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Author: S. Hross, P. Olivares, R. Zagst.

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Tail approximations in credit portfolios using Large Deviations techniques

Sven Hroß, Pablo Olivares and Rudi Zagst

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Abstract

In this paper, it is analyzed how Large Deviations (LD) techniques can be used for practical credit portfolio management. Applications include the internal risk management for a large credit portfolio, or the overall need to meet external requirements imposed by Basel II. For this purpose the paper provides fast and reliable methods for the computation of Value at Risk (VaR) and Conditional Value at Risk (CVaR) in general factor models using LD. Recovery rate (RR) is modelled as a random variable, depending on the state of the economy, using different models from the literature. The applicability of LD is shown, above mentioned risk measures are calculated and compared to the case of deterministic RR, demonstrating the need of an adequate modelling of RRs. Optimal portfolios with regard to either VaR or CVaR under portfolio constraints are derived, showing that LD techniques can outperform Monte Carlo (MC) simulations with regard to computational effort without a significant lack of accuracy.

1 Introduction

External regulatory frameworks, like Basel I and Basel II, ask banks to meet certain requirements. Thereby, special emphasis is put on large losses and how much capital banks have to put aside to guard against severe financial and operational risks (see e.g. Basel Committee on Banking Supervision (2006)).

This paper provides LD approximations of the tail distribution of financial losses on a portfolio consisting of many positions such as, e.g., credit portfolios managed by big banks. It offers insights into the LD approximation of the tail of such credit portfolios. Dependence among obligor defaults is modelled using general factor models. LD techniques are used to compute risk measures, respectively VaR and CVaR. Here, VaR is the loss, which is only exceeded with a certain probability in a certain time horizon and CVaR is the expected loss, given that the loss is greater or equal than the VaR. Different types of companies are taken into account in the portfolio and the RR is modelled as a random variable under different models. Moreover, correlation between the

state of the market, respectively obligor defaults, and the RR is incorporated in all the models. Finally, portfolio optimization with respect to VaR and CVaR is studied.

The paper puts emphasis on the modelling of RR, as a deterministic constant introduces an underestimation of risk measures VaR and CVaR. Several studies state a negative correlation between default and RRs (see e.g. Altman *et al.* (2005), Cantor *et al.* (2003) and Höcht and Zagst (2010)). In the context of banking supervision, some experts already suggest to estimate loss given default, defined as $LGD=1-RR$, depending on the state of the economy, to better account for recovery risk (see Basel Committee on Banking Supervision (2004)). The intention is that the RR should be higher (lower), if the economy is in a good (bad) state.

Five different models are considered to describe RRs as a random variable, depending on the state of the economy. More precisely, RRs will be modelled as normally-distributed (see Frye (2000b)), lognormally-distributed (see Pykhtin (2003) and Düllmann and Trapp (2004)), beta-distributed (see Frye (2000a) and Miu and Ozdemir (2006)) and Kumaraswamy-distributed random variables (see Höcht and Zagst (2009)). Moreover, RRs are modelled using a logistic function for transformation (see Düllmann and Trapp (2004)).

In the paper it is shown that a LD technique is applicable to all of those models. LD turns out to be computationally more efficient than MC simulations without a considerable loss in accuracy. Results are compared to the case of deterministic and totally random RR. The results show that stochastic recovery depending on the state of the economy has indeed a dramatic impact on risk measures. Furthermore, it is shown that LD techniques can be used for portfolio optimization. Two optimization frameworks under realistic portfolio constraints are presented taking into account VaR and CVaR. Optimal portfolios are derived for an inhomogeneous loan portfolio in an illustrative example.

Several studies use LD techniques for tail approximations in the framework of credit portfolios. Dembo *et al.* (2004) approximate the tail of the loss distribution for the special case of exponentially-distributed default exposures (meaning the amount that would be lost in the event of default) in homogeneous portfolios. Moreover, the market factor is assumed to be a discrete variable to avoid problems with numerical integration, when it comes to calculating the unconditional tail probability. Glasserman *et al.* (2007) analyze limits for the tail of the loss distribution in two limiting regimes. The first regime deals with increasingly large loss thresholds and moderate default probabilities, and the second with letting the individual default probabilities decrease toward zero when loss thresholds are assumed to be moderate. The analysis reveals a qualitative distinction between the two cases: in the regime with decreasing default probabilities, the tail of the loss distribution decreases exponentially, but in the large-threshold regime the decay is consistent with a power law. However, Glasserman *et al.* (2007) do not take into account recovery risk depending on the state of the economy.

In the literature several comparable techniques are known to approximate the tail of the loss distribution. MC simulations are probably most widely used,

at least for comparison reasons. The problem is that they get computationally expensive, especially when it comes to smaller default probabilities, higher loss levels and larger portfolios. Grundke (2007) applies a Fourier-based approach to the well-known credit portfolio model CreditMetricsTM of J.P. Morgan (see Bhatia *et al.* (1997)), to calculate risk measures as the VaR for a credit portfolio. It is shown that the Fourier-based approach is not computationally superior to MC simulations. Another technique is the normal approximation, described in e.g. Martin (2004). It is a direct application of the Central Limit Theorem, which can be used for tail approximations. As it is an asymptotic approximation, it becomes more accurate when the portfolio size increases. However, Huang *et al.* (2007) state that normal approximation fails to deal with lower granularity of the portfolio. Moreover, the Central Limit Theorem approximation converges slowly, hence a very large number of companies is required to achieve a given quality of the approximation. Saddlepoint approximations are comparable to LD, as both methods rely on computing the cumulant generating function for the portfolio loss function. Saddlepoint approximations try to give a closed-form formula for the tail probability. Gordy (2002) showed for the credit risk model CreditRisk+TM that saddlepoint approximations can be fast and robust. However, saddlepoint approximations require the evaluation of higher derivatives, leading to numerical instabilities.

In summary, the main contributions of this paper are the following: First, from a numerical point of view, it is shown that LD techniques are more efficient than MC simulations to compute risk measures in large credit portfolios. Moreover, they are applied under more general settings than in previous works. Second, it is shown that stochastic recovery models depending on the state of the economy have a major impact on risk measures and portfolio optimization, as oppose to deterministic or purely stochastic recovery. The positive correlation between the state of the economy and RRs plays a crucial role for risk measures, which may not be neglected. Third, a portfolio optimization is performed using LD techniques, which proves the reliability of this method. In this context, efficient frontiers and optimal portfolio weights are calculated.

The paper is organised as follows. Section 2 gives the theoretical background. The portfolio setting and the application of LD is explained and risk measures are introduced. Section 3 introduces five different models for the RR, which are analysed in Section 4. Illustrative examples for the models are evaluated and models are compared. In Section 5, the portfolio optimization framework is explained and optimization results are shown. Section 6 concludes.

2 Theoretical background

In this section the general portfolio setting, the main result from LD used in the paper and its application are explained. Moreover, risk measures VaR and CVaR are introduced and it is shown how those can be computed using LD results.

2.1 Portfolio setting and the application of LD

Large credit portfolios with a finite number n of obligors are analyzed (see e.g. Huang *et al.* (2007)). Any obligor i has a certain exposure at default (EAD_i), a recovery rate (RR_i) and a probability of default (\mathbb{P}_{D_i}). The EAD as well as the probability of default on a given period T are deterministic, the RR may be stochastic with a certain distribution. An obligor may default at the end of the period T and only at that point in time. Without loss of generality a time horizon of one year ($T = 1$) is assumed throughout the paper. Default of the i -th company is modelled as a Bernoulli random variable D_i , such that

$$D_i = \begin{cases} 1 & \text{with probability } \mathbb{P}_{D_i} \\ 0 & \text{with probability } 1 - \mathbb{P}_{D_i} \end{cases}.$$

Then, the loss resulting from obligor i is given by

$$L_i = EAD_i (1 - RR_i) D_i,$$

and the relative portfolio loss for a portfolio of n obligors, denoted by $L_P(n)$, is given by

$$L_P(n) = \frac{\sum_{i=1}^n L_i}{\sum_{i=1}^n EAD_i} = \frac{\sum_{i=1}^n EAD_i (1 - RR_i) D_i}{\sum_{i=1}^n EAD_i}, \quad (1)$$

which is the total loss over all obligors divided by the total exposure.

For further portfolio calculations, without loss of generality, it is assumed that $\sum_{i=1}^n EAD_i$ is equal to n , i.e. the portfolio value at time 0 is n , for a portfolio of n obligors. For given $EAD_i, i = 1, \dots, n$, it can be obtained replacing EAD_i by

$$EAD_i^{norm} = \frac{EAD_i}{\frac{1}{n} \sum_{i=1}^n EAD_i},$$

which can be regarded as a normalization as each EAD is divided by the mean EAD over all obligors. As the paper focuses on the relative portfolio loss $L_P(n)$ the normalization does not change the setting, as it holds that

$$\begin{aligned} L_P(n) &= \frac{\sum_{i=1}^n \frac{EAD_i}{\frac{1}{n} \sum_{i=1}^n EAD_i} (1 - RR_i) D_i}{n} \\ &= \frac{\sum_{i=1}^n EAD_i^{norm} (1 - RR_i) D_i}{n}. \end{aligned}$$

For the ease of exposition, EAD_i^{norm} is denoted by EAD_i in the sequel.

Dependence between obligor defaults is a well-known phenomenon in credit portfolios (see e.g. Bluhm *et al.* (2003)), which should not be neglected. Similar to Iscoe *et al.* (1999), a creditworthiness index Y_i^* for obligor i determines the “healthiness” of an obligor. It is assumed that Y_i^* is a random variable, related to some common credit drivers $Y_j, j = 1, \dots, k$, affecting all obligors, and to an idiosyncratic term ϵ_i , individual for each obligor, through the linear multi-factor model given by

$$Y_i^* = \beta_{i1}Y_1 + \dots + \beta_{ik}Y_k + \sigma_i\epsilon_i,$$

where β_{ij} is the sensitivity of Y_i^* to the j -th common factor and σ_i the sensitivity to the idiosyncratic term.

$Y_j, j = 1, \dots, k$, are independent random variables with joint cumulative distribution function (cdf) $\bar{\mathbb{F}}$. They are independent from $\epsilon_i, i = 1, \dots, n$, which are themselves independent and identically distributed. Let $\vec{Y} = (Y_1, \dots, Y_k)$ denote the vector of common factors. Correlation between defaults comes from the common factors in the model. Most studies deal with one common factor with Gaussian (see e.g. Vasicek (1987) and Finger (2004)) or more general distributions (see e.g. Schlösser (2010), Escobar *et al.* (2010) and Schönbucher (2000)).

Knowing the cdf $\mathbb{F}_{i,*}$ of the creditworthiness index Y_i^* , it is possible to link the default probability \mathbb{P}_{D_i} of obligor i to the creditworthiness index, i.e. the default variables D_i can be described by

$$D_i = \begin{cases} 1, & \text{if } Y_i^* \leq \mathbb{F}_{i,*}^{-1}(\mathbb{P}_{D_i}) \\ 0, & \text{else} \end{cases}.$$

This means that company i defaults if its creditworthiness index Y_i^* falls below the threshold $\mathbb{F}_{i,*}^{-1}(\mathbb{P}_{D_i})$, where $\mathbb{F}_{i,*}^{-1}$ is the inverse cdf defined in the generalized sense. Thus, default here is comparable to the Merton model (see Merton (1974)), where a predetermined threshold determines the default.

Next the main LD result used in the paper is enunciated. This theorem, introduced by Gärtner (1977) and Ellis (1984), gives a LD result for non-equally-distributed random variables.

Theorem 1. *Let $L_P(n)$ be defined as in Equation 1 and its ln-m.g.f. (moment generating function) given by $\Lambda_n(s) = \ln \mathbb{E}[e^{sL_P(n)}]$. Assume that for every $s \in \mathbb{R}$*

$$\lim_{n \rightarrow \infty} \frac{\Lambda_n(ns)}{n} = \Lambda(s)$$

for some function $\Lambda(s)$, such that $D_\Lambda = \{s \in \mathbb{R} : \Lambda(s) < \infty\}$ contains a neighborhood of zero and $\Lambda(s)$:

- (a) *is differentiable at the interior of D_Λ ,*
- (b) *is a step function or $D_\Lambda = \mathbb{R}$,*
- (c) *is lower semi-continuous.*

Then, a Large Deviation principle is satisfied and for $l > \mathbb{E}[L_P(n)]$ it holds that:

$$\lim_{n \rightarrow \infty} \frac{1}{n} \ln \mathbb{P}(L_P(n) \geq l) = -\Lambda^*(l),$$

where $\Lambda^*(l)$ is the Legendre transform of $\Lambda(\cdot)$ defined by

$$\Lambda^*(l) = \sup_{s \in \mathbb{R}} [sl - \Lambda(s)],$$

and $[l, \infty)$ is a set of continuity for Λ^* .

Proof. The proof is an easy adaptation of the Gärtner-Ellis theorem. See den Hollander (2000), p. 54 together with p. 30/31. \square

More detailed information about LD can be found in e.g. den Hollander (2000) and Dembo and Zeitouni (1998).

In the setting of the paper, company defaults and RRs are not a priori independent as the creditworthiness indices and – as will get clear later – RRs are depending on the common factors. However, conditional on the common factors, defaults and RRs are independent and the application of LD considerably simplifies. For this reason, the ln-m.g.f. Λ_n , the function Λ and the Legendre transform Λ^* conditional on the common factors are introduced. They are defined as

$$\Lambda_n(s|\vec{Y}) = \ln \mathbb{E}[e^{sL_P(n)}|\vec{Y}],$$

$$\Lambda(s|\vec{Y}) = \lim_{n \rightarrow \infty} \frac{\Lambda_n(ns|\vec{Y})}{n}$$

and

$$\Lambda^*(l|\vec{Y}) = \sup_{s \in \mathbb{R}} [sl - \Lambda(s|\vec{Y})].$$

Applying LD to the portfolio loss $L_P(n)$ according to Theorem 1 in the conditional form, $\Lambda(s|\vec{Y})$ is given by

$$\begin{aligned} \Lambda(s|\vec{Y}) &= \lim_{n \rightarrow \infty} \frac{1}{n} \ln \mathbb{E}[e^{nsL_P(n)}|\vec{Y}] \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \ln \mathbb{E}[e^{s \sum_{i=1}^n EAD_i (1-RR_i) D_i}|\vec{Y}] \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \ln \mathbb{E}[e^{sEAD_i (1-RR_i) D_i}|\vec{Y}] \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{1}{n} \ln(1 - \mathbb{P}_{D_i}(\vec{Y}) + \mathbb{P}_{D_i}(\vec{Y}) \mathbb{E}[e^{sEAD_i (1-RR_i)}|\vec{Y}]) \\ &\approx \sum_{i=1}^n \frac{1}{n} \ln(1 - \mathbb{P}_{D_i}(\vec{Y}) + \mathbb{P}_{D_i}(\vec{Y}) \mathbb{E}[e^{sEAD_i (1-RR_i)}|\vec{Y}]), \end{aligned}$$

provided the limit exists and, for the last statement, that n is large enough which will be assumed from now on. $\mathbb{P}_{D_i}(\vec{Y})$ is the probability of default of obligor

i , conditional on the common factors \vec{Y} . Denoting the cdf of the idiosyncratic term ϵ_i by \mathbb{F}_ϵ , $\mathbb{P}_{D_i}(\vec{Y})$ can be expressed as

$$\begin{aligned}\mathbb{P}_{D_i}(\vec{Y}) &= \mathbb{P}(D_i = 1 | \vec{Y}) \\ &= \mathbb{F}_\epsilon \left(\frac{\mathbb{F}_{i,*}^{-1}(\mathbb{P}_{D_i}) - (\beta_{i1}Y_1 + \dots + \beta_{ik}Y_k)}{\sigma_i} \right).\end{aligned}$$

If the derivative of $\Lambda(s|\vec{Y})$ with respect to s exists, then the solution $s = s(l|\vec{Y})$ for $\Lambda^*(l|\vec{Y})$ can be obtained by solving the equation

$$l - \frac{d}{ds}\Lambda(s|\vec{Y}) = 0,$$

which is a root search problem. The Legendre transform can then be set to

$$\Lambda^*(l|\vec{Y}) = l \cdot s(l|\vec{Y}) - \Lambda(s(l|\vec{Y})|\vec{Y}).$$

Using this result, according to Theorem 1, the probability of a portfolio loss greater or equal than l , conditional on the common factors, provided that n is large enough, is given by the LD approximation

$$\mathbb{P}(L_P(n) \geq l | \vec{Y}) \approx e^{-n\Lambda^*(l|\vec{Y})}.$$

The unconditional probability is obtained by integrating over the common factors, which yields

$$\mathbb{P}(L_P(n) \geq l) \approx \int_{\mathbb{R}^k} e^{-n\Lambda^*(l|\vec{y})} d\vec{\mathbb{F}}(\vec{y}).$$

In the following, a mild form of granularity is asked for the portfolio. It is assumed that the large enough number of n companies can be classified into C types of companies (denoted by c , $c \in \{1, \dots, C\}$). Companies of the same type share similar characteristics. Namely, these are the EAD, the probability of default (\mathbb{P}_D), the RR distribution with mean and standard deviation and the influence of the common factor on the creditworthiness index (β). Let portfolio weights be defined by the fraction of companies of type c in the portfolio, denoted by $x = (x_1, \dots, x_C)$.

Corollary 1. *In the setting with C types of companies and the RR models considered later in this paper in Section 3, Theorem 1 can be applied in its conditional form. Especially $\Lambda(s|\vec{Y})$ is (approximately) given by*

$$\Lambda(s|\vec{Y}) = \sum_{c=1}^C x_c \ln(1 - \mathbb{P}_{D_c}(\vec{Y}) + \mathbb{P}_{D_c}(\vec{Y}) \mathbb{E}[e^{sEAD_c(1-RR_c)} | \vec{Y}])$$

and its derivative with respect to s is

$$\begin{aligned} \frac{d}{ds}\Lambda(s|\vec{Y}) &= \sum_{c=1}^C x_c \frac{\mathbb{P}_{D_c}(\vec{Y}) \frac{d}{ds}\mathbb{E}[e^{s \cdot EAD_c (1-RR_c)}|\vec{Y}]}{1 - \mathbb{P}_{D_c}(\vec{Y}) + \mathbb{P}_{D_c}(\vec{Y})\mathbb{E}[e^{s \cdot EAD_c (1-RR_c)}|\vec{Y}]} \\ &= \sum_{c=1}^C x_c \frac{\mathbb{P}_{D_c}(\vec{Y})EAD_c \mathbb{E}[(1 - RR_c) e^{s \cdot EAD_c (1-RR_c)}|\vec{Y}]}{1 - \mathbb{P}_{D_c}(\vec{Y}) + \mathbb{P}_{D_c}(\vec{Y})\mathbb{E}[e^{s \cdot EAD_c (1-RR_c)}|\vec{Y}]}, \end{aligned}$$

where the last equation holds if the derivatives exist and the derivation and expectation operators can be exchanged.

If only one type of company is present ($C = 1$) and the RR is deterministic then the solution for $s(l|\vec{Y})$ is given in closed form by

$$s(l|\vec{Y}) = \frac{\ln \frac{l \cdot (1 - \mathbb{P}_D(\vec{Y}))}{\mathbb{P}_D(\vec{Y}) (EAD (1 - RR) - l)}}{EAD (1 - RR)},$$

otherwise $s(l|\vec{Y})$ is computed numerically.

2.2 Computation of VaR and CVaR using LD

The risk measures considered in this paper are the Value at Risk (VaR) and the Conditional Value at Risk (CVaR), as they capture the behaviour of the tail of a portfolio distribution.

Value at Risk (VaR) is a risk measure, used in the Basel II Accords for the calculation of a bank's capital requirements (see Basel Committee on Banking Supervision (2006) and Jorion (2000)). The VaR, for the relative losses of a portfolio $L_P(n)$, is calculated for a given confidence level α and a predetermined time horizon T , such that

$$VaR_\alpha(L_P(n)) = \inf\{l : \mathbb{P}(L_P(n) > l) \leq 1 - \alpha\}.$$

Thus, VaR gives the highest relative loss, which is not exceeded with probability α in the time horizon T . If $L_P(n)$ is continuous, it holds that $VaR_\alpha(L_P(n)) = \inf\{l : \mathbb{P}(L_P(n) \geq l) \leq 1 - \alpha\}$. In the following, $L_P(n)$ is assumed to be a continuous random variable, which is reasonable as n is assumed to be large.

The Conditional Value at Risk (CVaR) is another risk measure closely related to the VaR (see e.g. Frey and McNeil (2002)). The CVaR captures the behavior of losses in a portfolio, by calculating the mean loss conditional that the loss exceeds the VaR. Thus, the CVaR, for the relative losses of a portfolio $L_P(n)$, for a given confidence level α and a predetermined time horizon T , is given by

$$CVaR_\alpha(L_P(n)) = \mathbb{E}[L_P(n)|L_P(n) \geq VaR_\alpha(L_P(n))]. \quad (2)$$

Calculation of the VaR using LD is straightforward, because LD applied to $L_P(n)$ for a given portfolio loss l yields $\mathbb{P}(L_P(n) \geq l)$. Thus the problem of finding the VaR_α is equal to finding the smallest l , such that $\mathbb{P}(L_P(n) \geq l) \leq 1 - \alpha$.

Using a LD result the CVaR can be computed as follows: Rewriting Equation 2, the CVaR for the relative portfolio loss $L_P(n)$ is given by

$$\begin{aligned}
CVaR_\alpha(L_P(n)) &= \mathbb{E}[L_P(n)|L_P(n) \geq VaR_\alpha(L_P(n))] \\
&= \int_{VaR_\alpha(L_P(n))}^1 l d\mathbb{F}_{L_P(n)}^{VaR}(l) \\
&\stackrel{\text{int. by parts}}{=} 1 - \int_{VaR_\alpha(L_P(n))}^1 \mathbb{F}_{L_P(n)}^{VaR}(l) dl \\
&= VaR_\alpha(L_P(n)) + \int_{VaR_\alpha(L_P(n))}^1 1 - \mathbb{F}_{L_P(n)}^{VaR}(l) dl \\
&= VaR_\alpha(L_P(n)) + \frac{1}{1-\alpha} \int_{VaR_\alpha(L_P(n))}^1 \mathbb{P}(L_P(n) \geq l) dl,
\end{aligned} \tag{3}$$

where $\mathbb{F}_{L_P(n)}^{VaR}(l)$ is the cdf of the portfolio loss $L_P(n)$ at point l , conditional on the event, that the relative loss is greater equal than $VaR_\alpha(L_P(n))$.

Using LD techniques to compute the CVaR for a given confidence level α , one has to take into account several things, as LD only yields $\mathbb{P}(L_P(n) \geq l)$. Discretisation is inevitable to solve the integral in Equation 3, as the problem cannot be solved in closed form. First, one computes the VaR for the confidence level α . Second, one has to define a discrete grid, to capture the rest of the tail, such that the CVaR can be computed. The computation should be efficient, which means, it should take into account the computational costs of evaluating $\mathbb{P}(L_P(n) \geq l)$ and the accuracy of the CVaR estimation, compared to MC simulations.

Let g be the number of grid points for the discretisation. The “space” between the grid points is determined as follows: Additional to the VaR, another quantile – further to the tail of the loss distribution – is computed and the grid is taken equidistant between those 2 points. The motivation for this approach is, that the other quantile is needed to know how fast $\mathbb{P}(L_P(n) \geq l)$ decreases with increasing l . Choose Q such that $VaR_\alpha(L_P(n)) < Q \leq 1$. Moreover compute the equidistant discretisation E_g with g points on the interval $[VaR_\alpha(L_P(n)), Q]$. Thus, the distance between each two points is given by $\Delta_g = \frac{Q - VaR_\alpha(L_P(n))}{g-1}$. Then, the CVaR in Equation 3 can be computed numerically as follows:

$$\begin{aligned}
CVaR_\alpha(L_P(n)) &= VaR_\alpha(L_P(n)) + \frac{1}{1-\alpha} \int_{VaR_\alpha(L_P(n))}^1 \mathbb{P}(L_P(n) \geq l) dl \\
&\approx VaR_\alpha(L_P(n)) + \frac{1}{1-\alpha} \int_{VaR_\alpha(L_P(n))}^Q \mathbb{P}(L_P(n) \geq l) dl \\
&\approx VaR_\alpha(L_P(n)) + \frac{1}{1-\alpha} \sum_{l \in E_g} \mathbb{P}(L_P(n) \geq l) \Delta_g.
\end{aligned}$$

The computation of the CVaR, using LD techniques, is summarized in Algorithm 1.

Algorithm 1. (*Algorithm for the CVaR computation using LD techniques.*)

1. Compute $VaR_\alpha(L_P(n))$.
2. Choose $Q \in (VaR_\alpha(L_P(n)), 1]$.
3. Compute the equidistant discretisation E_g on $[VaR_\alpha(L_P(n)), Q]$ with distance $\Delta_g = \frac{Q - VaR_\alpha(L_P(n))}{g-1}$ for g grid points.
4. Compute $CVaR_\alpha(L_P(n)) \approx VaR_\alpha(L_P(n)) + \frac{1}{1-\alpha} \sum_{l \in E_g} \mathbb{P}(L_P(n) \geq l) \Delta_g$.

3 Stochastic recovery risk depending on the state of the economy

Systematic risk in RRs is considered now, which deals with the relationship between probabilities of default and RRs. In several studies, it is found that there is a negative correlation between default and RRs (see e.g. Frye (2000b), Altman *et al.* (2004), Altman *et al.* (2005) and Höcht and Zagst (2010)). If the economy is fine, default probabilities tend to be lower with on average higher RRs. In contrast, if the state of the economy is bad, default probabilities rise with on average lower RRs compared to good times (see e.g. Schuermann (2004) and Acharya *et al.* (2007)). Based on Altman *et al.* (2005), Figure 1 illustrates this link between default probabilities and RRs for corporate bonds, mainly of the US, for the years 1982–2001. The data utilized in this study comes from the NYU Salomon Center database on bond defaults and captures about 1,300 default events. Aggregate annual RRs are measured by the weighted average recovery of all corporate bonds. The weights are based on the market value of defaulting debt issues. The negative correlation between default and RRs is evident.

In the previous section it is assumed that the probability of default, depends on the common factors \vec{Y} through a linear multi-factor model. Now, RRs are modelled such that they capture the following – above mentioned – empirically observed properties: negative correlation between default and RRs and positive correlation between the state of the economy and RRs. The common factors can be seen as indicators for the state of the economy, which also influence RRs. To simplify, only one standard-normally distributed common factor Y is assumed throughout the rest of the paper, but theoretically it is possible to consider other distributions. To establish a link between the common factor and the RRs, a RR index Y_i^R is introduced. For company i let Y_i^R be defined by

$$Y_i^R = \beta_i^R Y + \sqrt{1 - (\beta_i^R)^2} \tilde{\epsilon}_i. \quad (4)$$

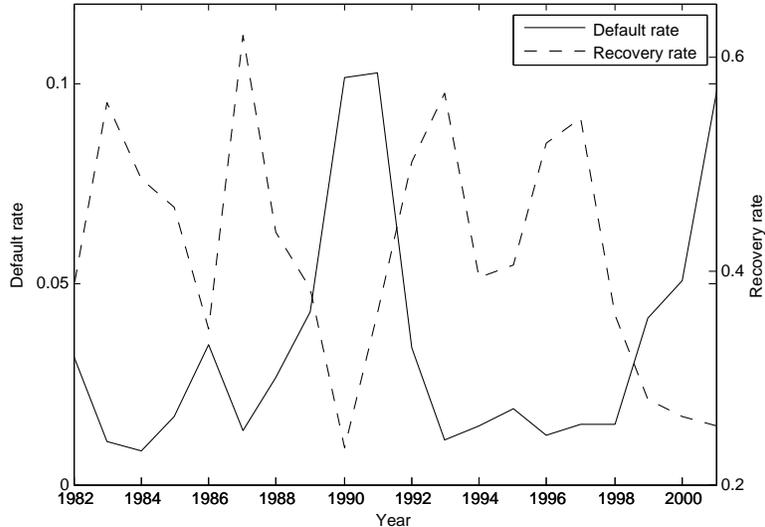


Figure 1: Default vs. RRs are shown for corporate bonds for the years 1982–2001 (Data: Altman *et al.* (2005)).

The value of the RR index is not directly interpretable, but the application will be such, that higher values of Y_i^R generally result in a higher RR. β_i^R is the sensitivity of the RR index of company i to the common factor Y . As the correlation between the common factor Y and RRs should be positive, $0 \leq \beta_i^R \leq 1$ is assumed. The parameter β_i^R plays an important role in Equation 4. It controls how much influence the common factor, respectively the general state of the economy, has on the RR index of each company. High values of β_i^R indicate a high influence of common factor Y and RRs will be very cyclical. $\beta_i^R = 1$ means, that the RR of company i is fully determined by the common factor Y . If the β_i^R are close to zero, RRs will not be very cyclical at all. For $\beta_i^R = 0$, the RR of company i does not depend on the common factor Y . The $\tilde{\epsilon}_i$ are idiosyncratic terms for each company, reflecting the randomness of RRs. They are assumed to be independent, identically distributed and independent of ϵ_i , with $\tilde{\epsilon}_i \sim N(0, 1)$. Thus, the RR index has a standard normal distribution. Moreover, the parameters β_i^R determine the correlation structure between the RR indices of two companies and the correlation between the common factor and the RR index. Companies of the same type c are assumed to have the same value for β^R .

Let $i \neq j$, then the correlation between the RR indices of two companies i and j is given by $Corr(Y_i^R, Y_j^R) = \beta_i^R \beta_j^R$ and $Corr(Y_i^R, Y) = \beta_i^R$.

Several ways to model RRs have been proposed in the literature. A closer look is taken at five different models for the RRs and their impact on the

portfolio VaR and CVaR. It is shown that LD techniques can be used efficiently for the calculation of risk measures within these models.

In Frye (2000b) recovery is modelled as a normally-distributed random variable depending on the common factor Y through the RR index Y_i^R . The recovery rate RR_i^N for obligor i is given by

$$\begin{aligned} RR_i^N &= \bar{\mu}_i + \bar{\sigma}_i Y_i^R \\ &= \bar{\mu}_i + \bar{\sigma}_i \beta_i^R Y + \bar{\sigma}_i \sqrt{1 - (\beta_i^R)^2} \tilde{\epsilon}_i. \end{aligned}$$

The parameter $\bar{\mu}_i$ denotes the average RR for firm i and $\bar{\sigma}_i$ is the standard deviation of the RR. The interpretability of the parameters is a strong advantage of this model. Moreover, the model can be fitted easily. The drawback is that RRs in this model are unbounded on both sides. This may be unrealistic as RRs smaller than zero cannot occur for ordinary loans or bonds and RRs greater than one are also unusual.

In the second model, the RR is modelled as a lognormally-distributed random variable. This approach was introduced by Pykhtin (2003) and further scrutinized by Düllmann and Trapp (2004). The recovery rate RR_i^{LN} for company i is given by

$$\begin{aligned} RR_i^{LN} &= e^{\hat{\mu}_i + \hat{\sigma}_i Y_i^R} \\ &= e^{\hat{\mu}_i + \hat{\sigma}_i \beta_i^R Y + \hat{\sigma}_i \sqrt{1 - (\beta_i^R)^2} \tilde{\epsilon}_i}. \end{aligned}$$

where $\hat{\mu}_i$ and $\hat{\sigma}_i$ are two parameters, which can be used to fit the mean and standard deviation of the RR. The lognormal distribution has two main advantages compared to the normal distribution. RRs are strictly non-negative and the distribution has a thicker tail (see Keisman and van de Castle (1999)).

In Frye (2000a) as well as in Miu and Ozdemir (2006) the RR is modelled as a beta-distributed random variable. The beta distribution has bounded support on $[0, 1]$, which is the expected range for a RR. The recovery rate RR_i^B for company i in this type of model is given by

$$\begin{aligned} RR_i^B &= \mathbb{F}_{Beta}^{-1}(\Phi(Y_i^R), a_B, b_B) \\ &= \mathbb{F}_{Beta}^{-1}(\Phi(\beta_i^R Y + \sqrt{1 - (\beta_i^R)^2} \tilde{\epsilon}_i), a_B, b_B). \end{aligned}$$

Here, \mathbb{F}_{Beta}^{-1} denotes the inverse of the cumulative distribution function of the beta distribution with parameters a_B and b_B . Mean and standard deviation for the RR can be fitted using the parameters a_B and b_B . However, \mathbb{F}_{Beta}^{-1} is not known in closed form which makes this model computationally very expensive as the incomplete beta function has to be evaluated numerically.

To overcome the drawback of the beta distribution, the latter is substituted by the Kumaraswamy distribution (see Kumaraswamy (1980) and Höcht and Zagst (2009), who apply the Kumaraswamy distribution for modelling LGD). The Kumaraswamy distribution is very flexible and has closed-form solutions for the cumulative distribution function, as well as for the inverse of the cumulative

distribution function. The Kumaraswamy distribution has bounded support on $[0, 1]$. The RR in this model is given by

$$\begin{aligned} RR_i^K &= \mathbb{F}_{Kum}^{-1}(\Phi(Y_i^R), a_K, b_K) \\ &= \mathbb{F}_{Kum}^{-1}(\Phi(\beta_i^R Y + \sqrt{1 - (\beta_i^R)^2} \tilde{\epsilon}_i), a_K, b_K), \end{aligned}$$

where \mathbb{F}_{Kum}^{-1} is the inverse of the cumulative distribution function of the Kumaraswamy distribution with parameters a_K and b_K . Mean and standard deviation for the RR can be fitted using the parameters a_K and b_K .

Düllmann and Trapp (2004) model the RR as a logistic function. A logistic function enables bounding the RR to $[0, 1]$. In this model, the RR is given by

$$\begin{aligned} RR_i^{LG} &= \frac{e^{\tilde{\mu}_i + \tilde{\sigma}_i Y_i^R}}{1 + e^{\tilde{\mu}_i + \tilde{\sigma}_i Y_i^R}} \\ &= \frac{e^{\tilde{\mu}_i + \tilde{\sigma}_i \beta_i^R Y + \tilde{\sigma}_i \sqrt{1 - (\beta_i^R)^2} \tilde{\epsilon}_i}}{1 + e^{\tilde{\mu}_i + \tilde{\sigma}_i \beta_i^R Y + \tilde{\sigma}_i \sqrt{1 - (\beta_i^R)^2} \tilde{\epsilon}_i}}, \end{aligned}$$

where $\tilde{\mu}_i$ and $\tilde{\sigma}_i$ are two parameters which can be chosen, e.g., such that the RR has a required level of mean and standard deviation.

Remarks (Checking the hypothesis in Corollary 1)

- For the RR models in the paper it holds that conditional on the common factors \vec{Y} , default events and RRs are independent.
- As can be easily verified, $D_\Lambda = [0, \infty)$ for the lognormal RR and $D_\Lambda = \mathbb{R}$ for all other models.
- Applying the theorem of dominated convergence, the derivative of Λ exists and the derivation and expectation operators can be exchanged.
- For all RR models it is immediately clear from the previous remarks that Λ is lower semi-continuous on D_Λ .
- As $\mathbb{P}(L_P(n) \geq l)$ is obtained by integrating over the common factors \vec{Y} , the assumption that $l > \mathbb{E}[L_P(n) | \vec{Y}]$ must be checked permanently during the computations.
- For the continuity of Λ^* on $[l, \infty)$ see den Hollander (2000), p. 10 and p. 31.

4 Numerical results

In this section, the different models for the RR are analysed within a numerical framework. First, for comparison, the parameters for the RR models are adjusted. Second, two illustrative examples show the differences in the models and the importance of dependent RRs.

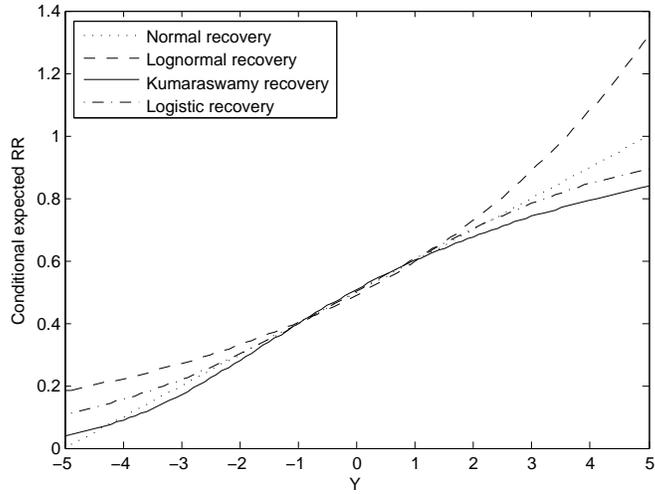
The parameters for the different models are adjusted such that the expected RR and its standard deviation are equal for all models. Moreover, this yields a recovery correlation of $Corr(RR_i, RR_j) = \beta_i^R \beta_j^R$ for two companies i and j ($i \neq j$) in all models. Not conditioning on the common factor Y , the Y_i^R have a standard normal distribution, independent of the value of β_i^R . For normally-distributed RR, the fitting is quite easy, as the parameters $\bar{\mu}_i$ and $\bar{\sigma}_i$ can be chosen as the required mean and standard deviation. The other models are fitted using numerical techniques. Parameters for an expected RR of 0.5 and 0.3 with a standard deviation of 0.1 are given in Table 1.

Parameters	$\mathbb{E}[RR] = 30\%$	$\mathbb{E}[RR] = 50\%$
Normal recovery	$\bar{\mu} = 0.3$ $\bar{\sigma} = 0.1$	$\bar{\mu} = 0.5$ $\bar{\sigma} = 0.1$
Lognormal recovery	$\hat{\mu} = -1.258$ $\hat{\sigma} = 0.325$	$\hat{\mu} = -0.713$ $\hat{\sigma} = 0.198$
Beta recovery	$a_B = 6$ $b_B = 14$	$a_B = 12$ $b_B = 12$
Kumaraswamy recovery	$a_K = 3.256$ $b_K = 34.632$	$a_K = 5.725$ $b_K = 33.326$
Logistic recovery	$\tilde{\mu} = -0.894$ $\tilde{\sigma} = 0.496$	$\tilde{\mu} = 0.008$ $\tilde{\sigma} = 0.425$

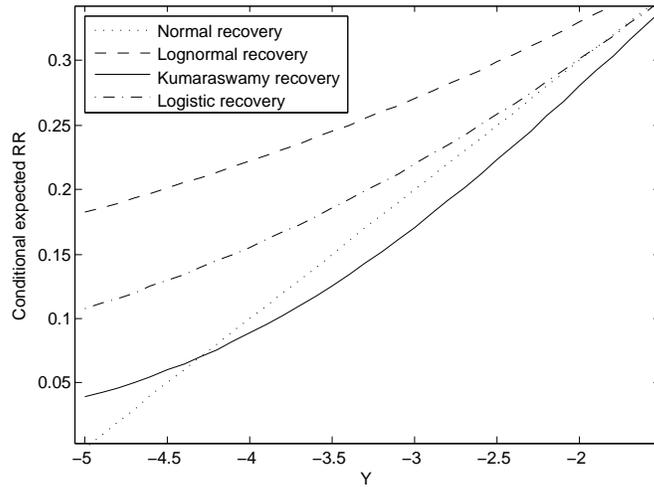
Table 1: Parameters for the different RR models. Parameters are fitted to an expected RR of 30% and 50% with a standard deviation of 10%.

Figure 2(a) shows RRs with an expected value of 0.5 and standard deviation of 0.1 for the different models with $\beta_i^R = 1$ depending on the common factor Y . As $\beta_i^R = 1$, the common factor determines the RR in all models completely. In general, RRs are higher for increasing values of Y , which indicate a better state of the economy. It can be seen that especially the tail behaviour of the RR differs for the different models. Comparing the RRs for the different models in a bad state of the economy, Figure 2(b) shows RRs for $Y \in [-5, -1.5]$. The model with the lowest recovery in a bad state of the economy is the one with a normally-distributed RR, then Kumaraswamy-distributed, beta-distributed, logistic and lognormally-distributed RR.

Furthermore, the main feature of the models is the correlation structure between default and RRs through their dependence on the common factor Y . Figure 3 illustrates the negative correlation between default and RRs for $\mathbb{P}_D = 5\%$, $\beta = 0.5$, $\beta^R = 1$ and a RR with Kumaraswamy distribution with mean 0.5



(a) Dependence between the common factor (Y) and RR.



(b) Dependence between the common factor (Y) and RR, at the “left” side of the tail of the common factor Y , the tail which indicates a bad state of the economy.

Figure 2: The dependence between the common factor Y and RR is shown. Parameters for the different models are chosen, such that the expected RR equals 50% with standard deviation 10% and $\beta_i^R = 1$. The model with beta-distributed RR is not shown because it is very similar to the logistic RR.

and standard deviation 0.1. It can be seen that RRs tend to be lower if default rates go up and vice versa.

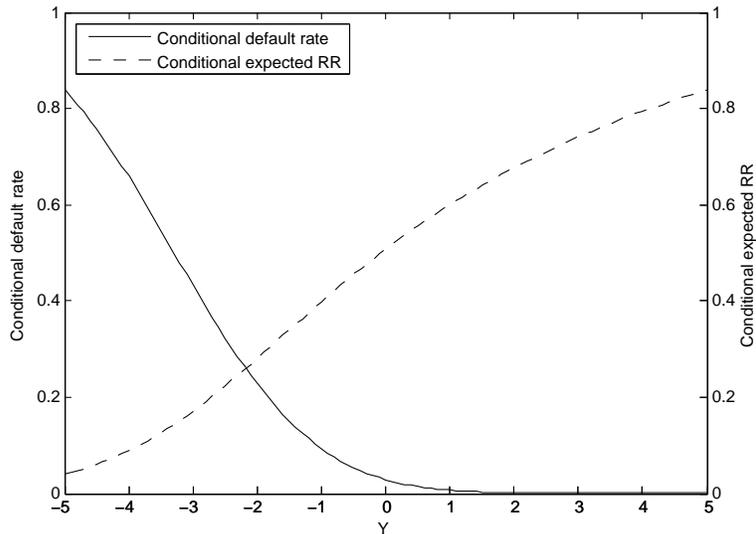


Figure 3: Default and RRs depending on the common factor Y .

An illustrative numerical example shall give insights to the use of LD techniques for the evaluation of credit risk models with systematic risk in RRs. The setting of the example is as follows.

The numerical example deals with $n = 10,000$ companies, which are classified into $C = 2$ types of companies. The exposures are $EAD_1 = 6$ for companies of type 1 and $EAD_2 = 4$ for companies of type 2. As the currency of the exposure does not play a role for the paper, the unit is suppressed. The RRs for firms of type 1 are distributed with an expected value of 0.5 and a standard deviation of 0.1, for firms of type 2, those values are 0.3 and 0.1. The influence of the common factor on the creditworthiness index for the two types of companies is assumed to be $\beta_1 = \beta_2 = 0.5$. Moreover, σ is set to $\sigma_c = \sqrt{1 - \beta_c^2} = \sqrt{0.75}$.

Together with a standard normal distribution for the idiosyncratic terms of the creditworthiness index, this yields that the Y_i^* are distributed standard-normally again. The unconditional probabilities of default for the two types of companies are $\mathbb{P}_{D_1} = 1\%$ and $\mathbb{P}_{D_2} = 5\%$. The portfolio weights are $x_1 = 50\%$ and $x_2 = 50\%$, i.e. the portfolio consists of 5,000 companies of each type. The influence of the common factor on the RR index is given by β_1^R and β_2^R and will be specified later. LD is used for the computation of VaR and CVaR for the 99% confidence level. For comparison, MC simulations with 100,000 runs are performed. For LD as well as MC simulations the computation time is

recorded.¹ For the computation of the CVaR using LD the parameters Q and g have to be specified. Let $Q = VaR_{99.9\%}(L_P(n))$ and $g = 17$. Thus the CVaR is computed within the interval $[VaR_{99\%}(L_P(n)), VaR_{99.9\%}(L_P(n))]$ using 17 evaluation points.

A closer look shall be taken at two cases for β_c^R . First, $\beta_c^R = 1, \forall c$, which means that the common factor determines the RR completely. Second, $\beta_c^R = 0, \forall c$, thus the common factor has no influence on the RR and RRs for the different companies are independent and totally random.

For $\beta_1^R = \beta_2^R = 1$:

Results for VaR and CVaR for all RR models are shown in Table 2 in the upper half. Comparing LD estimations to MC simulations, it can be seen that the estimations for VaR as well as for CVaR are very accurate. VaR is always slightly overestimated using LD techniques, whereas CVaR is slightly underestimated (except for the case of deterministic RR, where the CVaR is overestimated). To give an example, for normally-distributed RRs, the true VaR (CVaR) is 15.07% (20.46%) compared to 15.11% (20.34%) for the LD estimations. Moreover, LD approximations for VaR and CVaR can be computed much faster compared to MC simulations for all models. What catches eye is, that the model with beta-distributed RRs has by far the highest computational effort for LD estimations (345 sec) but also for MC simulations (29,980 sec). The reason for this is, that for the RRs the inverse of the cumulative distribution function of the beta distribution (\mathbb{F}_{Beta}^{-1}) has to be computed, which is very time-consuming. Comparing risk measures for the five RR models to the case of deterministic RR, it can be seen that risk measures are strongly underestimated if a deterministic RR is assumed. E.g. for Kumaraswamy-distributed RRs, the VaR (CVaR) is underestimated nearly 37% (42%) using a deterministic RR defined as the expected value of the random RR. The highest risk measures in descending order are attained for: normally-distributed, Kumaraswamy-distributed, beta-distributed, logistic and lognormally-distributed RR. The findings are consistent with Figure 2(b), as e.g. the highest risk measures are attained for the normal distribution, which shows the lowest average RR in a bad state of the economy.

For $\beta_1^R = \beta_2^R = 0$:

Results for VaR and CVaR are shown in Table 2 in the lower half. As before, LD techniques give very accurate estimations for risk measures VaR and CVaR. VaR as well as CVaR are slightly overestimated for all models, e.g. for lognormally-distributed RR, the true (MC) VaR (CVaR) is 11.00% (14.23%) compared to 11.07% (14.24%) for LD approximations. As $\beta_1^R = \beta_2^R = 0$, RRs are not influenced by the common factor and each company has its own random RR. Thus, there is no systematic risk in the RRs. LD approximations get computationally much more expensive compared to the previous case, as the conditional moment generating function has to be computed by numerical integration. In the case with $\beta_1^R = \beta_2^R = 1$, the numerical integration is not necessary because

¹Based on a computer with an Intel Celeron dual-core CPU with 1.73GHz and 2GB RAM.

the common factor Y fully determined the RRs and the conditional moment generating function was just the expectation over a constant. Now, only the model with normally-distributed RR is hardly affected, because the conditional moment generating function is known in closed form. With respect to computational time, MC simulation is about the same for all models considered as before. Except for the case of beta-distributed RRs, LDs still computationally outperform MC simulations. Comparing risk measures to the results in Table 2, it can be seen that VaR and CVaR are much lower now. Moreover, for all models, including the deterministic RR model, VaR and CVaR are quite the same. Consequently, the positive correlation between the state of the economy (the common factor Y) and RRs plays a crucial role for risk measures, which should not be neglected.

Some studies suggest higher values for the expected RR and especially for the standard deviation of the RR (see e.g. Höcht and Zagst (2010)). Let the setting be the same as before, only RRs have a different distribution, with a higher expected RR and a higher standard deviation of the RR. RRs are considered to have an expectation of 65% (for firms of type 1) and 50% (for firms of type 2) with a standard deviation of 30%. Table 3 shows risk measures VaR and CVaR for all RR models. It can be seen that LD techniques are still efficient and accurate in the modified setting with only slightly higher errors for VaR and CVaR approximations.

In the RR models, the RR index is influenced by the common factor Y . Having seen the results for $\beta_c^R = 1$ and $\beta_c^R = 0$, the question is, what happens to risk measures for $\beta_c^R \in (0, 1)$, which is not clear on the spot.

Figure 4 shows VaR and CVaR for different values of β_c^R . For this, the setting of the example is left unchanged and for Kumaraswamy-distributed RRs β_c^R is varied between zero and one. β_c^R is assumed to be the same for both types of companies. It can be seen that VaR and CVaR are strictly increasing with increasing β_c^R . This means that a higher influence of the common factor on the RR yields higher risk measures. On the one side, the highest risk measures are attained if the RR is fully determined by the common factor ($\beta_c^R = 1$). On the opposite, risk measures are lowest, if the common factor has no influence on the RR ($\beta_c^R = 0$). In this case, each firm has its own random RR, independent of the state of the economy.

Summing up the findings, it gets clear that LD approximations are a fast and reliable method for computing risk measures VaR and CVaR within the presented RR models. Except for the model with beta-distributed RRs, LD by far computationally outperforms MC simulations. In the case that RRs are fully determined by the common factor, LD approximations yield their best performance. In this case, also the risk measures have the highest values. If the RR is not fully fixed by the common factor then LD computations get slower, but are still faster than MC simulations (again, except for the model with beta-distributed RR). Moreover, it gets clear that systematic risk in RRs plays a crucial role for risk measures VaR and CVaR. On the one hand, if the model does not incorporate systematic risk, then there is no need for incorporating random recovery because risk measures for the random and the deterministic

		LD	MC	LD time	MC time
$\beta_1^R = \beta_2^R = 1$	NORMAL RECOVERY				
	$VaR_{99\%}$	15.11%	15.07%	8 sec	710 sec
	$CVaR_{99\%}$	20.34%	20.46%		
	LOGNORMAL RECOVERY				
	$VaR_{99\%}$	14.09%	14.04%	9 sec	810 sec
	$CVaR_{99\%}$	18.58%	18.62%		
	BETA RECOVERY				
	$VaR_{99\%}$	14.65%	14.61%	345 sec	29,980 sec
	$CVaR_{99\%}$	19.46%	19.51%		
	KUMARASWAMY RECOVERY				
	$VaR_{99\%}$	15.08%	15.03%	12 sec	1,550 sec
	$CVaR_{99\%}$	20.13%	20.19%		
	LOGISTIC RECOVERY				
	$VaR_{99\%}$	14.53%	14.48%	8 sec	800 sec
$CVaR_{99\%}$	19.25%	19.30%			
DETERMINISTIC RECOVERY					
$VaR_{99\%}$	11.07%	11.01%	2 sec	350 sec	
$CVaR_{99\%}$	14.23%	14.22%			
$\beta_1^R = \beta_2^R = 0$	NORMAL RECOVERY				
	$VaR_{99\%}$	11.07%	11.00%	9 sec	730 sec
	$CVaR_{99\%}$	14.23%	14.22%		
	LOGNORMAL RECOVERY				
	$VaR_{99\%}$	11.07%	11.00%	654 sec	770 sec
	$CVaR_{99\%}$	14.24%	14.23%		
	BETA RECOVERY				
	$VaR_{99\%}$	11.07%	11.00%	40,181 sec	30,150 sec
	$CVaR_{99\%}$	14.24%	14.22%		
	KUMARASWAMY RECOVERY				
	$VaR_{99\%}$	11.07%	11.01%	1,201 sec	1,830 sec
	$CVaR_{99\%}$	14.23%	14.22%		
	LOGISTIC RECOVERY				
	$VaR_{99\%}$	11.06%	10.99%	621 sec	890 sec
$CVaR_{99\%}$	14.22%	14.21%			
DETERMINISTIC RECOVERY					
$VaR_{99\%}$	11.07%	11.01%	2 sec	350 sec	
$CVaR_{99\%}$	14.23%	14.22%			

Table 2: $VaR_{99\%}$ and $CVaR_{99\%}$ computations for LD vs. MC simulations for the different RR models in the settings with $\beta_1^R = \beta_2^R = 1$ (the common factor determines the RR completely) and $\beta_1^R = \beta_2^R = 0$ (the common factor has no influence on the RR, RRs for the different companies are independent and random).

		LD	MC	LD time	MC time
$\beta_1^R = \beta_2^R = 1$	NORMAL RECOVERY				
	$VaR_{99\%}$	20.00%	19.48%	8 sec	707 sec
	$CVaR_{99\%}$	28.43%	28.00%		
	LOGNORMAL RECOVERY				
	$VaR_{99\%}$	14.81%	14.63%	8 sec	750 sec
	$CVaR_{99\%}$	19.74%	19.44%		
	BETA RECOVERY				
	$VaR_{99\%}$	17.23%	16.81%	305 sec	29,049 sec
	$CVaR_{99\%}$	22.50%	22.04%		
	KUMARASWAMY RECOVERY				
	$VaR_{99\%}$	17.22%	16.80%	11 sec	1,699 sec
	$CVaR_{99\%}$	22.48%	22.02%		
	LOGISTIC RECOVERY				
	$VaR_{99\%}$	17.12%	16.68%	14 sec	946 sec
$CVaR_{99\%}$	22.39%	21.91%			
DETERMINISTIC RECOVERY					
$VaR_{99\%}$	7.86%	7.67%	3 sec	397 sec	
$CVaR_{99\%}$	10.10%	9.90%			
$\beta_1^R = \beta_2^R = 0$	NORMAL RECOVERY				
	$VaR_{99\%}$	7.92%	7.68%	8 sec	718 sec
	$CVaR_{99\%}$	10.17%	9.91%		
	LOGNORMAL RECOVERY				
	$VaR_{99\%}$	7.91%	7.69%	629 sec	790 sec
	$CVaR_{99\%}$	10.16%	9.97%		
	BETA RECOVERY				
	$VaR_{99\%}$	7.71%	7.68%	39,274 sec	31,593 sec
	$CVaR_{99\%}$	9.90%	9.91%		
	KUMARASWAMY RECOVERY				
	$VaR_{99\%}$	7.75%	7.68%	1,151 sec	1,722 sec
	$CVaR_{99\%}$	9.90%	9.90%		
	LOGISTIC RECOVERY				
	$VaR_{99\%}$	7.69%	7.68%	629 sec	895 sec
$CVaR_{99\%}$	9.85%	9.91%			
DETERMINISTIC RECOVERY					
$VaR_{99\%}$	7.86%	7.67%	3 sec	397 sec	
$CVaR_{99\%}$	10.10%	9.90%			

Table 3: $VaR_{99\%}$ and $CVaR_{99\%}$ computations for LD vs. MC simulations for the different RR models in the settings with $\beta_1^R = \beta_2^R = 1$ (the common factor determines the RR completely) and $\beta_1^R = \beta_2^R = 0$ (the common factor has no influence on the RR, RRs for the different companies are independent and random).

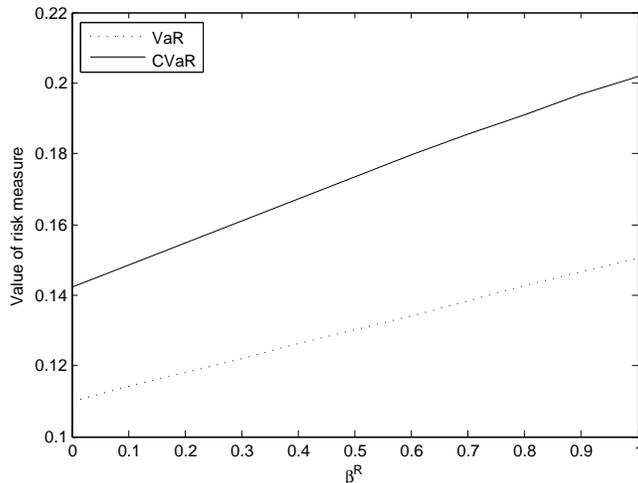


Figure 4: Risk measures VaR and CVaR are shown for the model with Kumaraswamy-distributed RR for different values of β_c^R .

case will be approximately the same. On the other hand, if systematic risk is incorporated, risk measures will be higher compared to the deterministic recovery case.

5 Portfolio optimization

In this section, a portfolio optimization is done to find optimal portfolios with regard to VaR and CVaR.

First, the optimization framework is introduced. Second, the numerical example from Section 4 is extended and a portfolio optimization is performed. Optimization is done using LD techniques, and MC simulations for comparison.

There are again C types of companies. Two constraints for the portfolio weights $x = (x_1, \dots, x_C)$ – chosen as the relative fractions of companies of type c – ensure a full investment ($\sum_{c=1}^C x_c = 1$), and avoid short-selling ($x_c \geq 0, c = 1, \dots, C$). Moreover the expected return of the portfolio shall at least meet a value of μ_{min} .

The return constraint may be expressed as

$$\sum_{c=1}^C x_c EAD_c \mu_c \geq \mu_{min}, \quad (5)$$

where μ_c are the expected returns for the different types of companies. It may seem odd that EAD_c appears in the left hand side of Equation 5, but this gets clear as x_c is not the fraction of cash investment into companies of type

c , but the fraction of companies of type c within the portfolio. So the different exposures for the different types of companies have to be taken into account.

For the optimization the question arises, how the values of the expected returns μ_c for the different types of companies should be determined. Market or expert estimates would probably be the best source.

The optimization problem is finally given by

$$\begin{aligned}
 & \min_{x \in \mathbb{R}^C} VaR_\alpha(L_P(n)) / \min_{x \in \mathbb{R}^C} CVaR_\alpha(L_P(n)) \\
 & \hspace{15em} \text{s.t.} \\
 & \hspace{10em} \sum_{c=1}^C x_c = 1 \\
 & \hspace{10em} x_c \geq 0, c = 1, \dots, C \\
 & \hspace{10em} \sum_{c=1}^C x_c EAD_c \mu_c \geq \mu_{min}.
 \end{aligned} \tag{6}$$

The numerical example from Section 4 is extended to 3 types of companies. Companies of type 1 and 2 have the same characteristics as before. Companies of type 3, which are the new investment possibility, have an exposure of 3 with a 3% probability of default and a RR of 50% with standard deviation 10%. The parameters β_c^R are specified as $\beta_c^R = 1$ for all three types of companies. Companies of type 3 share the same influence of the common factor on the creditworthiness index ($\beta_3 = 0.5$) as companies of type 1 and 2. The expected returns for companies of type 1, 2 and 3 are assumed to be 1%, 4% and 2.5%. The portfolio optimization is done for Kumaraswamy-distributed RRs because of several reasons. RRs are bounded between zero and one. Moreover, compared to the logistic function model, this model is less complicated from its setup. Furthermore, the beta distribution is computationally too slow. The whole setting for the optimization is summarized in Table 4.

Variable	Company type		
	1	2	3
EAD	6	4	3
β	0.5	0.5	0.5
\mathbb{P}_D	1%	5%	3%
Recovery (distribution)	Kumaraswamy		
Recovery (mean)	50%	30%	50%
Recovery (std)	10%	10%	10%
β^R	1	1	1
μ	1%	4%	2.5%

Table 4: Overview of the setting for the portfolio optimization.

To give an overview of the risk characteristics of a portfolio consisting of these 3 types of companies, Figure 5 shows VaR for the 99% confidence level

for different portfolio weights. It can be seen on the one hand that the lowest VaR is adopted for a full investment into companies of type 1. On the other hand, VaR is highest for a full investment into companies of type 2. Thus, companies of type 1 comprise the lowest and companies of type 2 the highest risk. Companies of type 3 have a risk somewhere in between. This makes sense as companies of type 1 are assumed to have the lowest expected return with 1% and companies of type 2 the highest with 4%.

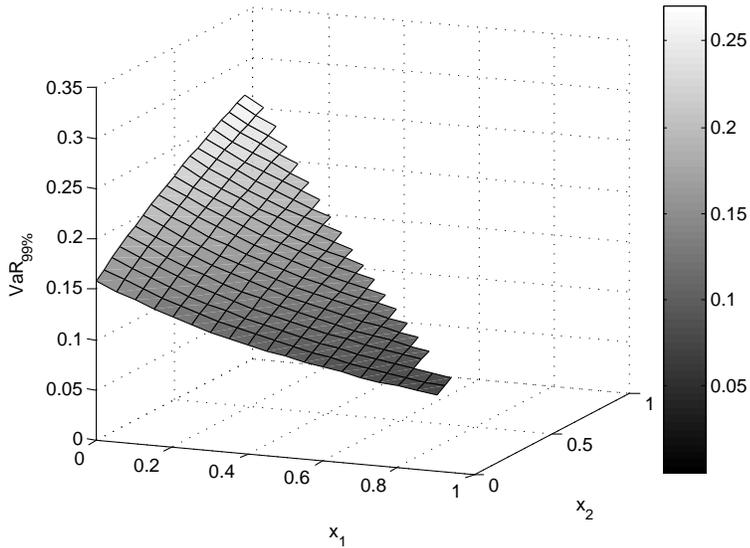
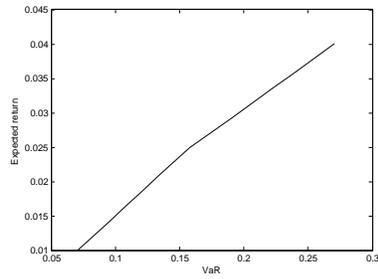


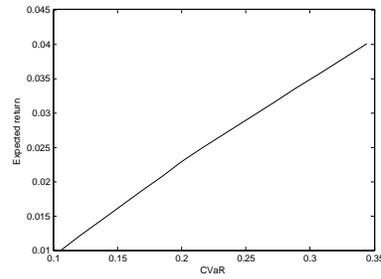
Figure 5: $VaR_{99\%}$ for the numerical example with 3 types of companies. The surface of the plot shows the $VaR_{99\%}$ for the different portfolios. x - and y -Axis show fractions of firms of type 1 and 2 in the portfolio, respectively the fraction of firms of type 3 in the portfolio is given by $1 - x_1 - x_2$.

Building up a portfolio of the 3 types of companies, each portfolio return between 1% and 4% is reachable. Computing optimal portfolios for VaR and CVaR with respect to the optimization problem given in (6) for changing μ_{min} yields the corresponding efficient frontiers. They are illustrated in Figure 6 together with the optimal portfolio weights. VaR and CVaR are the employed measures of risk plotted versus the expected return of the portfolio. It can be seen that the VaR and CVaR increase steadily with increasing expected return of the portfolio. The VaR (CVaR) for the 99% confidence level is 7.06% (10.57%) for a portfolio with an expected return of 1% and goes up to 27.07% (34.42%) for a portfolio with 4% expected return. When looking at the portfolio weights, it can be seen that the optimal portfolio in the VaR framework with 1% expected return consists of 100% companies of type 1. With increasing expected return companies of type 1 get replaced by companies of type 3, until for an expected

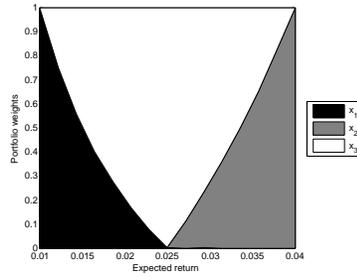
return of 2.5% all money is invested into companies of type 3. Further increasing the minimal expected return yields a replacement of companies of type 3 through companies of type 2. Finally, for a 100% investment into companies of type 2, the expected return is 4%. The optimal portfolio weights for the CVaR optimization are similar to those of the VaR optimization.



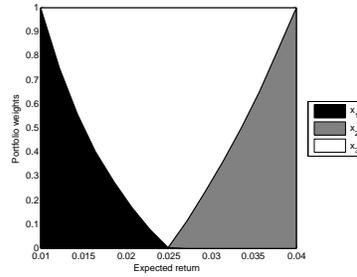
(a) Efficient frontier for 3 types of companies. VaR is used as a measure of risk versus the expected return.



(b) Efficient frontier for 3 types of companies. CVaR is used as a measure of risk versus the expected return.



(c) Portfolio weights for the optimal VaR portfolios.



(d) Portfolio weights for the optimal CVaR portfolios.

Figure 6: Efficient frontiers and portfolio weights for optimal portfolios.

6 Conclusion

The paper analyzes how LD techniques can be used in credit portfolio management. Five different models for the RR are evaluated, which account for systematic risk in RRs. The applicability of LD techniques turns out to be very flexible with a high accuracy in all models. Moreover, risk measures VaR and CVaR are computed efficiently using LDs and the computation is faster compared to simple MC simulations (except for beta-distributed RRs). Furthermore, it is proved that the positive correlation between the state of the economy and RRs plays a crucial role for risk measures, which may not be neglected. Thus, deterministic RRs may not be suitable for managing a portfolio of credit risky assets. Moreover, risk measures turn out to be higher with increasing influence of the common factor on the RR index. At last, a portfolio optimization is performed and optimal portfolios are derived using LD techniques. Future work may deal with generalized factor models and take into account more than one common factor, influencing the creditworthiness index and the RR index.

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