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On the Expected Discounted Penalty Function for a Risk Model Driven by a Spectrally Negative Lévy Process*

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Abstract

The Expected Discounted Penalty Function (EDPF) was introduced in a series of now classical papers [Gerber and Shiu (1997), (1998a), (1998b) and Gerber and Landry (1998)]. Several authors have studied this function in more general settings [for instance Tsai and Willmot (2002), Li and Garrido (2005) and Garrido and Morales (2006)]. In Morales (2007) we find results for the EDPF in a perturbed risk process with a subordinator as the model for the aggregate claims and a Brownian motion as the model for the perturbation. In this note, we study the EDPF for a model where the aggregate claims are driven by a subordinator and the perturbation is modeled by a spectrally negative Lévy process. This generalizes results found in Morales (2007).

KEYWORDS

Risk Theory, Expected Discounted Penalty Function, Lévy Processes, Renewal Equations
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1 Introduction

We consider a generalization of the perturbed risk model of Dufresne and Gerber (1991). We substitute the compound Poisson process by a subordinator and the perturbation by an independent spectrally negative Lévy process. This substitution yields

$$R(t) = u + c\,t - S(t) + Z(t)\ , \qquad t \geq 0\ , \tag{1}$$

where S is a subordinator with zero drift and Z is a zero-mean spectrally Lévy process. The parameter u is the initial surplus and c is a constant premium rate defined as $c = (1+\theta)\,\mathbb{E}\,[S(1)]$, where θ is the security loading factor. If we assume the net-profit condition $\theta > 0$ then $\lim_{t\to\infty} R(t) = \infty$. This is the model in Furrer and Schmidli (1994) and Yang and Zhang (2001). For an account on the classical risk model we refer to Asmussen (2000) or Kaas *et al.* (2001).

The purpose of this note is to show that the Expected Discounted Penalty Function (EDPF) for the risk model in (1) satisfies a Defective Renewal Equation (DRE) that generalizes equivalent results in Gerber and Landry (1998), Tsai and Willmot (2002) and Morales (2007). We note that (1) is a process with independent and stationary increments, i.e. a Lévy process, it is a subclass of the family of models discussed in Bertoin and Doney (1994), Morales and Schoutens (2003), Huzak *et al.* (2004) and Doney and Kyprianou (2006). It is slightly more general than models discussed in Dufresne *et al.* (1991), Morales (2004) and Garrido and Morales (2006) since we are adding a spectrally negative Lévy process to perturb the aggregate claim process. Yet, expressions for the EDPF equivalent to those found in Tsai and Willmot (2002) and Morales (2007) remain unexplored in setting (1).

In Morales (2007) we studied the EDPF for a perturbed risk model where the compound Poisson sum in the aggregate claims process has been substituted by a subordinator. Here, we work out an extension for a risk process driven by a subordinator and perturbed by a spectrally negative Lévy motion.

We start by giving a more detailed description of the model. The Laplace exponent of the subordinator in (1) is given by

$$\Psi_S(\xi) = \frac{1}{t} \ln\ \mathbb{E}\left[e^{-\xi\,S(t)}\right] = \int_0^\infty \left(e^{-\xi\,x} - 1\right)\,\nu_S(dx)\ , \qquad \xi > 0\ , \tag{2}$$

where ν_S is the Lévy mesure of S, i.e. it is a positive measure on $\mathbb{R}_+$ satisfying $\int_0^\infty (1 \wedge x)\,\nu_S(dx) < \infty$.

We define the perturbation Z in (1) through the dual process $-Z$. Notice that the perturbation has only negative jumps and therefore the Laplace exponent of the dual process $-Z$ can be written as

$$\Psi_{-Z}(\xi) = \frac{1}{t}\ln\ \mathbb{E}\left[e^{\xi\, Z(t)}\right] = \frac{\sigma^2}{2}\,\xi^2 + \int_0^\infty \left(e^{-\xi\, x} - 1 + \xi\, x\right)\,\nu_Z(dx)\ , \qquad \xi > 0\ , \tag{3}$$

where $\sigma \geqslant 0$ and ν_Z is a positive measure on $\mathbb{R}_+$ satisfying the usual condition $\int_0^\infty (1\wedge x^2)\,\nu_Z(dx) < \infty$ and the additional condition $\int_0^\infty x\,\nu_Z(dx) < \infty$. This last condition ensures finite expectation of Z. Notice that Z might contain a Brownian component (if $\sigma > 0$) as well as a pure jump component and so it generalizes the model in Morales (2007). It reduces to the lattest if ν_Z is the null measure. In this paper, we often have to refer to each component of Z separately and so we use the convention

$$Z(t) = W(t) + J(t) \qquad t \geqslant 0\ , \tag{4}$$

where W is a zero-mean Brownian motion and J is a zero-mean pure jump process defined through

$$\begin{aligned}
\Psi_{-W}(\xi) &= \frac{\sigma^2}{2}\,\xi^2\ , \qquad \xi > 0\ , \\
\Psi_{-J}(\xi) &= \int_0^\infty \left(e^{-\xi\, x} - 1 + \xi\, x\right)\,\nu_Z(dx)\ , \qquad \xi > 0\ ,
\end{aligned}$$

such that $\Psi_{-Z}(\xi) = \Psi_{-W}(\xi) + \Psi_{-J}(\xi)$.

For a comprehensive account on subordinators and spectrally Lévy processes we refer to Bertoin (1996) and Sato (1999). For a more detailed discussion on subordinators and Lévy processes in risk theory we refer to Morales and Schoutens (2003) and Garrido and Morales (2006).

Before continuing with or discussion we need to state some definitions.

Definition 1.1 *We define the ruin time τ associated with the risk process (1) as* $\tau = \inf\{t \geqslant 0\ ;\ R(t) \leqslant 0\}$.

Definition 1.2 *We define the ruin probability ψ associated with the risk process (1) as* $\psi(u) = \mathbb{P}\{\tau < \infty \mid R(0) = u\}$.

Since (1) is a Lévy process, we notice that our model belongs to the class studied in Huzak *et al.* (2004) and Doney and Kyprianou (2006). Exit problems for Lévy processes have been extensively studied in the last decade [we refer to Kyprianou (2006) for a comprehensive treatment], in particular fluctuation theory for spectrally negative processes is well understood. This recent developments will allow us to obtain expressions for the EDPF in a model like (1).

Gerber and Shiu (1998a) introduced the concept of discounted penalty function as a mean to study the distribution of the time to ruin, the amount at and immediately prior to ruin [we refer to the original paper of Gerber and Shiu (1998a) for a thorough discussion on all the relevant features of the EDPF]. Later, Gerber and Landry (1998), Tsai and Willmot (2002) and Morales (2007) considered a generalized EDPF that distinguished between the two possible sources of ruin: ruin by a claim and ruin due to the continuous perturbation. In this note we have no use for such a distiction and, in consequence, we use the following alternative definition of the EDPF:

Definition 1.3 *We define the EDPF ϕ_P associated with the risk process (1) as*

$$\phi_{P_\delta}(u) = \mathbb{E}\left[w\left(R(\tau_-), |R(\tau)|\right) e^{-\delta\tau} \mathbb{I}_{\{\tau<\infty\}} | R(0) = u\right] . \tag{5}$$

where the function w is a non-negative function defined on $\mathbb{R}^2_+ \cup \{(0,0)\}$ and τ is the time of ruin.

Remark 1.1 *We remark that ϕ_P depends on δ. For convenience we have not incorporated this dependance into the notation.*

We know that spectrally negative Lévy processes only creep downwards if they have a Brownian component [see Kyprianou (2006) for a discussion on path properties of Lévy processes] and therefore if $R(\tau) = 0$, ruin was necessarily caused by the perturbation. The definition in (5) implies that there is a constant penalty w_0 if ruin is caused by the perturbation (i.e. when $R(\tau) = 0$ which can only happen if there is a Brownian component in Z) and this constnat is given in fact by $w_0 = w(0, 0)$.

When the aggregate claims in (1) is a compound Poisson process, Tsai and Willmot (2002) showed that the EDPF ϕ follows a DRE [Theorem 2 in their paper]. The same result holds in a slightly more general form when the aggregate claims in (1) is a subordinator [Morales (2007)].

In this note, we show that the EDPF associated with a risk model as in (1) satisfies a DRE that generalizes the equivalent equation in Morales (2007). In Section 2, we first derive an integro-differential equation that has to be satisfied by the associated EDPF (5). The main result and its derivation are then presented in Section 3.

2 An Integro-Differential Equation for the EDPF in the General Perturbed Model

In this section we will derive an integro-differential equation (IDE) for the EDPF. First, we notice that using Ito's lemma we can identify the infinitesimal generator of the process R in (1) as

$$\begin{aligned} \mathcal{A}g(u) &= \frac{\sigma^2}{2}\, g''(u) + c\, g'(u) + \int_0^\infty [g(u-y) - g(u)]\, \nu_S(dy) \\ &\quad + \int_0^\infty \left[g(u-y) - g(u) + y\, g'(u)\right] \nu_Z(dy)\ . \end{aligned} \tag{6}$$

We can now prove the following result that will allow us to write an IDE for the EDPF.

Theorem 2.1 *Assume that ϕ_P is bounded and twice continuously differentiable on $[0,\infty)$ where the derivative at $u=0$ means the right-hand derivative. If ϕ_P solves*

$$\mathcal{A}\phi_P(u) = \delta\phi_P(u)\ , \qquad u > 0\ , \tag{7}$$

together with the following conditions

$$\lim_{u\to\infty} \phi_P(u) = 0 \tag{8}$$

$$\phi_P(u-z) = \begin{cases} \phi_P(u-z) & \text{if} \quad u-z > 0\ , \\ w(u, z-u) & \text{if} \quad u-z \leqslant 0\ , \end{cases} \tag{9}$$

where w is a non-negative function on $\mathbb{R}_+^2 \cup \{(0,0)\}$ with $w(u,0) = w_0$ for some constant w_0, then ϕ_P is the EDPF associated with (1), i.e.

$$\phi_P(u) = \mathbb{E}\left[w\left(R(\tau_-), |R(\tau)|\right) e^{-\delta\tau}\mathbb{I}_{\{\tau<\infty\}} | R(0) = u\right]\ .$$

Proof. Define the following function on $\mathbb{R}_+^2 \cup \{(0,0)\}$:

$$\bar{g}_\delta(u,z) = \phi_P(u-z)\ , \qquad u, z \geqslant 0\ .$$

If we make a change of variables, $u = x+y$ and $z = y$, we can write

$$g_\delta(x,y) = \bar{g}_\delta(x+y, y) = \phi_P(x+y-y)\ , \qquad x, y \geqslant 0\ ,$$

and condition (9) can be written as

$$g_\delta(x,y) = \begin{cases} \phi_P(x) & \text{if} \quad x > 0 \ , \\ \\ w(x+y,-x) & \text{if} \quad x \leqslant 0 \ , \end{cases} \tag{10}$$

Now, following a similar argument as in Paulsen and Gjessing (1997), we can use Ito's formula on the function $h_\delta(t,x,y) = e^{-\delta t} g_\delta(x,y)$ applied to the 3-dimensional process $(t, R(t), \Delta R(t))$ where we have defined $\Delta R(t) = R(t-) - R(t)$. Notice that the function h might not be continuous at $x = 0$ so we have to define the function $h_n(t,x,y) = h(t,x,y)\mathbb{I}_{\{(-\infty,-1/n]\cup[0,\infty)\}}(x)$ and a stopping time $T_b = \inf\{t \mid R(t) \geqslant b\}$ for $b > u$ (notice that $\tau < T_b$ a.s.).

Using Ito's lemma on $h_n(t\wedge\tau\wedge T_b, R(t\wedge\tau\wedge T_b), \Delta R(t\wedge\tau\wedge T_b))$, along with the condition on the infinitesimal generator (7), show that

$$h_n(t\wedge\tau\wedge T_b, R(t\wedge\tau\wedge T_b), \Delta R(t\wedge\tau\wedge T_b)) - h_n(0, R(0), \Delta R(0))$$

is a zero-mean martingale yielding

$$\mathbb{E}[h_n(t\wedge\tau\wedge T_b, R(t\wedge\tau\wedge T_b), \Delta R(t\wedge\tau\wedge T_b))] = \mathbb{E}[h_n(0, R(0), \Delta R(0))] \ .$$

Letting $b \to \infty$ (i.e. $T_b \to \infty$) and $n \to \infty$, bounded convergence yields

$$\mathbb{E}[h(t\wedge\tau, R(t\wedge\tau), \Delta R(t\wedge\tau))] = \mathbb{E}[h(0, R(0), \Delta R(0))] \ .$$

Because of (10) and since $R(0) = u$ and $\Delta R(0) = 0$, the right-hand side becomes,

$$\mathbb{E}[g_\delta(R(0), \Delta R(0))] = g_\delta(u, 0) = \phi_P(u) \ ,$$

and finally we have the following expression

$$\mathbb{E}[e^{-\delta(t\wedge\tau)} g_\delta(R(t\wedge\tau), \Delta R(t\wedge\tau))] = \phi_P(u) \ . \tag{11}$$

The left-hand side of this last equation can be written as

$$\begin{aligned} \mathbb{E}[e^{-\delta(t\wedge\tau)} g_\delta(R(t\wedge\tau), \Delta R(t\wedge\tau))] \quad &= \quad \mathbb{E}[e^{-\delta\tau} g_\delta(R(\tau), \Delta R(\tau))\mathbb{I}_{\{\tau\leqslant t\}}] \\ &\quad + \mathbb{E}[e^{-\delta t} g_\delta(R(t), \Delta R(t))\mathbb{I}_{\{\tau>t\}}] \ . \end{aligned}$$

Using the fact that $R(\tau) < 0$ and $R(t) > 0$ (on $\{t < \tau\}$) along with condition (10) we have

$$\begin{aligned} \mathbb{E}[e^{-\delta(t\wedge\tau)} g_\delta(R(t\wedge\tau), \Delta R(t\wedge\tau))] \quad &= \quad \mathbb{E}[e^{-\delta\tau} w(R(\tau^-), |R(\tau)|)\mathbb{I}_{\{\tau\leqslant t\}}] \\ &\quad + \mathbb{E}[e^{-\delta t} \phi_P(R(t))\mathbb{I}_{\{\tau>t\}}] \ . \end{aligned}$$

Substituting in (11) yields

$$\mathbb{E}[e^{-\delta\tau} w(R(\tau^-), |R(\tau)|)\mathbb{I}_{\{\tau\leqslant t\}}] + \mathbb{E}[e^{-\delta t}\phi_P(R(t))\mathbb{I}_{\{\tau>t\}}] \quad = \quad \phi_P(u) \; .$$

Taking limits when $t \to \infty$ and using the fact that the net-profit condition assures $\lim_{t\to\infty} R_t = \infty$ along with condition (8), we have that the second term in the left-hand side vanishes yielding

$$\mathbb{E}[e^{-\delta\tau} w(R(\tau^-), |R(\tau)|)\mathbb{I}_{\{\tau\leqslant\infty\}}] \quad = \quad \phi_P(u) \; ,$$

which completes the proof ■

The following corollary gives an IDE for the EDPF associated with (1).

Corollary 2.1 *If ϕ_P is the EDPF associated with the risk process (1), it satisfies conditions (8) and (9) along with the following integro-differential equation (IDE)*

$$\begin{aligned} &\frac{\sigma^2}{2}\,\phi''_P(u) + c\,\phi'_P(u) + \int_0^u [\phi_P(u-y) - \phi_P(u)]\,\nu_S(dy) + \chi_S(u) \\ &+ \int_0^\infty \left[\phi_P(u-y) - \phi_P(u) + y\,\phi'_P(u)\right]\nu_Z(dy) = [\Pi_S(u) + \delta]\,\phi_P(u) \; , \end{aligned} \tag{12}$$

where $\Pi_S(u) = \int_0^u \nu_S(dy)$ is the integrated tail of the Lévy measure ν_S and $\chi_S(u) \equiv \int_u^\infty w(u, y-u)\nu_S(dy)$.

Proof. A simple substitution of the expression for the infinitesimal generator (6) and condition (9) into (7) shows that the EDPF satisfies the above mentioned IDE. ■

Notice that equation (12) reduces to the IDE in Tsai and Willmot (2002) when the measure ν_Z is the null measure.

3 A Defective Renewal Equation for EDPF in the General Perturbed Model

We start our discussion by defining a generalized Lundberg equation and the corresponding Lundberg coefficient for the model in (1). This will be important in the derivations of the main result of this section.

Definition 3.1 *We define the generalized Lundberg coefficient ρ as the non-negative solution of*

$$c\,r + \Psi_{S-Z}(r) = \delta\ , \tag{13}$$

where Ψ_{S_Z} is the Laplace exponent of the process $S - Z$, i.e.

$$\Psi_{S-Z}(r) = \frac{1}{t} \ln\ \mathbb{E}\left[e^{-r\,(S(t)-Z(t))}\right]\ .$$

Notice that because of the independence between S and Z we have that $\mathbb{E}\left[e^{-r\,(S(t)-Z(t))}\right] = \exp\left[t(\Psi_S(r) + \Psi_{-Z}(r))\right]$. This implies that the Laplace exponent of $S - Z$ can be written in terms of the exponents defined in (2) and (3), i.e. $\Psi_{S-Z}(r) = \Psi_S(r) + \Psi_{-Z}(r)$. In view of this, we can write out in more detail an expression for the Lundberg equation (13), namely

$$c\,r + \int_0^\infty \left(e^{-r\,x} - 1\right)\,\nu_S(dx) + \frac{\sigma^2}{2}\,r^2 + \int_0^\infty \left(e^{-r\,x} - 1 + r\,x\right)\,\nu_Z(dx) = \delta\ . \tag{14}$$

One can verify that this last equation reduces to classical forms of the Lundberg equation for particular choices of S and Z.

In order to solve the IDE (12), we use standard Laplace transform techniques and a weak convergence argument. We first define a family of spectrally negative Lévy processes $\{R_\epsilon\}_{\epsilon>0}$ converging weakly to (1). If we set

$$\nu_Z^\epsilon(dx) = \mathbb{I}_{(\epsilon,\infty)}(x)\,\nu_Z(dx)\ , \quad x \geqslant 0 \tag{15}$$

in (3), we are defining the following family of processes:

$$R_\epsilon(t) = u + c\,t - S(t) + Z_\epsilon(t)\ , \qquad t \geq 0\ . \tag{16}$$

Standard techniques show that R_ϵ converges weakly to R since $\lim_{\epsilon\to 0} \nu_Z^\epsilon(dx) = \nu_Z(dx)$[see Morales (2007)]. Let us denote by ϕ_P^ϵ its associated EDPF and by ρ_ϵ the non-negative solution of the associated Lundberg equation (14). Using the definition (15) and the weak convergence of $R_\epsilon \xrightarrow{d} R$ we notice that $\rho_\epsilon \to \rho$ and $\mathcal{A}\phi_P^\epsilon \to \mathcal{A}\phi_P$ as $\epsilon \to 0$, finally implying that $\phi_P^\epsilon \to \phi_P$. In order to avoid an integrability problem, we will work with the process R_ϵ to derive an expression for the Laplace transform ϕ_P. We will then take the limit of $\widehat{\phi_P^\epsilon}$ as $\epsilon \to 0$ in order to find an expression for $\widehat{\phi_P}$. The following two lemmas will be needed in the proof of the main result:

Lemma 3.1 *Given a risk model like (16) and its associated EDPF ϕ_P^ϵ, we have that, for any $\epsilon > 0$*

$$\frac{\sigma^2}{2}\,\rho_\epsilon\,{\phi_P^\epsilon}'(0) = \widehat{\chi_S}(\rho_\epsilon)+\widehat{\chi_Z^\epsilon}(\rho_\epsilon)-\frac{\sigma^2}{2}\,\rho_\epsilon\,\phi_P^\epsilon(0)-c\,\phi_P^\epsilon(0)-\phi_P^\epsilon(0)\int_0^\infty y\,\mathbb{I}_{(\epsilon,\infty)}(y)\,\nu_Z(dy)\ , \tag{17}$$

where $\widehat{f}$ denotes the Laplace transform of the function f and

$$\chi_Z^\epsilon(u) \equiv \int_u^\infty w(u,y-u)\,\mathbb{I}_{(\epsilon,\infty)}(y)\nu_Z(dy)\ .$$

Proof.

For any $\epsilon > 0$, the process R_ϵ in (16) is spectrally negative and so equation (12) holds for ϕ_P^ϵ. By taking Laplace transforms in (12) we get

$$\begin{aligned}
&\frac{\sigma^2}{2}\left[\xi^2\,\widehat{\phi}_P^\epsilon(\xi)-\xi\,\phi_P^\epsilon(0)-{\phi_P^\epsilon}'(0)\right]+c\left[\xi\,\widehat{\phi}_P^\epsilon(\xi)-\phi_P^\epsilon(0)\right]\\
&+\int_0^\infty e^{-\xi u}\int_0^u\left[\phi_P^\epsilon(u-y)-\phi_P^\epsilon(u)\right]\nu_S(dy)\,du+\widehat{\chi}_S(\xi)\\
&+\int_0^\infty e^{-\xi u}\int_0^\infty\left[\phi_P^\epsilon(u-y)-\phi_P^\epsilon(u)+y\,{\phi_P^\epsilon}'(u)\right]\nu_Z^\epsilon(dy)\,du\\
=\ &\int_0^\infty e^{-\xi u}\left[\Pi_S(u)+\delta\right]\phi_P^\epsilon(u)\,du\ .
\end{aligned} \tag{18}$$

Now, if we set $\xi = \rho_\epsilon$ and we use the fact that ρ_ϵ is the solution of (14), then this last equation becomes

$$\begin{aligned}
&\frac{\sigma^2}{2}\left[\rho_\epsilon^2\,\widehat{\phi}_P^\epsilon(\rho_\epsilon)-\rho_\epsilon\,\phi_P^\epsilon(0)-{\phi_P^\epsilon}'(0)\right]+c\left[\rho_\epsilon\,\widehat{\phi}_P^\epsilon(\rho_\epsilon)-\phi_P^\epsilon(0)\right]\\
&+\int_0^\infty e^{-\rho_\epsilon u}\int_0^u\left[\phi_P^\epsilon(u-y)-\phi_P^\epsilon(u)\right]\nu_S(dy)\,du+\widehat{\chi}_S(\rho_\epsilon)\\
&+\int_0^\infty e^{-\rho_\epsilon u}\int_0^\infty\left[\phi_P^\epsilon(u-y)-\phi_P^\epsilon(u)+y\,{\phi_P^\epsilon}'(u)\right]\nu_Z^\epsilon(dy)\,du\\
=\ &\int_0^\infty e^{-\rho_\epsilon u}\left\{\begin{array}{c}\Pi_S(u)+c\,\rho_\epsilon+\frac{\sigma^2}{2}\,\rho_\epsilon^2\\ +\int_0^\infty\left(e^{-\rho_\epsilon x}-1\right)\,\nu_S(dx)\\ +\int_0^\infty\left(e^{-\rho_\epsilon x}-1+\rho_\epsilon x\right)\,\nu_Z^\epsilon(dx)\end{array}\right\}\phi_P^\epsilon(u)\,du\ .
\end{aligned}$$

After a straight forward cancelation of terms we get

$$\begin{aligned}
&\frac{\sigma^2}{2}\left[-\rho_\epsilon\,\phi_P^\epsilon(0)-{\phi_P^\epsilon}'(0)\right]+c\left[-\phi_P^\epsilon(0)\right]+\widehat{\chi}_S(\rho_\epsilon)\\
&+\int_0^\infty e^{-\rho_\epsilon u}\int_0^u \phi_P^\epsilon(u-y)\nu_S(dy)\,du\\
&+\int_0^\infty e^{-\rho_\epsilon u}\int_0^\infty \left[\phi_P^\epsilon(u-y)-\phi_P^\epsilon(u)+y\,{\phi_P^\epsilon}'(u)\right]\nu_Z^\epsilon(dy)\,du\\
=\;&\int_0^\infty e^{-\rho_\epsilon u}\left\{\begin{array}{c}\int_0^\infty e^{-\rho_\epsilon\,x}\,\nu_S(dx)\\ +\int_0^\infty\left(e^{-\rho_\epsilon\,x}-1+\rho_\epsilon\,x\right)\,\nu_Z^\epsilon(dx)\end{array}\right\}\phi_P^\epsilon(u)\,du\;.
\end{aligned} \tag{19}$$

Now, a change of variable shows that

$$\int_0^\infty e^{-\rho_\epsilon u}\int_0^u \phi_P^\epsilon(u-y)\nu_S(dy)\,du=\int_0^\infty e^{-\rho_\epsilon u}\int_0^\infty e^{-\rho_\epsilon\,x}\,\nu_S(dx)\phi_P^\epsilon(u)\,du\;. \tag{20}$$

In view of which we can cancel one more term and equation (19) can be rewritten as

$$\begin{aligned}
&\frac{\sigma^2}{2}\left[-\rho_\epsilon\,\phi_P^\epsilon(0)-{\phi_P^\epsilon}'(0)\right]+c\left[-\phi_P^\epsilon(0)\right]+\widehat{\chi}_S(\rho_\epsilon)\\
&+\int_0^\infty e^{-\rho_\epsilon u}\int_0^u \left[\phi_P^\epsilon(u-y)-\phi_P^\epsilon(u)+y\,{\phi_P^\epsilon}'(u)\right]\nu_Z^\epsilon(dy)\,du\\
&+\int_0^\infty e^{-\rho_\epsilon u}\int_u^\infty \left[\phi_P^\epsilon(u-y)-\phi_P^\epsilon(u)+y\,{\phi_P^\epsilon}'(u)\right]\nu_Z^\epsilon(dy)\,du\\
=\;&\int_0^\infty e^{-\rho_\epsilon u}\left\{\int_0^\infty\left(e^{-\rho_\epsilon\,x}-1+\rho_\epsilon\,x\right)\,\nu_Z^\epsilon(dx)\right\}\phi_P^\epsilon(u)\,du\;.
\end{aligned} \tag{21}$$

Once again, a change of variable shows that

$$\int_0^\infty e^{-\rho_\epsilon u}\int_0^u \phi_P^\epsilon(u-y)\nu_Z^\epsilon(dy)\,du=\int_0^\infty e^{-\rho_\epsilon u}\int_0^\infty e^{-\rho_\epsilon\,x}\,\nu_Z^\epsilon(dx)\phi_P^\epsilon(u)\,du\;, \tag{22}$$

and so, after further cancellations, we can rewrite (21) as

$$
\begin{aligned}
&\frac{\sigma^2}{2}\left[-\rho_\epsilon\,\phi_P^\epsilon(0)-\phi_P^{\epsilon\,\prime}(0)\right]+c\left[-\phi_P^\epsilon(0)\right]+\widehat{\chi}_S(\rho_\epsilon)\\
&+\int_0^\infty e^{-\rho_\epsilon u}\int_0^\infty y\,\phi_P^{\epsilon\,\prime}(u)\nu_Z^\epsilon(dy)\,du\\
&+\int_0^\infty e^{-\rho_\epsilon u}\int_u^\infty \phi_P^\epsilon(u-y)\nu_Z^\epsilon(dy)\,du\\
=\;&\int_0^\infty e^{-\rho_\epsilon u}\int_0^\infty \rho_\epsilon\,x\,\nu_Z(dx)\phi_P^\epsilon(u)\,du\;.
\end{aligned} \tag{23}
$$

Now, we notice that for the second term we have

$$
\int_0^\infty e^{-\rho_\epsilon u}\int_0^\infty y\,\phi_P^{\epsilon\,\prime}(u)\nu_Z^\epsilon(dy)\,du=\int_0^\infty y\,\nu_Z^\epsilon(dy)\int_0^\infty e^{-\rho_\epsilon u}\phi_P^{\epsilon\,\prime}(u)\,du\;,
$$

which can be rewritten as

$$
\int_0^\infty y\,\nu_Z^\epsilon(dy)\left[\rho_\epsilon\widehat{\phi}_P^\epsilon(\rho_\epsilon)-\phi_P^\epsilon(0)\right]=\int_0^\infty y\,\nu_Z^\epsilon(dy)\left[\rho_\epsilon\int_0^\infty e^{-\rho_\epsilon u}\phi_P^\epsilon(u)\,du-\phi_P^\epsilon(0)\right]\;, \tag{24}
$$

because $\widehat{f'}(\xi)=\xi\widehat{f}(\xi)-f(0)$. Notice that we are using the fact that $\int_0^\infty y\,\nu_Z^\epsilon(dy)<\infty$. This is why we work with R_ϵ. Substituiting (24) into (23) yields further cancellations and we can finally write

$$
\begin{aligned}
&\frac{\sigma^2}{2}\left[-\rho_\epsilon\,\phi_P^\epsilon(0)-\phi_P^{\epsilon\,\prime}(0)\right]+c\left[-\phi_P^\epsilon(0)\right]+\widehat{\chi}_S(\rho_\epsilon)\\
&\qquad-\phi_P^\epsilon(0)\int_0^\infty y\,\nu_Z^\epsilon(dy)+\widehat{\chi}_Z(\rho_\epsilon)=0\;,
\end{aligned} \tag{25}
$$

where $\chi_Z^\epsilon(u)\equiv\int_u^\infty\phi_P^\epsilon(u-y)\nu_Z^\epsilon(dy)=\int_u^\infty w(u,y-u)\,\nu_Z^\epsilon(dy)$. This concludes the proof. ■

The following lemma gives an expression for the Laplace transform of the EDPF ϕ_P^ϵ.

Lemma 3.2 *Consider a risk model as in (16) and its associated EDPF ϕ_P^ϵ, we have that, for any $\epsilon>0$*

$$
\widehat{\phi}_P^\epsilon(\xi)=\frac{\widehat{H}_w^\epsilon(\xi)+w_0\widehat{A}^\epsilon(\xi)}{1-\widehat{g}_P^\epsilon(\xi)}\;,\qquad \xi\geqslant 0\;, \tag{26}
$$

where the functions g_P^ϵ, H_w^ϵ and A^ϵ are defined through their Laplace transforms as follows:

$$\widehat{g}_P^\epsilon(\xi) \quad = \quad \frac{[\Psi_{S-J^\epsilon}(\xi) - \Psi_{S-J^\epsilon}(\rho_\epsilon)]}{c(\rho_\epsilon - \xi)}\widehat{k}(\rho_\epsilon + \xi) \; , \qquad \xi \geqslant 0 \; ,$$

$$\widehat{H}_w^\epsilon(\xi) \quad = \quad \frac{[\widehat{\chi}^\epsilon(\xi) - \widehat{\chi}^\epsilon(\rho_\epsilon)]}{c(\rho_\epsilon - \xi)}\widehat{k}(\rho_\epsilon + \xi) \; , \qquad \xi \geqslant 0 \; ,$$

with

$$\widehat{\chi}^\epsilon(\xi) = \int_0^\infty e^{-\xi x}\left[\int_x^\infty w(x, y-x)\nu_S(dy) + \int_x^\infty w(x, y-x)\nu_Z^\epsilon(dy)\right] \, dx \; , \qquad \xi \geqslant 0 \; ,$$

$$A^\epsilon(x) = e^{-\rho_\epsilon \, x}[1 - K(x)] \; , \quad x \geqslant 0 \; ,$$

$$K(x) = \int_0^x k(s)ds = \int_0^x \frac{2c}{\sigma^2} \, e^{-\frac{2c}{\sigma^2} \, s} \, ds \; , \quad x \geqslant 0 \; ,$$

Ψ_{S-J^ϵ} is the Laplace exponent of $S - J^\epsilon$ and J^ϵ is the pure-jump component of Z^ϵ.

Proof.

Let us go back to equation (12). We had established that taking Laplace transforms in (12) yields (18). Now, if we use the fact that ρ_ϵ is the solution of (14) then, after similar cancellations as those found in the proof of Lemma 3.1 and rearranging some terms, equation (18) can be written

$$\begin{aligned}
& \left[\frac{\sigma^2}{2}\,\xi^2 + c\,\xi\right]\widehat{\phi}_P^\epsilon(\xi) + \frac{\sigma^2}{2}\,\left[-\xi\,\phi_P^\epsilon(0) - {\phi_P^\epsilon}'(0)\right] - c\,\phi_P^\epsilon(0) \\
& + \int_0^\infty e^{-\xi u}\int_0^u \phi_P^\epsilon(u-y)\nu_S(dy)\,du + \widehat{\chi}_S(\xi) \\
& + \int_0^\infty e^{-\xi u}\int_0^u \left[\phi_P^\epsilon(u-y) + y\,{\phi_P^\epsilon}'(u)\right]\nu_Z^\epsilon(dy)\,du \\
& + \int_0^\infty e^{-\xi u}\int_u^\infty \left[\phi_P^\epsilon(u-y) + y\,{\phi_P^\epsilon}'(u)\right]\nu_Z^\epsilon(dy)\,du \\
= \; & \int_0^\infty e^{-\xi u}\left\{\begin{array}{c} c\,\rho_\epsilon + \frac{\sigma^2}{2}\,\rho_\epsilon^2 + \int_0^\infty e^{-\rho_\epsilon\,x}\,\nu_S(dx) \\ + \int_0^\infty \left(e^{-\rho_\epsilon\,x} + \rho_\epsilon\,x\right)\,\nu_Z^\epsilon(dx) \end{array}\right\}\phi_P^\epsilon(u)\,du \; .
\end{aligned}$$

Now, if we use our previous definition for χ_Z^ϵ and we rearrange some more terms we can write

$$\left[\frac{\sigma^2}{2}\,\xi^2 + c\,\xi - \frac{\sigma^2}{2}\,\rho_\epsilon^2 - c\,\rho_\epsilon\right]\widehat{\phi}_P^\epsilon(\xi) + \frac{\sigma^2}{2}\,\left[-\xi\,\phi_P^\epsilon(0) - {\phi_P^\epsilon}'(0)\right] - c\,\phi_P^\epsilon(0)$$

$$+\int_0^\infty e^{-\xi u}\int_0^u \phi_P^\epsilon(u-y)\nu_S(dy)\,du + \widehat{\chi}_S(\xi)$$

$$+\int_0^\infty e^{-\xi u}\int_0^u \phi_P^\epsilon(u-y)\nu_Z^\epsilon(dy)\,du + \widehat{\chi}_Z^\epsilon(\xi)$$

$$+\int_0^\infty e^{-\xi u}\int_0^\infty y\,{\phi_P^\epsilon}'(u)\nu_Z^\epsilon(dy)\,du$$

$$= \int_0^\infty e^{-\xi u}\left\{\begin{array}{c}\int_0^\infty e^{-\rho_\epsilon\,x}\,\nu_S(dx)\\ +\int_0^\infty\left(e^{-\rho_\epsilon\,x}+\rho_\epsilon\,x\right)\,\nu_Z^\epsilon(dx)\end{array}\right\}\phi_P^\epsilon(u)\,du\;.$$

If we use (20) and (22), the above equation becomes

$$\begin{aligned}&\left[\frac{\sigma^2}{2}\,\xi^2 + c\,\xi - \frac{\sigma^2}{2}\,\rho_\epsilon^2 - c\,\rho_\epsilon\right]\widehat{\phi}_P^\epsilon(\xi) + \frac{\sigma^2}{2}\,\left[-\xi\,\phi_P^\epsilon(0) - {\phi_P^\epsilon}'(0)\right] - c\,\phi_P^\epsilon(0)\\ &+\int_0^\infty e^{-\xi u}\int_0^\infty e^{-\xi\,x}\,\nu_S(dx)\phi_P^\epsilon(u)\,du + \widehat{\chi}_S(\xi)\\ &+\int_0^\infty e^{-\xi u}\int_0^\infty e^{-\xi\,x}\,\nu_Z^\epsilon(dx)\phi_P^\epsilon(u)\,du + \widehat{\chi}_Z^\epsilon(\xi)\\ &+\int_0^\infty e^{-\xi u}\int_0^\infty y\,{\phi_P^\epsilon}'(u)\nu_Z^\epsilon(dy)\,du\\ =&\int_0^\infty e^{-\xi u}\left\{\begin{array}{c}\int_0^\infty e^{-\rho_\epsilon\,x}\,\nu_S(dx)\\ +\int_0^\infty\left(e^{-\rho_\epsilon\,x}+\rho_\epsilon\,x\right)\,\nu_Z^\epsilon(dx)\end{array}\right\}\phi_P^\epsilon(u)\,du\;.\end{aligned}\tag{27}$$

Moreover, because of equation (24), the last term in the left-hand side of the above equation can be written as

$$\int_0^\infty e^{-\xi u}\int_0^\infty y\,{\phi_P^\epsilon}'(u)\nu_Z^\epsilon(dy)\,du = \int_0^\infty y\,\nu_Z^\epsilon(dy)\int_0^\infty e^{-\xi u}{\phi_P^\epsilon}'(u)\,du\;,$$

$$= \int_0^\infty y\,\nu_Z^\epsilon(dy)\left[\xi\int_0^\infty e^{-\xi u}\phi_P^\epsilon(u)\,du - \phi_P^\epsilon(0)\right]\;. \tag{28}$$

Substituiting (28) into (27) and rearranging we get

$$\begin{aligned} & \left[\frac{\sigma^2}{2}\,\xi^2 + c\,\xi - \frac{\sigma^2}{2}\,\rho_\epsilon^2 - c\,\rho_\epsilon\right]\widehat{\phi}_P^\epsilon(\xi) + \frac{\sigma^2}{2}\,\left[-\xi\,\phi_P^\epsilon(0) - {\phi_P^\epsilon}'(0)\right] - c\,\phi_P^\epsilon(0) \\ & + \left[\int_0^\infty e^{-\xi\,x}\,\nu_S(dx) - \int_0^\infty e^{-\rho_\epsilon\,x}\,\nu_S(dx)\right]\widehat{\phi}_P^\epsilon(\xi) + \widehat{\chi}_S(\xi) \\ & + \left[\int_0^\infty e^{-\xi\,x}\,\nu_Z^\epsilon(dx) - \int_0^\infty e^{-\rho_\epsilon\,x}\,\nu_Z^\epsilon(dx)\right]\widehat{\phi}_P^\epsilon(\xi) + \widehat{\chi}_Z^\epsilon(\xi) \qquad (29) \\ & \left[\xi\int_0^\infty x\,\nu_Z^\epsilon(dx) - \rho_\epsilon\int_0^\infty x\,\nu_Z^\epsilon(dx)\right]\widehat{\phi}_P^\epsilon(\xi) \\ = \quad & \phi_P^\epsilon(0)\int_0^\infty x\,\nu_Z^\epsilon(dx)\;. \end{aligned}$$

If we now substitute $\frac{\sigma^2}{2}\,{\phi_P^\epsilon}'(0)$ by equation (17) from Lemma 3.1 into (29), we get after further simplifications

$$\begin{aligned} & \left[\frac{\sigma^2}{2}(\xi^2-\rho_\epsilon^2) + c(\xi-\rho_\epsilon) + \int_0^\infty e^{-\xi\,x}\,\nu_S(dx) - \int_0^\infty e^{-\rho_\epsilon\,x}\,\nu_S(dx)\right]\widehat{\phi}_P^\epsilon(\xi) \\ & + \left[\int_0^\infty e^{-\xi\,x}\,\nu_Z^\epsilon(dx) - \int_0^\infty e^{-\rho_\epsilon\,x}\,\nu_Z^\epsilon(dx) + (\xi-\rho_\epsilon)\int_0^\infty x\,\nu_Z^\epsilon(dx)\right]\widehat{\phi}_P^\epsilon(\xi) \\ = \quad & [\widehat{\chi}_Z^\epsilon(\rho_\epsilon) + \widehat{\chi}_S(\rho_\epsilon)] - [\widehat{\chi}_Z^\epsilon(\xi) + \widehat{\chi}_S(\xi)] + \frac{\sigma^2}{2}(\xi-\rho_\epsilon)\phi_P^\epsilon(0)\;. \qquad (30) \end{aligned}$$

Notice that the left-hand side in (30) can be simplified further yielding

$$\begin{aligned} & \left[\frac{\sigma^2}{2}(\xi^2-\rho_\epsilon^2) + c(\xi-\rho_\epsilon) + \int_0^\infty \left(e^{-\xi\,x}-1\right)\,\nu_S(dx) - \int_0^\infty \left(e^{-\rho_\epsilon\,x}-1\right)\,\nu_S(dx)\right]\widehat{\phi}_P^\epsilon(\xi) \\ & + \left[\int_0^\infty \left(e^{-\xi\,x}-1+\xi\,x\right)\,\nu_Z^\epsilon(dx) - \int_0^\infty \left(e^{-\rho_\epsilon\,x}-1+\rho_\epsilon\,x\right)\,\nu_Z^\epsilon(dx)\right]\widehat{\phi}_P^\epsilon(\xi) \\ = \quad & \left[\frac{\sigma^2}{2}(\xi^2-\rho_\epsilon^2) + c(\xi-\rho_\epsilon) + \Psi_S(\rho_\epsilon) - \Psi_S(\xi) + \Psi_{-J^\epsilon}(\rho_\epsilon) - \Psi_{-J^\epsilon}(\xi)\right]\widehat{\phi}_P^\epsilon(\xi)\;, \qquad (31) \end{aligned}$$

where J^ϵ is the pure-jump part of Z^ϵ as defined through (4).

Finally, using (31) we can rewrite equation (30) in the following simpler form:

$$\left[1 - \frac{\Psi_{S-J^\epsilon}(\xi) - \Psi_{S-J^\epsilon}(\rho_\epsilon)}{\frac{\sigma^2}{2}(\rho_\epsilon^2-\xi^2) + c(\rho_\epsilon-\xi)}\right]\widehat{\phi}_P^\epsilon(\xi) = \frac{\widehat{\chi}^\epsilon(\xi) - \widehat{\chi}^\epsilon(\rho_\epsilon) + \frac{\sigma^2}{2}(\rho_\epsilon-\xi)\phi_P^\epsilon(0)}{\frac{\sigma^2}{2}(\rho_\epsilon^2-\xi^2) + c(\rho_\epsilon-\xi)}\;, \qquad (32)$$

where $\chi^\epsilon(u) \equiv \chi_S(u) + \chi_Z^\epsilon(u)$.

This equation generalizes equivalent results in Tsai and Willmot (2002). From equation (32) we can isolate $\widehat{\phi}_P^\epsilon$. But before doing this, we notice that

$$\begin{aligned} \frac{\sigma^2}{2}(\rho_\epsilon^2 - \xi^2) + c(\rho_\epsilon - \xi) &= \left[\frac{\sigma^2}{2}(\rho_\epsilon + \xi) + c\right](\rho_\epsilon - \xi) \\ &= \frac{\left[\frac{\sigma^2}{2}(\rho_\epsilon + \xi)^2 + c(\rho_\epsilon + \xi)\right]}{c(\rho_\epsilon + \xi)}\, c(\rho_\epsilon - \xi) = \frac{\Psi_{-c-W}(\rho_\epsilon + \xi)}{c(\rho_\epsilon + \xi)}\, c(\rho_\epsilon - \xi), \end{aligned} \tag{33}$$

where we can recognize the Laplace transform of the density k appearing in the ladder-height decomposition for a model like (1) with a Brownian perturbation [Huzak *et al.*(2004)]. From Morales (2007) we have that

$$\widehat{k}(\xi) = \frac{c\,\xi}{\Psi_{-c-W}(\xi)}\,, \qquad \xi \geqslant 0\,,$$

and moreover, simple calculations show that k is an exponential density with mean $\frac{\sigma^2}{2\,c}$. Therefore, equation (32) can be written as

$$\begin{aligned} &\left[1 - \frac{\Psi_{S-J^\epsilon}(\xi) - \Psi_{S-J^\epsilon}(\rho_\epsilon)}{c(\rho_\epsilon - \xi)}\,\widehat{k}(\rho_\epsilon + \xi)\right]\widehat{\phi}_P^\epsilon(\xi) \\ &= \frac{\widehat{\chi}^\epsilon(\xi) - \widehat{\chi}^\epsilon(\rho_\epsilon)}{c(\rho_\epsilon - \xi)}\,\widehat{k}(\rho_\epsilon + \xi) + \frac{\frac{\sigma^2}{2}(\rho_\epsilon - \xi)}{\frac{\sigma^2}{2}(\rho_\epsilon^2 - \xi^2) + c(\rho_\epsilon - \xi)}\phi_P^\epsilon(0)\,, \end{aligned} \tag{34}$$

As for the second term in the right-hand side, using (33) again, we can easily see that

$$\begin{aligned} \frac{\frac{\sigma^2}{2}(\rho_\epsilon - \xi)}{\frac{\sigma^2}{2}(\rho_\epsilon^2 - \xi^2) + c(\rho_\epsilon - \xi)} &= \frac{\frac{\sigma^2}{2}(\rho_\epsilon + \xi)}{\frac{\sigma^2}{2}(\rho_\epsilon + \xi)^2 + c(\rho_\epsilon + \xi)} \\ &= \frac{\Psi_{-c-W}(\rho_\epsilon + \xi) - c(\rho_\epsilon + \xi)}{(\rho_\epsilon + \xi)\Psi_{-c-W}(\rho_\epsilon + \xi)} = \frac{1 - \frac{c(\rho_\epsilon + \xi)}{\Psi_{-c-W}(\rho_\epsilon + \xi)}}{(\rho_\epsilon + \xi)} \\ &= \frac{1 - \widehat{k}(\rho_\epsilon + \xi)}{\rho_\epsilon + \xi}\,. \end{aligned} \tag{35}$$

Substituting (35) into (34) yields

$$\left[1-\frac{\Psi_{S-J^\epsilon}(\xi)-\Psi_{S-J^\epsilon}(\rho_\epsilon)}{c(\rho_\epsilon-\xi)}\,\widehat{k}(\rho_\epsilon+\xi)\right]\widehat{\phi}^\epsilon_P(\xi)$$

$$= \frac{\widehat{\chi}^\epsilon(\xi)-\widehat{\chi}^\epsilon(\rho_\epsilon)}{c(\rho_\epsilon-\xi)}\,\widehat{k}(\rho_\epsilon+\xi)+\frac{1-\widehat{k}(\rho_\epsilon+\xi)}{\rho_\epsilon+\xi}\,\phi^\epsilon_P(0)\ , \tag{36}$$

from where we can identify $\widehat{g}^\epsilon_P$ and $\widehat{H}^\epsilon_w$ in (26). As for the function $\widehat{A}^\epsilon$, since k is an exponential density with mean $\frac{\sigma^2}{2\,c}$, a straight forward computation shows that $\widehat{A}^\epsilon(\xi)=\frac{1-\widehat{k}(\xi)}{\xi}$. Finally, we notice that $\phi_P(0)=w_0$. Isolating $\widehat{\phi}_P$ concludes the proof. ■

Now we are in a position to state the main result of the paper. In the following theorem we give explicit expressions for the functions $\widehat{g}^\epsilon_P$ and $\widehat{H}^\epsilon_w$ and their limits as $\epsilon\to 0$.

Theorem 3.1 *Consider the risk process defined in (1) and let ϕ_P denote its associated EDPF. Then, ϕ_P satisfies conditions (8) and (9) along with the DRE*

$$\phi_P(u)=\int_0^u \phi_P(u-y)g_P(y)dy+w_0e^{-\rho u}[1-K(u)]+H_w(u)\ ,\quad u\geqslant 0\ ,$$

where

$$g_P(y)=\frac{1}{c}\int_0^y e^{-\rho(y-s)}k(y-s)\left[\int_s^\infty e^{-\rho(x-s)}\nu_S(dx)+G_\rho(s)\right]ds\ ,$$

$$H_w(u)=\frac{1}{c}\int_0^u e^{-\rho(u-s)}k(u-s)\int_s^\infty e^{-\rho(x-s)}\chi(x)\ dx\ ds\ ,$$

$$\chi(x)=\int_x^\infty w(x,y-x)\nu_S(dy)+\int_x^\infty w(x,y-x)\nu_Z(dy)\ ,\qquad x\geqslant 0\ ,$$

the function G_ρ is defined through its Laplace transform

$$\int_0^\infty e^{-\xi x}G_\rho(x)dx=\frac{\Psi_{-J}(\xi)-\Psi_{-J}(\rho)}{\rho-\xi}\ ,\qquad \xi\geqslant 0\ ,$$

ρ is the unique non-negative solution of the generalized Lundberg equation

$$cr+\Psi_{S-Z}(r)=\delta\quad \textit{with}\ \ \rho=0\ \ \textit{when}\ \ \delta=0\ ,$$

and K (k) is an exponential distribution (density) with mean $\sigma^2/2c$, i.e. $K(x)=1-e^{-(2c/\sigma^2)u}$.

Proof. Let $\{R_\epsilon\}_{\epsilon>0}$ be the family of classical risk processes as defined in (16) and let their associated family of EDPF's be denoted by $\{\phi_P^\epsilon\}_{\epsilon>0}$. Using the same arguments as in Morales (2007), we can show that $R_\epsilon \xrightarrow{d} R$ as $\epsilon \to 0$. We now show that the family of Laplace transforms of the EDPF's $\{\widehat{\phi}_P^\epsilon\}_{\epsilon>0}$ converge to the Laplace transform of the function ϕ_P in Theorem 3.1, i.e. we show that $\widehat{\phi}_P^\epsilon \to \widehat{\phi}_P$ as $\epsilon \to 0$.

We start by showing that the Laplace transforms of the functions in Theorem 3.1 are given by

$$\widehat{\phi}_P(\xi) = \frac{\widehat{H}_w(\xi)}{1-\widehat{g}_P(\xi)} + \frac{w_0\,\widehat{A}(\xi)}{1-\widehat{g}_P(\xi)}\,, \qquad \xi \geqslant 0\,,$$

$$\widehat{g}_P(\xi) = \frac{[\Psi_{S-J}(\xi)-\Psi_{S-J}(\rho)]}{c(\rho-\xi)}\widehat{k}(\rho+\xi)\,, \qquad \xi \geqslant 0\,,$$

$$\widehat{H}_w(\xi) = \frac{[\widehat{\chi}(\xi)-\widehat{\chi}(\rho)]}{c\,(\rho-\xi)}\widehat{k}(\rho+\xi)\,, \qquad \xi \geqslant 0\,,$$

and

$$\widehat{\chi}(\xi) = \int_0^\infty e^{-\xi x}\left[\int_x^\infty w(x,y-x)\nu_S(dy) + \int_x^\infty w(x,y-x)\nu_Z(dy)\right] dx\,, \qquad \xi \geqslant 0\,,$$

where

$$A(x) = e^{-\rho\,x}[1-K(x)]\,, \quad x \geqslant 0\,,$$

$$K(x) = \int_0^x k(s)ds = \int_0^x \frac{2c}{\sigma^2}\, e^{-\frac{2c}{\sigma^2}\,s}\,ds\,, \quad x \geqslant 0\,,$$

Ψ_{S-J} is the Laplace exponent of $S-J$ and J is the pure-jump component of Z.

These are obtained by a straight-forward calculation. In a bottom-to-top order we have:

i) The expression for $\widehat{\chi}$ holds by the definition of the Laplace transform of χ.

ii) Notice that H_w can be written as the convolution of two functions,

$$H_w(x) = \frac{1}{c}\gamma * \eta(x)\,, \qquad x \geqslant 0\,,$$

where $\gamma(x) = e^{-\rho x}k(x)$ and $\eta(x) = \int_x^\infty e^{-\rho(z-x)}\chi(z)dz$. This implies that

$$\widehat{H}_w(\xi) = \frac{1}{c}\widehat{\gamma}(\xi)\widehat{\eta}(\xi)\,, \qquad \xi \geqslant 0\,. \tag{37}$$

Now, we only have to compute the Laplace transforms of these functions. By changing the order of integration we can see that

$$\begin{aligned}\widehat{\eta}(\xi) &= \int_0^\infty e^{-\xi\,x}\left[\int_x^\infty e^{-\rho(z-x)}\chi(z)dz\right]\,dx\\ &= \int_0^\infty \left[\int_0^z e^{-x(\xi-\rho)}dx\right]e^{-\rho z}\chi(z)\,dz = \frac{\widehat{\chi}(\xi)-\widehat{\chi}(\rho)}{\rho-\xi}\,, \qquad \xi \geqslant 0\,.\end{aligned}$$

In a straight forward way we have that

$$\widehat{\gamma}(\xi) = \int_0^\infty e^{-\xi\,x}\left[e^{-\rho x}k(x)\right]\,dx = \widehat{k}(\xi+\rho)$$

If we substitute $\widehat{\gamma}$ and $\widehat{\eta}$ into (37) we find the expression for $\widehat{H}_w$.

iii) Notice that g_P can also be written as the convolution of two functions,

$$g_P(x) = \frac{1}{c}\gamma * \zeta(x)\,, \qquad x \geqslant 0\,,$$

where $\gamma(x)$ is the one defined in ii) and $\zeta(x) = \int_x^\infty e^{-\rho(s-x)}\nu_S(ds) + G_\rho(x)$. This implies that

$$\widehat{g}_P(\xi) = \frac{1}{c}\widehat{\gamma}(\xi)\widehat{\zeta}(\xi)\,, \qquad \xi \geqslant 0\,. \tag{38}$$

Now, we only have to compute the Laplace transforms of function ζ. By changing the order of integration we can see that

$$\begin{aligned}\widehat{\zeta}(\xi) &= \int_0^\infty e^{-\xi\,x}\left[\int_x^\infty e^{-\rho(z-x)}\nu_S(dz) + G_\rho(x)\right]\,dx\\ &= \int_0^\infty \left[\int_0^z e^{-x(\xi-\rho)}dx\right]e^{-\rho z}\nu_S(dz) + \widehat{G}_\rho(\xi)\,, \qquad \xi \geqslant 0\,.\end{aligned}$$

Now, we know that for a subordinator S, its associated ladder-height density m appearing in the ladder-height decomposition for its ruin probability [see Morales (2007)] is defined as through its Laplace transform as $\widehat{m}(\xi) = \frac{\Psi_S(\xi)}{\xi\,\mathbb{E}[S(1)]} = \frac{(1+\theta)\Psi_S(\xi)}{c\,\xi}$ and it satisfies $dm(x) = \frac{1+\theta}{c}\nu_S(dx)$.

Moreover, the function G_ρ in Theorem 3.1 is defined through its Laplace transform and so, by substituting $dm(x)$ and $\widehat{G}_\rho$, we can write

$$\begin{aligned}\widehat{\zeta}(\xi) &= \frac{c}{1+\theta}\frac{\xi\widehat{m}(\xi)-\rho\widehat{m}(\rho)}{\rho-\xi}+\frac{\Psi_{-J}(\xi)-\Psi_{-J}(\rho)}{\rho-\xi}\\ &= \frac{\Psi_S(\xi)-\Psi_S(\rho)}{\rho-\xi}+\frac{\Psi_{-J}(\xi)-\Psi_{-J}(\rho)}{\rho-\xi}\\ &= \frac{\Psi_{S-J}(\xi)-\Psi_{S-J}(\rho)}{\rho-\xi}\ , \qquad \xi\geqslant 0\ .\end{aligned}$$

If we substitute $\widehat{\gamma}$ and $\widehat{\zeta}$ into (38) we find the expression for $\widehat{g}_P$.

iv) Finally, notice that ϕ_P can also be written in terms of convolutions. We first remark that the DRE for ϕ_P in Theorem 3.1 has a unique solution given by

$$\phi_P(x)=\left[H_w(x)+w_0e^{-\rho x}[1-K(x)]\right]*\sum_{n=0}^{\infty}g_P^{*n}(x)\ , \qquad x\geqslant 0\ ,$$

i.e. it is the convolution of two functions $a(x)=H_w(x)+w_0e^{-\rho x}[1-K(x)]$ and $b(x)=\sum_{n=0}^{\infty}g_P^{*n}(x)$. This implies that

$$\widehat{\phi}_P(\xi)=\widehat{a}(\xi)\widehat{b}(\xi)\ , \qquad \xi\geqslant 0\ . \tag{39}$$

Now, we only have to compute the Laplace transforms of the functions a and b. Clearly, we have

$$\begin{aligned}\widehat{a}(\xi) &= \int_0^\infty e^{-\xi x}\left[H_w(x)+w_0e^{-\rho x}[1-K(x)]\right]\ dx\\ &= \widehat{H}_w(\xi)+w_0\ \widehat{A}(\xi)\ , \qquad \xi\geqslant 0\ ,\end{aligned}$$

and

$$\begin{aligned}\widehat{b}(\xi) &= \int_0^\infty e^{-\xi x}\left[\sum_{n=0}^{\infty}g_P^{*n}(x)\right]\ dx=\sum_{n=0}^{\infty}[\widehat{g}_P(\xi)]^n\\ &= \frac{1}{1-\widehat{g}_P(\xi)}\ , \qquad \xi\geqslant 0\ .\end{aligned}$$

If we substitute $\widehat{a}$ and $\widehat{b}$ into (39) we find the expression for $\widehat{\phi}_P$.

Now, using Lemma 3.2, we can show that $\widehat{\chi}_\epsilon \to \widehat{\chi}$, $\widehat{H}^\epsilon_w \to \widehat{H}_w$, $\widehat{g}^\epsilon_P \to \widehat{g}_P$ and finally give an expression for $\lim_{\epsilon\to 0} \widehat{\phi}^\epsilon_P$.

We start with the functions $\widehat{\chi}$ and $\widehat{A}$. We clearly have that

$$\begin{aligned}\lim_{\epsilon\to 0} \widehat{\chi}_\epsilon(\xi) &= \lim_{\epsilon\to 0} \int_0^\infty e^{-\xi x} \left[\int_x^\infty w(x, y-x)\nu_S(dy) + \int_x^\infty w(x, y-x)\nu^\epsilon_Z(dy)\right] dx \\ &= \widehat{\chi}(\xi)\ , \qquad \xi \geqslant 0\ ,\end{aligned}$$

and

$$\lim_{\epsilon\to 0} \widehat{A}^\epsilon(\xi) = \lim_{\epsilon\to 0} e^{-\rho_\epsilon\, x}[1 - K(x)] = \widehat{A}(\xi)\ , \qquad \xi \geqslant 0\ .$$

In a similar way, from our previous result on $\widehat{\chi}_\epsilon$ we have that

$$\lim_{\epsilon\to 0} \widehat{H}^\epsilon_w(\xi) = \lim_{\epsilon\to 0} \frac{[\widehat{\chi}^\epsilon(\xi) - \widehat{\chi}^\epsilon(\rho_\epsilon)]}{c(\rho_\epsilon - \xi)} \widehat{k}(\rho_\epsilon + \xi) = \widehat{H}_w(\xi)\ , \qquad \xi \geqslant 0\ .$$

The same applies to the function $\widehat{g}_P(\xi)$, using the fact that $J^\epsilon \xrightarrow{d} J$ as $\epsilon \to 0$, we have that

$$\lim_{\epsilon\to 0} \widehat{g}^\epsilon_P(\xi) = \lim_{\epsilon\to 0} \frac{[\Psi_{S-J^\epsilon}(\xi) - \Psi_{S-J^\epsilon}(\rho_\epsilon)]}{c(\rho_\epsilon - \xi)} \widehat{k}(\rho_\epsilon + \xi) = \widehat{g}_P(\xi)\ , \qquad \xi \geqslant 0\ .$$

Finally, by our previous results on $\widehat{A}^\epsilon$, $\widehat{\chi}^\epsilon$, $\widehat{H}^\epsilon_w$ and $\widehat{g}^\epsilon_P$ we have that

$$\lim_{\epsilon\to 0} \widehat{\phi}^\epsilon_P(\xi) = \lim_{\epsilon\to 0} \frac{\widehat{H}^\epsilon_w(\xi) + w_0 \widehat{A}^\epsilon(\xi)}{1 - \widehat{g}^\epsilon_P(\xi)} = \widehat{\phi}_P(\xi)\ , \qquad \xi \geqslant 0\ ,$$

where $\widehat{\phi}_P$ is the Laplace transform of the DRE in Theorem 3.1. Now, since $\mathcal{A}\phi^\epsilon_P \to \mathcal{A}\phi_P$ as $\epsilon \to 0$, we have that $\phi^\epsilon_P \to \phi_P$ and therefore the limit of $\widehat{\phi}^\epsilon_P$ as $\epsilon \to 0$ is indeed equal to the Laplace transform $\widehat{\phi}_P$ of the EDPF of the model in (1). This completes the proof.

■

4 Conclusions

We have generalized results in Tsai and Willmot (2002) and Morales (2007) for the EDPF in the perturbed case. We substituted the compound Poisson process by a subordinator and the perturbation by a spectrally negative Lévy process and showed that still satisfies a DRE (Theorem 3.1). Our results are more general and allow for a wider range of models for the aggregate claims process and the perturbation, in particular those for which closed-form expressions are available like the gamma and inverse Gaussian processes.

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