# Cops and Robbers on Graphs Based on Designs

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Joint work with Anthony Bonato

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Known Meyniel extremal families of graphs arise from:

- Incidence graphs of projective planes.
- Incidence graphs of affine planes with $k \geq 0$ parallel classes removed. (Baird and Bonato, 2012+)

# Graphs we have studied

1. Polarity graphs
2. $t$-orbit graphs
3. Incidence graphs of:
   - balanced incomplete block designs (projective and affine planes, oval designs, Denniston designs)
   - group divisible designs (transversal designs, truncated transversal designs)
4. $m$-subset incidence graphs of $t$-designs
5. block intersection graphs
6. point graphs of partial geometries

(Red indicates Meyniel extremal families arise.)

# Tools

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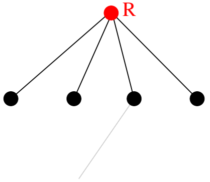

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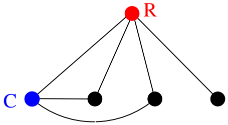

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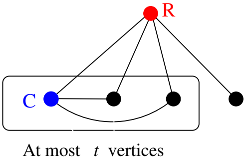

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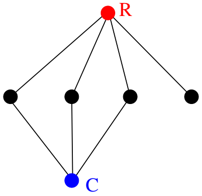

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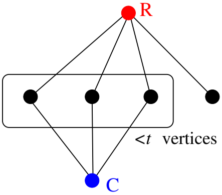


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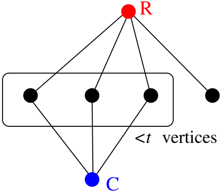


- If there are less than $\delta/t$ cops, the number of guarded neighbours is less than $(\delta/t)t = \delta$.
- So the robber is guaranteed an escape.

# Tools

Corollary

*If $G$ is $C_4$-free, then $c(G) \geq \delta(G)/2$.*

# Meyniel's conjecture and graphs of diameter 2

Theorem (Lu and Peng, 2012+)

*If $G$ has order $n$ and diameter 2, then $c(G) \leq 2\sqrt{n} - 1$.*

# Meyniel's conjecture and graphs of diameter 2

### Theorem (Lu and Peng, 2012+)

*If $G$ has order $n$ and diameter 2, then $c(G) \leq 2\sqrt{n} - 1$.*

### Question

*Is there an infinite family of graphs of diameter 2 with cop number $c\sqrt{n}$ for some constant $c$?*

# Polarity graphs

## Definition

Suppose $\mathrm{PG}(2, q)$ has point set $P$ and lines $L$. A **polarity** $\pi : P \to L$ is a bijection such that for all $p_1, p_2 \in P$, $p_1 \in \pi(p_2)$ if and only if $p_2 \in \pi(p_1)$.

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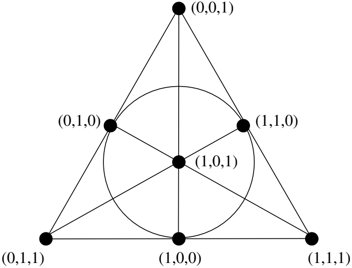

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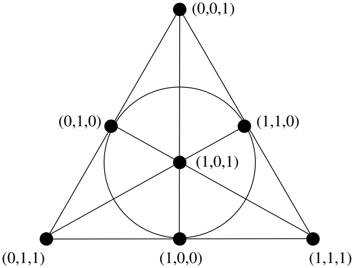


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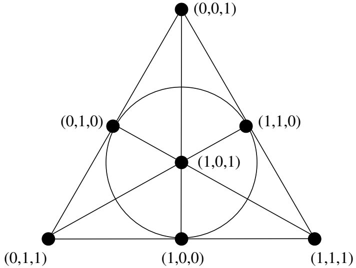


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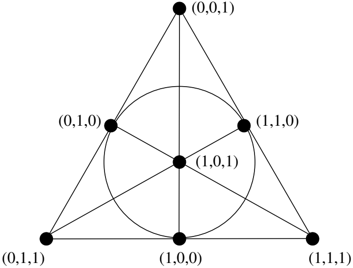


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This is the **orthogonal polarity**: a point is mapped to its orthogonal complement.

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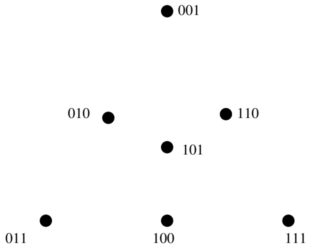

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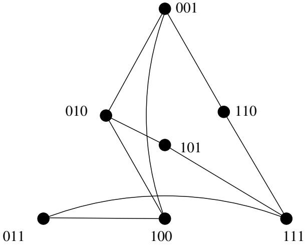

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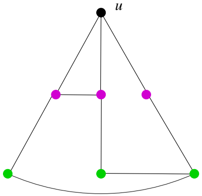
- is $(q, q+1)$-regular.
- is $C_4$-free.
- has diameter 2.
- has unbounded chromatic number as $q \to \infty$ (Godsil, Newman, 2008).

## Theorem

*If $G_q$ is a polarity graph of a $\mathrm{PG}(2,q)$, then $q/2 \leq c(G_q) \leq q+1$*



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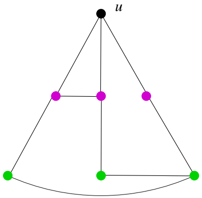


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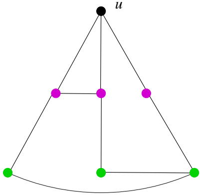


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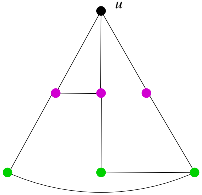


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(Fill in non-prime power orders by adding corners and using number theory.)

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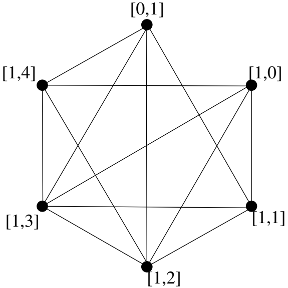
- Let $H = \{1, h, \ldots, h^{t-1}\}$.
- Form a graph $G$ as follows. Vertices of $G$ are the $t$-element orbits of $(\mathrm{GF}(q) \times \mathrm{GF}(q)) \setminus \{(0,0)\}$ under the action of multiplication by powers of $h$. Join $[a, b]$ and $[x, y]$ by an edge if $ax + by \in H$.

## Example

In $\mathrm{GF}(5)$, with $t = 4$, $h = 2$, $H = \{1, 2, 4, 3\}$

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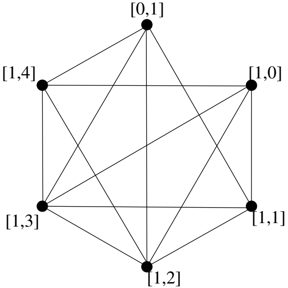


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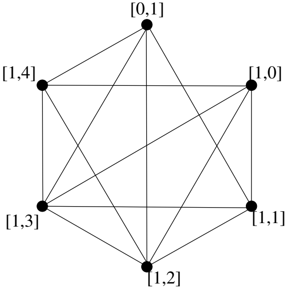


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Since the order is $(q^2 - 1)/t$, we get Meyniel extremal.

# Incidence graphs

## Definition

Let $(V, \mathcal{B})$ be an incidence structure (block design). Its **incidence graph** is the bipartite graph with vertex set $V \cup \mathcal{B}$ such that there is an edge between $x \in V$ and $B \in \mathcal{B}$ if and only if $x \in B$.

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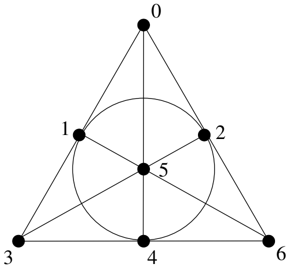


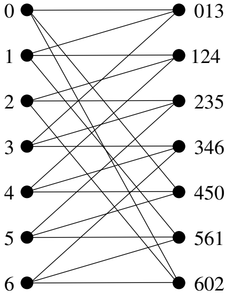

It is known that Meyniel extremal families can be derived from incidence graphs of:

- projective planes.
- affine planes with a fixed number of parallel classes removed (Baird and Bonato, 2012+)

# Balanced Incomplete Block Designs

### Definition

A $\mathbf{BIBD(v, k, \lambda)}$ is a pair $(V, \mathcal{B})$, where $V$ is a set of $v$ points, and $\mathcal{B}$ is a set of $k$-subsets of $V$, called blocks, such that each pair of points is contained in exactly $\lambda$ blocks.

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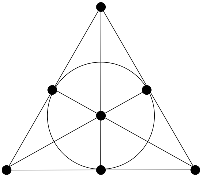
A projective plane of order $q$ is a $\mathrm{BIBD}(q^2+q+1,q+1,1)$, and an affine plane of order $q$ is a $\mathrm{BIBD}(q^2,q,1)$.



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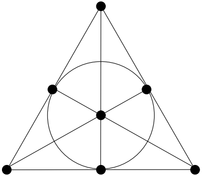

BIBD(7,3,1)

The **replication number** $r$ of a BIBD is the number of blocks containing a given point. Note: $r = \lambda(v - 1)/(k - 1)$.

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**Idea of proof. $\mathbf{c(G) \geq k}$ :** $G$ has girth at least 6, so

$$c(G) \geq \delta(G) = \min\{k, r\} = k.$$

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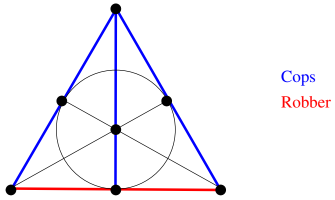

The incidence graph of a $\mathrm{BIBD}(v, k, 1)$ has order

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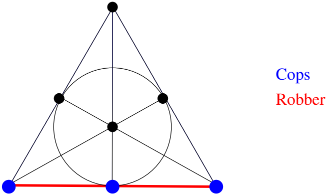

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- *An oval design, i.e.* $\mathrm{BIBD}(2^{i-1}(2^i-1), 2^{i-1}, 1)$ *(Bose and Shrikhande, 1960)*
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Oval designs and Denniston designs (with $j = i + \alpha$) give new Meyniel extremal families.

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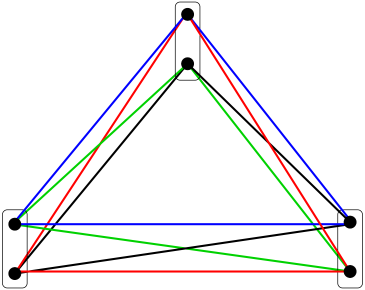
Example

A $\mathrm{TD}(3, 2)$.



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$$c(G) \geq \min\left\{k, \frac{(m-1)n}{k-1}\right\}.$$

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- So the families of graphs involved are those studied by Baird and Bonato.
- But we now know their exact cop number.

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A **truncated transversal design**, $\mathbf{TTD(k, n, u)}$ is a $\{k, k+1\}$-GDD with $k$ parts of size $n$ and of size $u$, such that each point in the group of size $u$ appears only in blocks of size $k+1$.

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*If $n$ is a prime power, $3 \leq k \leq n$ and $1 \leq u \leq n$, then there exists a $\mathrm{TTD}(k, n, u)$.*

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### Theorem

*If $G$ is the incidence graph of a $\mathrm{TTD}(k, n, u)$, then $\min\{k, n\} \leq c(G) \leq \min\{k+1, n\}$.*

$G$ has order $n^2 + kn + u$, so we obtain Meyniel extremal families from $\mathrm{TTD}(n, n, u)$ and $\mathrm{TTD}(n-\alpha, n, u)$ for fixed $\alpha$ and $u$.

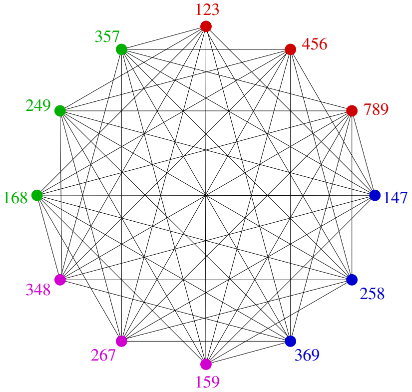
# Block intersection graphs

## Definition

Let $(V, \mathcal{B})$ be a block design. Its **block intersection graph** has vertex set $\mathcal{B}$, with blocks $B_1$ and $B_2$ adjacent if and only if $B_1 \cap B_2 \neq \emptyset$.

## Example

| | | | |
|---|---|---|---|
| 123 | 147 | 159 | 168 |
| 456 | 258 | 267 | 249 |
| 789 | 369 | 348 | 357 |



**Theorem**

*Let $G$ be the block intersection graph of a $\mathrm{BIBD}(v, k, 1)$. Then $c(G) \leq k$. Moreover, if $v > k(k-1)^2 + 1$, then $c(G) = k$.*

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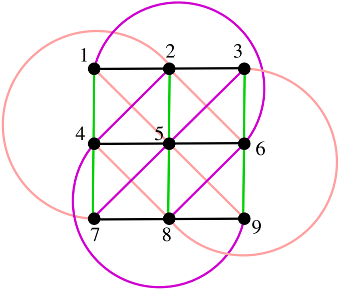


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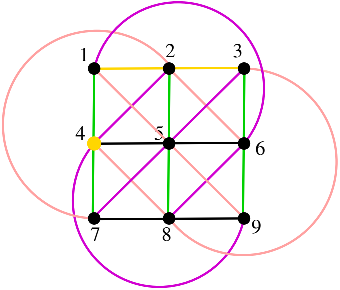


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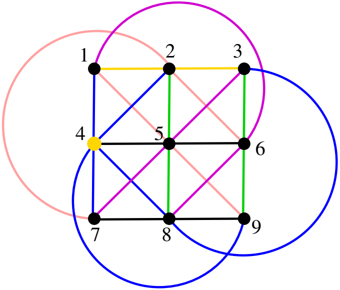


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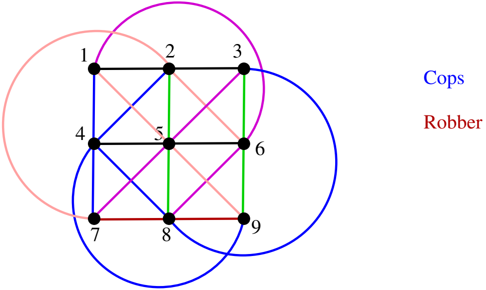

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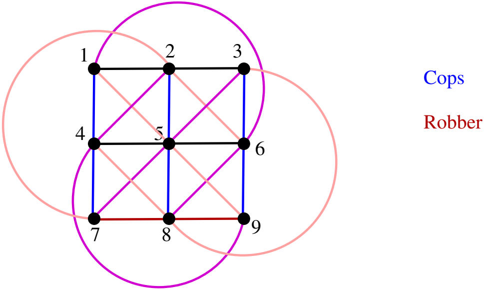

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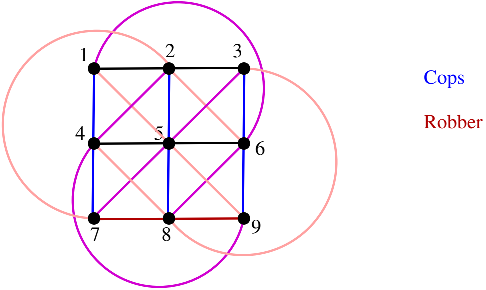
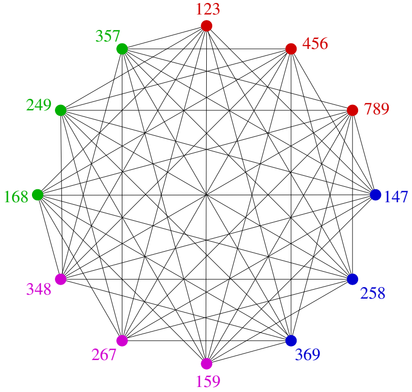



The lower bound is by a counting argument.

## Lemma

- *The block intersection graph of a projective plane is complete, and so has cop number 1.*
- *The block intersection graph of an affine plane is complete multipartite, and so has cop number 2.*



# Other questions:

- Are there other Meyniel extremal families based on designs? Not based on designs?
- Other graphs based on designs
- Other designs
- $t$-designs
- Higher index