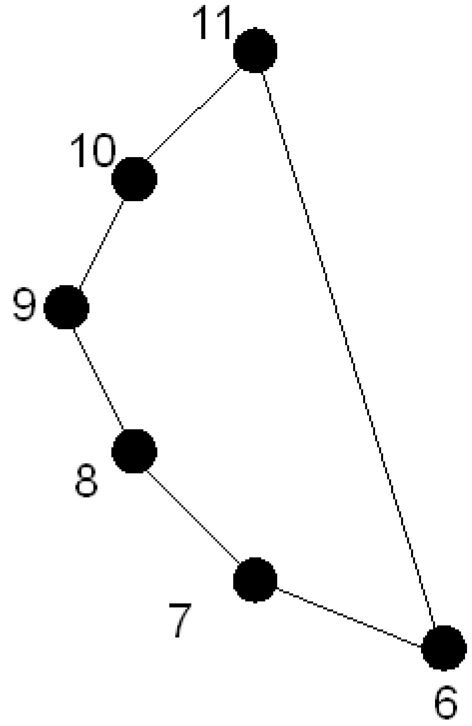
1 is an o-corner

$c(C_{12;1,5}) = 2$



2 is an o-corner

$c(C_{12;1,5}) = 2$



Repeatedly removing o-corners results in a cycle with copnumber 2.

“Theorem”

Suppose gcd(n,k)=1 or gcd(n,m)=1. Then $c(C_{n;m,k}) = 2$ if and only if $C_{n;m,k}$ is isomorphic to one of

$$C_{n;1,2}$$

$$C_{n;1,3}$$

$$C_{2k;1,k}$$

$$C_{2k+2;1,k}$$

Thank-you