```
> restart;
> h:=proc(x);x*log(x) end proc;
```

$$h := \mathbf{proc}(x)\ \ x*\log(x)\ \ \mathbf{end\ proc} \tag{1}$$

```
> f := h(b) + h(9/2 - b) + log((7*9*4)*3) - 9/2*log(9) - (z211 +
  z200 + z210 + z201)*log(7) - (z211 + z200 + z210 + z201 + z111 +
  z100 + z110 + z101)*log(3) + (z111 + z100 + z110 + z101)*log(2) -
  h(z211) - h(z111) - h(z - z211 - z111) - h(z200) - h(z100) - h(z
  - z200 - z100) - h(z210) - h(z110) - h(1/2 - z - z210 - z110) - h
  (z201) - h(z101) - h(1/2 - z - z201 - z101)
```

$$f := b\ln(b) + \left(\frac{9}{2} - b\right)\ln\left(\frac{9}{2} - b\right) + \ln(756) - 9\ln(3) - (z211 + z200 + z210 + z201)\ln(7) - (z211 + z200 + z210 + z201 + z111 + z100 + z110 + z101)\ln(3) + (z111 + z100 + z110 + z101)\ln(2) - z211\ln(z211) - z111\ln(z111) - (z - z211 - z111)\ln(z - z211 - z111) - z200\ln(z200) - z100\ln(z100) - (z - z200 - z100)\ln(z - z200 - z100) - z210\ln(z210) - z110\ln(z110) - \left(\frac{1}{2} - z - z210 - z110\right)\ln\left(\frac{1}{2} - z - z210 - z110\right) - z201\ln(z201) - z101\ln(z101) - \left(\frac{1}{2} - z - z201 - z101\right)\ln\left(\frac{1}{2} - z - z201 - z101\right) \tag{2}$$

```
> g := b(9 - 2*b)^(1/2)*(((z211*z111*(z - z211 - z111)*z200*z100*(z
  - z200 - z100)*z210*z110*(1/2 - z - z210 - z110)*z201*z101*(1/2 -
  z - z201 - z101))*(pi*n)^9 *2^7)^(-1/2))
```

$$g := \left(\sqrt{b(9 - 2b)}\,\sqrt{128}\right) \Big/ \left(128\left(z211\,z111\,(z - z211 - z111)\,z200\,z100\,(z - z200 - z100)\,z210\,z110\left(\frac{1}{2} - z - z210 - z110\right)z201\,z101\left(\frac{1}{2} - z - z201 - z101\right)\pi^9 n^9\right)^{1/2}\right) \tag{3}$$

```
> g := subs(b = 2*z211 + 2*z200 + z111 + z100 - z110 - z101 + 2 + z
  - 2*z210 - 2*z201, g);
```

$$g := \Big(\big((2\,z211 + 2\,z200 + z111 + z100 - z110 - z101 + 2 + z - 2\,z210 - 2\,z201)(5 - 4\,z211 - 4\,z200 - 2\,z111 - 2\,z100 + 2\,z110 + 2\,z101 - 2\,z + 4\,z210 + 4\,z201)\big)^{1/2}\sqrt{128}\Big) \Big/ \tag{4}$$

$$\left(128 \left(z211\, z111\, (z - z211 - z111)\, z200\, z100\, (z - z200 - z100)\, z210\, z110 \left(\frac{1}{2} - z - z210 - z110\right) z201\, z101 \left(\frac{1}{2} - z - z201 - z101\right) \pi^9 n^9\right)^{1/2}\right)$$

```
> f := subs(b = 2*z211 + 2*z200 + z111 + z100 - z110 - z101 + 2 + z
  - 2*z210 - 2*z201, f);
```

$$f := (2\, z211 + 2\, z200 + z111 + z100 - z110 - z101 + 2 + z - 2\, z210 - 2\, z201) \ln(2\, z211 + 2\, z200 + z111 + z100 - z110 - z101 + 2 + z - 2\, z210 - 2\, z201) + \left(\frac{5}{2} - 2\, z211 - 2\, z200 - z111 - z100 + z110 + z101 - z + 2\, z210 + 2\, z201\right) \ln\left(\frac{5}{2} - 2\, z211 - 2\, z200 - z111 - z100 + z110 + z101 - z + 2\, z210 + 2\, z201\right) + \ln(756) - 9 \ln(3) - (z211 + z200 + z210 + z201) \ln(7) - (z211 + z200 + z210 + z201 + z111 + z100 + z110 + z101) \ln(3) + (z111 + z100 + z110 + z101) \ln(2) - z211 \ln(z211) - z111 \ln(z111) - (z - z211 - z111) \ln(z - z211 - z111) - z200 \ln(z200) - z100 \ln(z100) - (z - z200 - z100) \ln(z - z200 - z100) - z210 \ln(z210) - z110 \ln(z110) - \left(\frac{1}{2} - z - z210 - z110\right) \ln\left(\frac{1}{2} - z - z210 - z110\right) - z201 \ln(z201) - z101 \ln(z101) - \left(\frac{1}{2} - z - z201 - z101\right) \ln\left(\frac{1}{2} - z - z201 - z101\right) \tag{5}$$

We now calculate the partial derivatives of f

```
> diff(f, z): zeq := simplify(exp(%)): Pz := simplify(numer(%) -
  denom(%))
```

$$Pz := -8 z^3 + (-4\, z100 + 4\, z101 + 4\, z110 - 4\, z111 - 12\, z200 + 12\, z201 + 12\, z210 - 12\, z211 + 6)\, z^2 + \big(4\, z101^2 + (-8\, z100 + 4\, z110 - 8\, z111 - 12\, z200 + 12\, z201 + 8\, z210 - 12\, z211 - 10)\, z101 + 4\, z110^2 + (-8\, z100 - 8\, z111 - 12\, z200 + 8\, z201 + 12\, z210 - 12\, z211 - 10)\, z110 + 8\, z201^2 + (-12\, z100 - 12\, z111 - 16\, z200 + 12\, z210 - 16\, z211 - 14)\, z201 + 8\, z210^2 + (-12\, z100 - 12\, z111 - 16\, z200 - 16\, z211 - 14)\, z210 + 4\, z100^2 + (4\, z111 + 12\, z200 + 8\, z211 - 6)\, z100 + 4\, z111^2 + (8\, z200 + 12\, z211 - 6)\, z111 + 8\, z200^2 + (12\, z211 - 2)\, z200 + 8\, z211^2 - 2\, z211 + 7\big)\, z + (4\, z110 + 4\, z210 - 2)\, z101^2 + \big(4\, z110^2 + (-4\, z100 - 4\, z111 - 8\, z200 + 12\, z201 + 12\, z210 - 8\, z211 - 12)\, z110 + (12\, z210 - 6)\, z201 + 8\, z210^2 + (-4\, z100 - 4\, z111 - 8\, z200 - 8\, z211 - 14)\, z210 \tag{6}$$

$$
\begin{aligned}
&+ (4\, z111 + 4\, z211 + 2)\, z100 + (4\, z200 + 2)\, z111 + (4\, z211 + 4)\, z200 + 4\, z211 + 5\big) \\
&z101 + (4\, z201 - 2)\, z110^2 + \big(8\, z201^2 + (-4\, z100 - 4\, z111 - 8\, z200 + 12\, z210 - 8\, z211 \\
&- 14)\, z201 - 6\, z210 + (4\, z111 + 4\, z211 + 2)\, z100 + (4\, z200 + 2)\, z111 + (4\, z211 \\
&+ 4)\, z200 + 4\, z211 + 5)\, z110 + (8\, z210 - 4)\, z201^2 + \big(8\, z210^2 + (-4\, z100 - 4\, z111 \\
&- 8\, z200 - 8\, z211 - 16)\, z210 + (8\, z111 + 8\, z211 + 2)\, z100 + (8\, z200 + 2)\, z111 \\
&+ (8\, z211 + 4)\, z200 + 4\, z211 + 6\big)\, z201 - 4\, z210^2 + ((8\, z111 + 8\, z211 + 2)\, z100 \\
&+ (8\, z200 + 2)\, z111 + (8\, z211 + 4)\, z200 + 4\, z211 + 6)\, z210 + (-4\, z111 \\
&- 4\, z211)\, z100^2 + \big(-4\, z111^2 + (-12\, z200 - 12\, z211 + 10)\, z111 - 12\, z200\, z211 \\
&- 8\, z211^2 + 10\, z211 - 1\big)\, z100 - 4\, z111^2\, z200 + \big(-1 - 8\, z200^2 + (-12\, z211 \\
&+ 10)\, z200\big)\, z111 - 8\, z200^2\, z211 + \big(-8\, z211^2 + 10\, z211 - 2\big)\, z200 - 2\, z211 - 2
\end{aligned}
$$

```
> diff(f, z211): z211eq := simplify(exp(%)): Pz211 := simplify
  (numer(%) - denom(%))
```

$$
\begin{aligned}
Pz211 :=& -352\, z211^3 + (-336\, z - 352\, z100 + 352\, z101 + 352\, z110 - 368\, z111 - 704\, z200 \\
&+ 704\, z201 + 704\, z210 + 808)\, z211^2 + \big(-104\, z111^2 + (-176\, z - 192\, z100 + 192\, z101 \\
&+ 192\, z110 - 384\, z200 + 384\, z201 + 384\, z210 + 372)\, z111 - 72\, z^2 + (-160\, z100 \\
&+ 160\, z101 + 160\, z110 - 320\, z200 + 320\, z201 + 320\, z210 + 436)\, z - 88\, z100^2 \\
&+ (176\, z101 + 176\, z110 - 352\, z200 + 352\, z201 + 352\, z210 + 404)\, z100 - 88\, z101^2 + (\\
&-176\, z110 + 352\, z200 - 352\, z201 - 352\, z210 - 404)\, z101 - 88\, z110^2 + (352\, z200 \\
&- 352\, z201 - 352\, z210 - 404)\, z110 - 352\, z200^2 + (704\, z201 + 704\, z210 + 808)\, z200 \\
&- 352\, z201^2 + (-704\, z210 - 808)\, z201 - 352\, z210^2 - 808\, z210 - 541\big)\, z211 + 4\, (z \\
&- z111)\, (2\, z200 + z111 + z100 - z110 - z101 + 2 + z - 2\, z210 - 2\, z201)^2
\end{aligned}
\qquad \textbf{(7)}
$$

```
> diff(f, z200):z200eq := simplify(exp(%)): Pz200 := simplify(numer
  (%) - denom(%))
```

$$
\begin{aligned}
Pz200 :=& -352\, z200^3 + (-336\, z - 368\, z100 + 352\, z101 + 352\, z110 - 352\, z111 + 704\, z201 \\
&+ 704\, z210 - 704\, z211 + 808)\, z200^2 + \big(-104\, z100^2 + (-176\, z + 192\, z101 + 192\, z110 \\
&- 192\, z111 + 384\, z201 + 384\, z210 - 384\, z211 + 372)\, z100 - 72\, z^2 + (160\, z101 \\
&+ 160\, z110 - 160\, z111 + 320\, z201 + 320\, z210 - 320\, z211 + 436)\, z - 88\, z101^2 + (\\
&-176\, z110 + 176\, z111 - 352\, z201 - 352\, z210 + 352\, z211 - 404)\, z101 - 88\, z110^2 \\
&+ (176\, z111 - 352\, z201 - 352\, z210 + 352\, z211 - 404)\, z110 - 88\, z111^2 + (352\, z201 \\
&+ 352\, z210 - 352\, z211 + 404)\, z111 - 352\, z201^2 + (-704\, z210 + 704\, z211 - 808)\, z201 \\
&- 352\, z210^2 + (704\, z211 - 808)\, z210 - 352\, z211^2 + 808\, z211 - 541\big)\, z200 + 4\, (z
\end{aligned}
\qquad \textbf{(8)}
$$

$$- z100)\,(2\,z211 + z111 + z100 - z110 - z101 + 2 + z - 2\,z210 - 2\,z201)^2$$

```
> diff(f, z210): z210eq := simplify(exp(%)): Pz210 := simplify
  (numer(%) - denom(%))
```

$$Pz210 := -704\,z210^3 + (672\,z + 704\,z100 - 704\,z101 - 736\,z110 + 704\,z111 + 1408\,z200 - 1408\,z201 + 1408\,z211 + 1280)\,z210^2 + \big(-208\,z110^2 + (352\,z + 384\,z100 - 384\,z101 + 384\,z111 + 768\,z200 - 768\,z201 + 768\,z211 + 568)\,z110 - 144\,z^2 + (-320\,z100 + 320\,z101 - 320\,z111 - 640\,z200 + 640\,z201 - 640\,z211 - 728)\,z - 176\,z100^2 + (352\,z101 - 352\,z111 - 704\,z200 + 704\,z201 - 704\,z211 - 648)\,z100 - 176\,z101^2 + (352\,z111 + 704\,z200 - 704\,z201 + 704\,z211 + 648)\,z101 - 176\,z111^2 + (-704\,z200 + 704\,z201 - 704\,z211 - 648)\,z111 - 704\,z200^2 + (1408\,z201 - 1408\,z211 - 1296)\,z200 - 704\,z201^2 + (1408\,z211 + 1296)\,z201 - 704\,z211^2 - 1296\,z211 - 682\big)\,z210 - 8\left(-\frac{5}{2} + 2\,z211 + 2\,z200 + z111 + z100 - z110 - z101 + z - 2\,z201\right)^2\left(-\frac{1}{2} + z + z110\right) \quad \textbf{(9)}$$

```
> diff(f, z201):z201eq := simplify(exp(%)): Pz201 := simplify(numer
  (%) - denom(%))
```

$$Pz201 := -704\,z201^3 + (672\,z + 704\,z100 - 736\,z101 - 704\,z110 + 704\,z111 + 1408\,z200 - 1408\,z210 + 1408\,z211 + 1280)\,z201^2 + \big(-208\,z101^2 + (352\,z + 384\,z100 - 384\,z110 + 384\,z111 + 768\,z200 - 768\,z210 + 768\,z211 + 568)\,z101 - 144\,z^2 + (-320\,z100 + 320\,z110 - 320\,z111 - 640\,z200 + 640\,z210 - 640\,z211 - 728)\,z - 176\,z100^2 + (352\,z110 - 352\,z111 - 704\,z200 + 704\,z210 - 704\,z211 - 648)\,z100 - 176\,z110^2 + (352\,z111 + 704\,z200 - 704\,z210 + 704\,z211 + 648)\,z110 - 176\,z111^2 + (-704\,z200 + 704\,z210 - 704\,z211 - 648)\,z111 - 704\,z200^2 + (1408\,z210 - 1408\,z211 - 1296)\,z200 - 704\,z210^2 + (1408\,z211 + 1296)\,z210 - 704\,z211^2 - 1296\,z211 - 682\big)\,z201 - 8\left(-\frac{5}{2} + 2\,z211 + 2\,z200 + z111 + z100 - z110 - z101 + z - 2\,z210\right)^2\left(-\frac{1}{2} + z + z101\right) \quad \textbf{(10)}$$

```
> diff(f, z111): z111eq := simplify(exp(%)): Pz111 := simplify
  (numer(%) - denom(%))
```

$$Pz111 := -4\,z^2 + (-4\,z100 + 4\,z101 + 4\,z110 - 6\,z111 - 8\,z200 + 8\,z201 + 8\,z210 - 4\,z211 - 8)\,z - 2\,z111^2 + (-2\,z100 + 2\,z101 + 2\,z110 - 4\,z200 + 4\,z201 + 4\,z210 + 23)\,z111 \quad \textbf{(11)}$$

$$+4\,z211\,(z100-z101-z110+2\,z200-2\,z201-2\,z210+2\,z211+2)$$

```
> diff(f, z100): z100eq := simplify(exp(%)): Pz100 := simplify
  (numer(%) - denom(%))
```

$$Pz100 := -4\,z^2+(-6\,z100+4\,z101+4\,z110-4\,z111-4\,z200+8\,z201+8\,z210-8\,z211-8)\,z-2\,z100^2+(2\,z101+2\,z110-2\,z111+4\,z201+4\,z210-4\,z211+23)\,z100-4\,z200\,(z101+z110-z111-2\,z200+2\,z201+2\,z210-2\,z211-2) \tag{12}$$

```
> diff(f, z110): z110eq := simplify(exp(%)): Pz110 := simplify
  (numer(%) - denom(%))
```

$$Pz110 := 4\,z^2+(4\,z100-4\,z101-6\,z110+4\,z111+8\,z200-8\,z201-4\,z210+8\,z211-12)\,z+2\,z110^2+(-2\,z100+2\,z101-2\,z111-4\,z200+4\,z201-4\,z211-20)\,z110+4\left(-2\,z210-\frac{5}{2}+2\,z211+2\,z200+z111+z100-z101-2\,z201\right)\left(z210-\frac{1}{2}\right) \tag{13}$$

```
> diff(f, z101): z101eq := simplify(exp(%)): Pz101 := simplify
  (numer(%) - denom(%))
```

$$Pz101 := 4\,z^2+(4\,z100-6\,z101-4\,z110+4\,z111+8\,z200-4\,z201-8\,z210+8\,z211-12)\,z+2\,z101^2+(-2\,z100+2\,z110-2\,z111-4\,z200+4\,z210-4\,z211-20)\,z101+4\left(z201-\frac{1}{2}\right)\left(-2\,z201-\frac{5}{2}+2\,z211+2\,z200+z111+z100-z110-2\,z210\right) \tag{14}$$

From algebraic manipulations we know that z211z100 = z200z111 and z210z110=z201z101.

```
> factor(simplify(resultant(Pz211, Pz200, z101)));
```

$$46294416\,(z200\,z-z211\,z+z100\,z211-z111\,z200)^2 \tag{15}$$

Thus, for a critical point in the interior, it must be the case that z200 = z211. This implies that z100= z111.

```
> factor(simplify(resultant(Pz100, Pz111, z210)))
```

$$-216\,z100\,z+216\,z111\,z+216\,z100\,z211-216\,z111\,z200 \tag{16}$$

This implies that z100 = z111, which in turn implies that z200=z211. We let z111=z100=c1, z211= z200=c2=c, z110=z101=c3, z210=z201=c4.

fz111eq=fz100eq=fc1

```
> fc1 := subs(z200 = z211, z201 = z210, z100 = z111, z101 = z110,
  z111eq): fc1 := subs(z111 = c1, z211 = c, z110 = c3, z210 = c4,
  fc1)
```

$$fc1 := -\frac{2\,(4\,c+2\,c1-2\,c3+2+z-4\,c4)\,(z-c-c1)}{3\,c1\left(-\frac{5}{2}+4\,c+2\,c1-2\,c3+z-4\,c4\right)} \tag{17}$$

```
> fc2 := subs(z200 = z211, z201 = z210, z100 = z111, z101 = z110,
  z211eq): fc2 := simplify(subs(z111 = c1, z211 = c, z110 = c3,
  z210 = c4, fc2))
```

(18)

$$fc2 := -\frac{4\,(4\,c+2\,c1-2\,c3+2+z-4\,c4)^2\,(-z+c+c1)}{21\,(-5+8\,c+4\,c1-4\,c3+2\,z-8\,c4)^2\,c} \quad \textbf{(18)}$$

```
> fc3 := subs(z200 = z211, z201 = z210, z100 = z111, z101 = z110,
  z110eq): fc3 := subs(z111 = c1, z211 = c, z110 = c3, z210 = c4,
  fc3)
```

$$fc3 := \frac{(-5+8\,c+4\,c1-4\,c3+2\,z-8\,c4)\,(-1+2\,z+2\,c4+2\,c3)}{6\,(4\,c+2\,c1-2\,c3+2+z-4\,c4)\,c3} \quad \textbf{(19)}$$

```
> fc4 := subs(z200 = z211, z201 = z210, z100 = z111, z101 = z110,
  z210eq): fc4 := subs(z111 = c1, z211 = c, z110 = c3, z210 = c4,
  fc4)
```

$$fc4 := -\frac{\left(-\frac{5}{2}+4\,c+2\,c1-2\,c3+z-4\,c4\right)^2\left(-\frac{1}{2}+z+c4+c3\right)}{21\,(4\,c+2\,c1-2\,c3+2+z-4\,c4)^2\,c4} \quad \textbf{(20)}$$

```
> fzc := subs(z200 = z211, z201 = z210, z100 = z111, z101 = z110,
  zeq):fzc := subs(z111 = c1, z211 = c, z110 = c3, z210 = c4, fzc)
```

$$fzc := -\frac{(4\,c+2\,c1-2\,c3+2+z-4\,c4)\left(-\frac{1}{2}+z+c4+c3\right)^2}{\left(-\frac{5}{2}+4\,c+2\,c1-2\,c3+z-4\,c4\right)(z-c-c1)^2} \quad \textbf{(21)}$$

Analytical calculations demonstrated that at a critical point in the interior of J, $z = c (21*((9/2-b)/b)^2 + 14((9/2-b)/b) + 1$, $c_1 = 14*c*((9/2-b)/b), c_2=c$, $c_3 = 14 ((9/2-b)/b)^{-1} * (1/2 - c*(21*((9/2-b)/b)^2 + 14 ((9/2-b)/b) + 1))/(21*((9/2-b)/b)^-2 + 14*((9/2-b)/b) + 1)$, $c_4 = (1/2 - c*(21*((9/2-b)/b)^2 + 14 ((9/2-b)/b) + 1))/(21*((9/2-b)/b)^-2 + 14*((9/2-b)/b) + 1)$.

```
>
  fc1 := subs(z = c*(21*((9/2 - b)/b)^2 + 14*(9/2 - b)/b + 1), c1 =
  14*c*(9/2 - b)/b, c3 = 14*(b/(9/2-b))*(1/2 - c*(21*((9/2 - b)/b)
  ^2 + 14*(9/2 - b)/b + 1))/(((21*(b/(9/2 - b))^2 + 14*b/(9/2 - b)
  + 1))), c4 = (1/2 - c*(21*((9/2 - b)/b)^2 + 14*(9/2 - b)/b + 1))/
  (21*(b/(9/2 - b))^2 + 14*b/(9/2 - b) + 1), fc1)
```

$$fc1 := -\Bigg(\Bigg(4\,c+\frac{28\,c\left(\frac{9}{2}-b\right)}{b} \quad \textbf{(22)}$$

$$
-\frac{28\,b\left(\frac{1}{2}-c\left(\frac{21\left(\frac{9}{2}-b\right)^{2}}{b^{2}}+\frac{14\left(\frac{9}{2}-b\right)}{b}+1\right)\right)}{\left(\frac{9}{2}-b\right)\left(\frac{21\,b^{2}}{\left(\frac{9}{2}-b\right)^{2}}+\frac{14\,b}{\frac{9}{2}-b}+1\right)}+2+c\left(\frac{21\left(\frac{9}{2}-b\right)^{2}}{b^{2}}
+\frac{14\left(\frac{9}{2}-b\right)}{b}+1\right)-\frac{4\left(\frac{1}{2}-c\left(\frac{21\left(\frac{9}{2}-b\right)^{2}}{b^{2}}+\frac{14\left(\frac{9}{2}-b\right)}{b}+1\right)\right)}{\frac{21\,b^{2}}{\left(\frac{9}{2}-b\right)^{2}}+\frac{14\,b}{\frac{9}{2}-b}+1}\Bigg)
$$

$$
\left(c\left(\frac{21\left(\frac{9}{2}-b\right)^{2}}{b^{2}}+\frac{14\left(\frac{9}{2}-b\right)}{b}+1\right)-c-\frac{14\,c\left(\frac{9}{2}-b\right)}{b}\right)b\Bigg)\Bigg/
$$

$$
\Bigg(21\,c\left(\frac{9}{2}-b\right)\Bigg(-\frac{5}{2}+4\,c+\frac{28\,c\left(\frac{9}{2}-b\right)}{b}
$$

$$
-\frac{28\,b\left(\frac{1}{2}-c\left(\frac{21\left(\frac{9}{2}-b\right)^{2}}{b^{2}}+\frac{14\left(\frac{9}{2}-b\right)}{b}+1\right)\right)}{\left(\frac{9}{2}-b\right)\left(\frac{21\,b^{2}}{\left(\frac{9}{2}-b\right)^{2}}+\frac{14\,b}{\frac{9}{2}-b}+1\right)}+c\left(\frac{21\left(\frac{9}{2}-b\right)^{2}}{b^{2}}
+\frac{14\left(\frac{9}{2}-b\right)}{b}+1\right)-\frac{4\left(\frac{1}{2}-c\left(\frac{21\left(\frac{9}{2}-b\right)^{2}}{b^{2}}+\frac{14\left(\frac{9}{2}-b\right)}{b}+1\right)\right)}{\frac{21\,b^{2}}{\left(\frac{9}{2}-b\right)^{2}}+\frac{14\,b}{\frac{9}{2}-b}+1}\Bigg)\Bigg)
$$

```
> simplify(%)
```

$$
\big((-9+2\,b)\,(5120\,b^{4}\,c-448\,b^{4}-46080\,b^{3}\,c-1008\,b^{3}+285120\,b^{2}\,c-816480\,b\,c \qquad \mathbf{(23)}
$$

$$-688905\,c)\big)\Big/\big(2\,b\,(5120\,b^4\,c+128\,b^4-46080\,b^3\,c+2880\,b^3+285120\,b^2\,c+1458\,b^2-816480\,b\,c-688905\,c)\big)$$

```
> fc2 := simplify(subs(z = c*(21*((9/2 - b)/b)^2 + 14*(9/2 - b)/b + 1), c1 = 14*c*(9/2 - b)/b, c3 = 14*(b/(9/2-b))*(1/2 - c*(21*((9/2 - b)/b)^2 + 14*(9/2 - b)/b + 1))/(((21*(b/(9/2 - b))^2 + 14*b/(9/2 - b) + 1))), c4 = (1/2 - c*(21*((9/2 - b)/b)^2 + 14*(9/2 - b)/b + 1))/(21*(b/(9/2 - b))^2 + 14*b/(9/2 - b) + 1), fc2))
```

$$fc2 := \big((4\,b^2-36\,b+81)\,(5120\,b^4\,c-448\,b^4-46080\,b^3\,c-1008\,b^3+285120\,b^2\,c-816480\,b\,c-688905\,c)^2\big)\Big/\big(4\,b^2\,(5120\,b^4\,c+128\,b^4-46080\,b^3\,c+2880\,b^3+285120\,b^2\,c+1458\,b^2-816480\,b\,c-688905\,c)^2\big) \quad \textbf{(24)}$$

```
> fc3 := simplify(subs(z = c*(21*((9/2 - b)/b)^2 + 14*(9/2 - b)/b + 1), c1 = 14*c*(9/2 - b)/b, c3 = 14*(b/(9/2-b))*(1/2 - c*(21*((9/2 - b)/b)^2 + 14*(9/2 - b)/b + 1))/(((21*(b/(9/2 - b))^2 + 14*b/(9/2 - b) + 1))), c4 = (1/2 - c*(21*((9/2 - b)/b)^2 + 14*(9/2 - b)/b + 1))/(21*(b/(9/2 - b))^2 + 14*b/(9/2 - b) + 1), fc3))
```

$$fc3 := \big(2\,b\,(5120\,b^4\,c+128\,b^4-46080\,b^3\,c+2880\,b^3+285120\,b^2\,c+1458\,b^2-816480\,b\,c-688905\,c)\big)\Big/\big((5120\,b^4\,c-448\,b^4-46080\,b^3\,c-1008\,b^3+285120\,b^2\,c-816480\,b\,c-688905\,c)\,(-9+2\,b)\big) \quad \textbf{(25)}$$

```
> fzc := simplify(subs(z = c*(21*((9/2 - b)/b)^2 + 14*(9/2 - b)/b + 1), c1 = 14*c*(9/2 - b)/b, c3 = 14*(b/(9/2-b))*(1/2 - c*(21*((9/2 - b)/b)^2 + 14*(9/2 - b)/b + 1))/(((21*(b/(9/2 - b))^2 + 14*b/(9/2 - b) + 1))), c4 = (1/2 - c*(21*((9/2 - b)/b)^2 + 14*(9/2 - b)/b + 1))/(21*(b/(9/2 - b))^2 + 14*b/(9/2 - b) + 1), fzc))
```

$$fzc := -\big(16\,(32\,b^2\,c-2\,b^2-504\,b\,c+1701\,c)^2\,b^4\,(5120\,b^4\,c-448\,b^4-46080\,b^3\,c-1008\,b^3+285120\,b^2\,c-816480\,b\,c-688905\,c)\big)\Big/\big((4\,b^2-36\,b+81)^2\,c^2\,(5120\,b^4\,c+128\,b^4-46080\,b^3\,c+2880\,b^3+285120\,b^2\,c+1458\,b^2-816480\,b\,c-688905\,c)\,(32\,b^2+216\,b+81)^2\big) \quad \textbf{(26)}$$

```
> solve([fc1 = fc3, 9/2 > b,b>0,c>0],c)
```

$$\begin{cases} [\,] \\ \left[\left\{c = -\frac{4\,b^3\,(32\,b^2+104\,b-171)}{5120\,b^4-46080\,b^3+285120\,b^2-816480\,b-688905}\right\}\right] \\ \left[\left\{c = \frac{4\,b^3\,(160\,b^2-1944\,b-2997)}{20480\,b^5-230400\,b^4+1555200\,b^3-5832000\,b^2+4592700\,b+6200145}\right\}, \left\{c = -\frac{4\,b}{5120\,b^4-46080}\right. \\ [\,] \end{cases}$$

```
> solve([fc1 = fc2, b < 9/2, 0 < b, 0 < c], c);
```

$$\begin{cases} [\,] & b \leq -\frac{13}{8}+\frac{\sqrt{511}}{8} \\ \left[\left\{c = -\frac{4\,b^3\,(32\,b^2+104\,b-171)}{5120\,b^4-46080\,b^3+285120\,b^2-816480\,b-688905}\right\}\right] & b < \frac{9}{2} \\ [\,] & \frac{9}{2} \leq b \end{cases} \tag{28}$$

```
> evalf(-(-13 + sqrt(511))^3*(-133 + 22*sqrt(511))/(19840*(4063*
  sqrt(511) - 89278)))
```

$$-0.006338448275 \tag{29}$$

Thus, we can assume that the only plausible solutions is $c_2^(*)=-(4 b^3 (32 b^2+104 b-171))/(5120 b^4-46080 b^3+285120 b^2-816480 b-688905)$ when $9/2>b>0$ and $b!=9/4$ and $c^(*3) = 1/144$ when $b=9/4$.

```
> simplify(subs(c = -4*b^3*(32*b^2 + 104*b - 171)/(5120*b^4 - 46080*b^3 + 285120*b^2 - 816480*b - 688905), fzc))
```

$$-\frac{8\,b^3\,\left(32\,b^2-392\,b+945\right)^2}{\left(2\,b-9\right)^3\,\left(32\,b^2+104\,b-171\right)^2} \tag{30}$$

```
> evalf(solve([%=1,9/2>b,b>0]))
```

$$\{b=0.8065779289\},\ \{b=2.250000000\},\ \{b=3.693422071\} \tag{31}$$

```
> subs(b = 0.8065779289, -4*b^3*(32*b^2 + 104*b - 171)/(5120*b^4 - 46080*b^3 + 285120*b^2 - 816480*b - 688905))
```

$$-0.0001175309606 \tag{32}$$

b = 0.8065779289, c = -0.0001175309606 do not satisfy our requirements for c since it must be positive.

```
> subs(b = 3.693422071, -4*b^3*(32*b^2 + 104*b - 171)/(5120*b^4 - 46080*b^3 + 285120*b^2 - 816480*b - 688905))
```

$$0.1105793451 \tag{33}$$

The associated upper-bound of c is:

```
> subs(b = 3.693422071, 1/(42*(9/2 - b)^2/b^2 + 28*(9/2 - b)/b + 2));
```

$$0.09883651395 \tag{34}$$

Since this value of c does not satisfy the upper bound,$(b=3.693422071,c=0.1105793451)$ cannot be a critical point as it is not situated within the feasible region of f. We now turn our attention to $b=2.25$.

```
> subs(b=9/4,-4*b^3*(32*b^2 + 104*b - 171)/(5120*b^4 - 46080*b^3 + 285120*b^2 - 816480*b - 688905))
```

$$\frac{1}{144} \tag{35}$$

This value is positive. The associated upper-bound of c is:

```
> subs(b = 9/4, 1/(42*(9/2 - b)^2/b^2 + 28*(9/2 - b)/b + 2));
```

$$\frac{1}{72} \tag{36}$$

c = 1/144 is both greater than zero and obeys our upper-bound. We now turn our attention towards possible maxima on the boundary. It has been demonstrated analytically that there are no global maximums on the boundary where $0 < z < 1/2$. We now consider the case where $z = 0$, which entails that $z_{211}=z_{200}=z_{111}=z_{100}=0$. Algebraic manipulations of the partials imply that $z_{210} = z_{201}$ and $z_{110}=z_{101}$ at critical points.

```
> f0 := simplify(subs(z = 0, z211 = 0, z111 = 0, z200 = 0, z100 = 0, f))
```

$$f0 := \frac{\left(-2\,z101-2\,z110-4\,z201-4\,z210+4\right)\ln\left(2-z110-z101-2\,z210-2\,z201\right)}{2} \tag{37}$$

$$+\frac{(5+2 z110+2 z101+4 z210+4 z201) \ln(5+2 z110+2 z101+4 z210+4 z201)}{2}$$
$$+\frac{(-1+2 z201+2 z101) \ln(1-2 z201-2 z101)}{2}$$
$$+\frac{(-1+2 z210+2 z110) \ln(1-2 z210-2 z110)}{2}$$
$$+\frac{(-2 z101-2 z110-6 z201-6 z210+1) \ln(2)}{2}$$
$$+\frac{(-2 z101-2 z110-2 z201-2 z210-12) \ln(3)}{2}+\frac{(2-2 z210-2 z201) \ln(7)}{2}$$
$$-z101 \ln(z101)-z201 \ln(z201)-z210 \ln(z210)-z110 \ln(z110)$$

```
> f0z210 := simplify(diff(f0, z210));
```

$$f0z210 := -2 \ln(2-z110-z101-2 z210-2 z201)+2 \ln(5+2 z110+2 z101+4 z210+4 z201)+\ln(1-2 z210-2 z110)-3 \ln(2)-\ln(3)-\ln(7)-\ln(z210) \quad \textbf{(38)}$$

```
> f0z210 := simplify(exp(%), symbolic);
```

$$f0z210 := -\frac{(5+2 z110+2 z101+4 z210+4 z201)^2 (-1+2 z210+2 z110)}{168 (-2+z110+z101+2 z210+2 z201)^2 z210} \quad \textbf{(39)}$$

```
> simplify(subs(z101 = z110, z201 = z210, f0z210));
```

$$-\frac{(5+4 z110+8 z210)^2 (-1+2 z210+2 z110)}{672 (-1+z110+2 z210)^2 z210} \quad \textbf{(40)}$$

This is only zero when $z210 + z110 = 1/2$.

```
> f0z210 := simplify(numer(%) - denom(%));
```

$$f0z210 := -32 z110^3+(-832 z210-64) z110^2+\left(-2944 z210^2+1168 z210-10\right) z110-2816 z210^3+2592 z210^2-642 z210+25 \quad \textbf{(41)}$$

```
> f0z110 := simplify(diff(f0, z110))
```

$$f0z110 := -\ln(2-z110-z101-2 z210-2 z201)+\ln(5+2 z110+2 z101+4 z210+4 z201)+\ln(1-2 z210-2 z110)-\ln(2)-\ln(3)-\ln(z110) \quad \textbf{(42)}$$

```
> f0z110 := simplify(exp(%), symbolic);
```

$$f0z110 := \frac{(5+2 z110+2 z101+4 z210+4 z201) (-1+2 z210+2 z110)}{6 (-2+z110+z101+2 z210+2 z201) z110} \quad \textbf{(43)}$$

```
> simplify(subs(z101 = z110, z201 = z210, f0z110));
```

$$\frac{(5+4 z110+8 z210) (-1+2 z210+2 z110)}{12 (-1+z110+2 z210) z110} \quad \textbf{(44)}$$

This is also only zero when $z210 + z110 = 1/2$.

```
> f0z110 := simplify(numer(%) - denom(%));
```

$$f0z110 := -4\, z110^2 + 16\, z210^2 + 18\, z110 + 2\, z210 - 5 \quad \mathbf{(45)}$$

```
> subs(z101 = z110, z201 = z210, f0);
```

$$\frac{(-4\, z110 - 8\, z210 + 4)\ \ln(2 - 2\, z110 - 4\, z210)}{2} + \frac{(5 + 4\, z110 + 8\, z210)\ \ln(5 + 4\, z110 + 8\, z210)}{2} + (-1 + 2\, z210 + 2\, z110)\ \ln(1 - 2\, z210 - 2\, z110) + \frac{(-4\, z110 - 12\, z210 + 1)\ \ln(2)}{2} + \frac{(-4\, z110 - 4\, z210 - 12)\ \ln(3)}{2} + \frac{(2 - 4\, z210)\ \ln(7)}{2} - 2\, z110 \ln(z110) - 2\, z210 \ln(z210) \quad \mathbf{(46)}$$

```
> subs(z210 = 1/2 - z110, %);
```

$$2\, z110 \ln(2\, z110) + \frac{(9 - 4\, z110)\ \ln(9 - 4\, z110)}{2} + \frac{(8\, z110 - 5)\ \ln(2)}{2} - 7\ \ln(3) + 2\, z110 \ln(7) - 2\, z110 \ln(z110) - 2\left(\frac{1}{2} - z110\right) \ln\left(\frac{1}{2} - z110\right) \quad \mathbf{(47)}$$

```
> diff(%,z110)
```

$$2\ \ln(2\, z110) - 2\ \ln(9 - 4\, z110) + 4\ \ln(2) + 2\ \ln(7) - 2\ \ln(z110) + 2\ \ln\left(\frac{1}{2} - z110\right) \quad \mathbf{(48)}$$

```
> solve(%)
```

$$\frac{19}{52} \quad \mathbf{(49)}$$

```
> evalf(subs(z110 = 19/52, z210 = 1/2 - 19/52, z101 = 19/52, z201 =
  1/2 - 19/52, f0));
```

$$1.672261141 \quad \mathbf{(50)}$$

```
> evalf(subs(z = 1/4, z100 = 7/72, z101 = 7/72, z110 = 7/72, z111 =
  7/72, z200 = 1/144, z201 = 1/144, z210 = 1/144, z211 = 1/144,f))
```

$$2.315007612 \quad \mathbf{(51)}$$

This implies that there is no local maximum when $z = 0$ and $z210 + z110 = 1/2$. We now consider the case where $z = 1/2$.

```
> f0 := simplify(subs(z = 1/2, z210 = 0, z110 = 0, z201 = 0, z101 =
  0, f))
```

$$f0 := \frac{(-2\, z100 - 2\, z111 - 4\, z200 - 4\, z211 + 4)\ \ln(2 - 2\, z211 - 2\, z200 - z111 - z100)}{2} \quad \mathbf{(52)}$$

$$+\frac{(4\,z211+4\,z200+2\,z111+2\,z100+5)\,\ln(4\,z211+4\,z200+2\,z111+2\,z100+5)}{2}$$
$$+\frac{(-1+2\,z200+2\,z100)\,\ln(1-2\,z200-2\,z100)}{2}$$
$$+\frac{(-1+2\,z211+2\,z111)\,\ln(1-2\,z211-2\,z111)}{2}$$
$$+\frac{(-2\,z100-2\,z111-6\,z200-6\,z211+1)\,\ln(2)}{2}$$
$$+\frac{(-2\,z100-2\,z111-2\,z200-2\,z211-12)\,\ln(3)}{2}+\frac{(2-2\,z211-2\,z200)\,\ln(7)}{2}$$
$$-z100\ln(z100)-z200\ln(z200)-z211\ln(z211)-z111\ln(z111)$$

```
> f0z211:=diff(f0,z211)
```

$$f0z211:=-2\ln(2-2\,z211-2\,z200-z111-z100)-\frac{-2\,z100-2\,z111-4\,z200-4\,z211+4}{2-2\,z211-2\,z200-z111-z100}+2\ln(4\,z211+4\,z200+2\,z111+2\,z100+5)+1+\ln(1-2\,z211-2\,z111)-\frac{-1+2\,z211+2\,z111}{1-2\,z211-2\,z111}-3\ln(2)-\ln(3)-\ln(7)-\ln(z211) \quad \textbf{(53)}$$

```
> f0z211:=simplify(exp(%))
```

$$f0z211:=-\frac{(4\,z211+4\,z200+2\,z111+2\,z100+5)^2\,(-1+2\,z211+2\,z111)}{168\,(-2+2\,z211+2\,z200+z111+z100)^2\,z211} \quad \textbf{(54)}$$

```
> f0z111:=diff(f0,z111)
```

$$f0z111:=-\ln(2-2\,z211-2\,z200-z111-z100)-\frac{-2\,z100-2\,z111-4\,z200-4\,z211+4}{2\,(2-2\,z211-2\,z200-z111-z100)}+\ln(4\,z211+4\,z200+2\,z111+2\,z100+5)+\ln(1-2\,z211-2\,z111)-\frac{-1+2\,z211+2\,z111}{1-2\,z211-2\,z111}-\ln(2)-\ln(3)-\ln(z111) \quad \textbf{(55)}$$

```
> f0z111:=simplify(exp(%))
```

$$f0z111:=\frac{(4\,z211+4\,z200+2\,z111+2\,z100+5)\,(-1+2\,z211+2\,z111)}{6\,(-2+2\,z211+2\,z200+z111+z100)\,z111} \quad \textbf{(56)}$$

```
> f0z200:=diff(f0,z200)
```

$$f0z200:=-2\ln(2-2\,z211-2\,z200-z111-z100)-\frac{-2\,z100-2\,z111-4\,z200-4\,z211+4}{2-2\,z211-2\,z200-z111-z100}+2\ln(4\,z211+4\,z200+2\,z111+2\,z100+5)+1+\ln(1-2\,z200-2\,z100)-\frac{-1+2\,z200+2\,z100}{1-2\,z200-2\,z100}-3\ln(2)-\ln(3)-\ln(7)-\ln(z200) \quad \textbf{(57)}$$

```
> f0z200:=simplify(exp(%))
```

$$f0z200 := -\frac{(4\,z211 + 4\,z200 + 2\,z111 + 2\,z100 + 5)^2\,(-1 + 2\,z200 + 2\,z100)}{168\,(-2 + 2\,z211 + 2\,z200 + z111 + z100)^2\,z200} \quad (58)$$

```
> f0z100:=diff(f0,z100)
```

$$f0z100 := -\ln(2 - 2\,z211 - 2\,z200 - z111 - z100) - \frac{-2\,z100 - 2\,z111 - 4\,z200 - 4\,z211 + 4}{2\,(2 - 2\,z211 - 2\,z200 - z111 - z100)} + \ln(4\,z211 + 4\,z200 + 2\,z111 + 2\,z100 + 5) + \ln(1 - 2\,z200 - 2\,z100) - \frac{-1 + 2\,z200 + 2\,z100}{1 - 2\,z200 - 2\,z100} - \ln(2) - \ln(3) - \ln(z100) \quad (59)$$

```
> f0z100:=simplify(exp(%))
```

$$f0z100 := \frac{(4\,z211 + 4\,z200 + 2\,z111 + 2\,z100 + 5)\,(-1 + 2\,z200 + 2\,z100)}{6\,(-2 + 2\,z211 + 2\,z200 + z111 + z100)\,z100} \quad (60)$$

```
> simplify(subs(z100 = z111, z200 = z211, f0z111));
```

$$\frac{(8\,z211 + 4\,z111 + 5)\,(-1 + 2\,z211 + 2\,z111)}{12\,(-1 + 2\,z211 + z111)\,z111} \quad (61)$$

This is only zero when $z211 + z111 = 1/2$.

```
> subs(z100 = z111, z200 = z211, f0);
```

$$\frac{(-4\,z111 - 8\,z211 + 4)\ln(2 - 4\,z211 - 2\,z111)}{2} + \frac{(8\,z211 + 4\,z111 + 5)\ln(8\,z211 + 4\,z111 + 5)}{2} + (-1 + 2\,z211 + 2\,z111)\ln(1 - 2\,z211 - 2\,z111) + \frac{(-4\,z111 - 12\,z211 + 1)\ln(2)}{2} + \frac{(-4\,z111 - 4\,z211 - 12)\ln(3)}{2} + \frac{(2 - 4\,z211)\ln(7)}{2} - 2\,z111\ln(z111) - 2\,z211\ln(z211) \quad (62)$$

```
> subs(z211 = 1/2 - z111, %);
```

$$2\,z111\ln(2\,z111) + \frac{(9 - 4\,z111)\ln(9 - 4\,z111)}{2} + \frac{(8\,z111 - 5)\ln(2)}{2} - 7\ln(3) + 2\,z111\ln(7) - 2\,z111\ln(z111) - 2\left(\frac{1}{2} - z111\right)\ln\left(\frac{1}{2} - z111\right) \quad (63)$$

```
> diff(%,z111)
```

$$2\ln(2\,z111) - 2\ln(9 - 4\,z111) + 4\ln(2) + 2\ln(7) - 2\ln(z111) + 2\ln\left(\frac{1}{2} - z111\right) \quad (64)$$

```
> solve(%)
```

$$\frac{19}{52} \quad (65)$$

```
> evalf(subs(z111 = 19/52, z211 = 1/2 - 19/52, z100 = 19/52, z200 =
  1/2 - 19/52, f0));
```

$$1.672261141 \quad (66)$$

This, as we have demonstrated earlier, is not greater than $f(\hat{z})$.

```
> with(VectorCalculus):
> B := Hessian(f, [z, z211, z111, z200, z100, z210, z110, z201,
  z101] = [1/4, 1/144, 7/72, 1/144, 7/72, 1/144, 7/72, 1/144,
  7/72], shape = symmetric);
```

$$B := \left[\left[-\frac{1672}{63}, \frac{544}{63}, \frac{488}{63}, \frac{544}{63}, \frac{488}{63}, -\frac{544}{63}, -\frac{488}{63}, -\frac{544}{63}, -\frac{488}{63}\right], \left[\frac{544}{63}, -\frac{9280}{63}, -\frac{320}{63}, \frac{32}{9}, \frac{16}{9}, -\frac{32}{9}, -\frac{16}{9}, -\frac{32}{9}, -\frac{16}{9}\right], \left[\frac{488}{63}, -\frac{320}{63}, -\frac{1024}{63}, \frac{16}{9}, \frac{8}{9}, -\frac{16}{9}, -\frac{8}{9}, -\frac{16}{9}, -\frac{8}{9}\right], \left[\frac{544}{63}, \frac{32}{9}, \frac{16}{9}, -\frac{9280}{63}, -\frac{320}{63}, -\frac{32}{9}, -\frac{16}{9}, -\frac{32}{9}, -\frac{16}{9}\right], \left[\frac{488}{63}, \frac{16}{9}, \frac{8}{9}, -\frac{320}{63}, -\frac{1024}{63}, -\frac{16}{9}, -\frac{8}{9}, -\frac{16}{9}, -\frac{8}{9}\right], \left[-\frac{544}{63}, -\frac{32}{9}, -\frac{16}{9}, -\frac{32}{9}, -\frac{16}{9}, -\frac{9280}{63}, -\frac{320}{63}, \frac{32}{9}, \frac{16}{9}\right], \left[-\frac{488}{63}, -\frac{16}{9}, -\frac{8}{9}, -\frac{16}{9}, -\frac{8}{9}, -\frac{320}{63}, -\frac{1024}{63}, \frac{16}{9}, \frac{8}{9}\right], \left[-\frac{544}{63}, -\frac{32}{9}, -\frac{16}{9}, -\frac{32}{9}, -\frac{16}{9}, \frac{32}{9}, \frac{16}{9}, -\frac{9280}{63}, -\frac{320}{63}\right], \left[-\frac{488}{63}, -\frac{16}{9}, -\frac{8}{9}, -\frac{16}{9}, -\frac{8}{9}, \frac{16}{9}, \frac{8}{9}, -\frac{320}{63}, -\frac{1024}{63}\right]\right] \tag{67}$$

```
> B1:=B*(1/2)
```

(68)

$$B1 := \begin{bmatrix} -\frac{836}{63} & \frac{272}{63} & \frac{244}{63} & \frac{272}{63} & \frac{244}{63} & -\frac{272}{63} & -\frac{244}{63} & -\frac{272}{63} & -\frac{244}{63} \\ \frac{272}{63} & -\frac{4640}{63} & -\frac{160}{63} & \frac{16}{9} & \frac{8}{9} & -\frac{16}{9} & -\frac{8}{9} & -\frac{16}{9} & -\frac{8}{9} \\ \frac{244}{63} & -\frac{160}{63} & -\frac{512}{63} & \frac{8}{9} & \frac{4}{9} & -\frac{8}{9} & -\frac{4}{9} & -\frac{8}{9} & -\frac{4}{9} \\ \frac{272}{63} & \frac{16}{9} & \frac{8}{9} & -\frac{4640}{63} & -\frac{160}{63} & -\frac{16}{9} & -\frac{8}{9} & -\frac{16}{9} & -\frac{8}{9} \\ \frac{244}{63} & \frac{8}{9} & \frac{4}{9} & -\frac{160}{63} & -\frac{512}{63} & -\frac{8}{9} & -\frac{4}{9} & -\frac{8}{9} & -\frac{4}{9} \\ -\frac{272}{63} & -\frac{16}{9} & -\frac{8}{9} & -\frac{16}{9} & -\frac{8}{9} & -\frac{4640}{63} & -\frac{160}{63} & \frac{16}{9} & \frac{8}{9} \\ -\frac{244}{63} & -\frac{8}{9} & -\frac{4}{9} & -\frac{8}{9} & -\frac{4}{9} & -\frac{160}{63} & -\frac{512}{63} & \frac{8}{9} & \frac{4}{9} \\ -\frac{272}{63} & -\frac{16}{9} & -\frac{8}{9} & -\frac{16}{9} & -\frac{8}{9} & \frac{16}{9} & \frac{8}{9} & -\frac{4640}{63} & -\frac{160}{63} \\ -\frac{244}{63} & -\frac{8}{9} & -\frac{4}{9} & -\frac{8}{9} & -\frac{4}{9} & \frac{8}{9} & \frac{4}{9} & -\frac{160}{63} & -\frac{512}{63} \end{bmatrix} \quad \textbf{(68)}$$

> $LinearAlgebra{:-}Determinant(\textbf{(68)})$

$$-\frac{46221064723759104}{117649} \quad \textbf{(69)}$$

```
> deter:=-%;
```

$$deter := \frac{46221064723759104}{117649} \quad \textbf{(70)}$$

```
> ifactor(%)
```

$$\frac{(2)^{30}\,(3)^{16}}{(7)^{6}} \quad \textbf{(71)}$$

We now begin to estimate E[Y^2]/E[Y]^2. Calculations indicate that we need to evaluate g at $\hat {\textbf{z}}$.

```
> simplify(subs(z = 1/4, z100 = 7/72, z101 = 7/72, z110 = 7/72,
  z111 = 7/72, z200 = 1/144, z201 = 1/144, z210 = 1/144, z211 =
  1/144,g))
```

$$\frac{15479341056\sqrt{2+\left(\frac{1}{4}\right)\left(\frac{9}{2}\right)}\,\sqrt{2}}{2401\sqrt{\pi^9 n^9}} \quad \textbf{(72)}$$

```
> ifactor(7739670528/2401)
```

(73)

$$\frac{(2)^{17}\,(3)^{10}}{(7)^{4}} \tag{73}$$

```
> f_expanded := subs(z = 1/4 + x*y, z00 = 1/20 + x*y00, z01 = 1/20
  + x*y01, z10 = 1/20 + x*y10, z11 = 1/20 + x*y11, f)
```

$$f\_expanded := \left(2\,z211+2\,z200+z111+z100-z110-z101+\frac{9}{4}+x\,y-2\,z210-2\,z201\right)\ln\left(2\,z211+2\,z200+z111+z100-z110-z101+\frac{9}{4}+x\,y-2\,z210-2\,z201\right)+\left(\frac{9}{4}-2\,z211-2\,z200-z111-z100+z110+z101-x\,y+2\,z210+2\,z201\right)\ln\left(\frac{9}{4}-2\,z211-2\,z200-z111-z100+z110+z101-x\,y+2\,z210+2\,z201\right)+\ln(756)-9\ln(3)-(z211+z200+z210+z201)\ln(7)-(z211+z200+z210+z201+z111+z100+z110+z101)\ln(3)+(z111+z100+z110+z101)\ln(2)-z211\ln(z211)-z111\ln(z111)-\left(\frac{1}{4}+x\,y-z211-z111\right)\ln\left(\frac{1}{4}+x\,y-z211-z111\right)-z200\ln(z200)-z100\ln(z100)-\left(\frac{1}{4}+x\,y-z200-z100\right)\ln\left(\frac{1}{4}+x\,y-z200-z100\right)-z210\ln(z210)-z110\ln(z110)-\left(\frac{1}{4}-x\,y-z210-z110\right)\ln\left(\frac{1}{4}-x\,y-z210-z110\right)-z201\ln(z201)-z101\ln(z101)-\left(\frac{1}{4}-x\,y-z201-z101\right)\ln\left(\frac{1}{4}-x\,y-z201-z101\right) \tag{74}$$

```
> f_expanded := subs(z = 1/4 + x*y, z100 = 7/72 + x*y100, z101 =
  7/72 + x*y101, z110 = 7/72, z111 = 7/72 + x*y111, z200 = 1/144 +
  x*y200, z201 = 1/144 + x*y201, z210 = 1/144 + x*y210, z211 =
  1/144 + x*y211, f)
```

$$f\_expanded := \left(\frac{9}{4}+2\,x\,y211+2\,x\,y200+x\,y111+x\,y100-x\,y101+x\,y-2\,x\,y210-2\,x\,y201\right)\ln\left(\frac{9}{4}+2\,x\,y211+2\,x\,y200+x\,y111+x\,y100-x\,y101+x\,y-2\,x\,y210-2\,x\,y201\right)+\left(\frac{9}{4}-2\,x\,y211-2\,x\,y200-x\,y111-x\,y100+x\,y101-x\,y+2\,x\,y210+2\,x\,y201\right)\ln\left(\frac{9}{4}-2\,x\,y211-2\,x\,y200-x\,y111-x\,y100+x\,y101-x\,y+2\,x\,y210+2\,x\,y201\right)+\ln(756)-9\ln(3)-\left(\frac{1}{36}+x\,y211+x\,y200+x\,y210+x\,y201\right)\ln(7) \tag{75}$$

$$-\left(\frac{5}{12}+x\,y211+x\,y200+x\,y210+x\,y201+x\,y111+x\,y100+x\,y101\right)\ln(3)+\left(\frac{7}{18}+x\,y111+x\,y100+x\,y101\right)\ln(2)-\left(\frac{1}{144}+x\,y211\right)\ln\left(\frac{1}{144}+x\,y211\right)-\left(\frac{7}{72}+x\,y111\right)\ln\left(\frac{7}{72}+x\,y111\right)-\left(\frac{7}{48}+x\,y-x\,y211-x\,y111\right)\ln\left(\frac{7}{48}+x\,y-x\,y211-x\,y111\right)-\left(\frac{1}{144}+x\,y200\right)\ln\left(\frac{1}{144}+x\,y200\right)-\left(\frac{7}{72}+x\,y100\right)\ln\left(\frac{7}{72}+x\,y100\right)-\left(\frac{7}{48}+x\,y-x\,y200-x\,y100\right)\ln\left(\frac{7}{48}+x\,y-x\,y200-x\,y100\right)-\left(\frac{1}{144}+x\,y210\right)\ln\left(\frac{1}{144}+x\,y210\right)-\frac{7\ln\left(\frac{7}{72}\right)}{72}-\left(\frac{7}{48}-x\,y-x\,y210\right)\ln\left(\frac{7}{48}-x\,y-x\,y210\right)-\left(\frac{1}{144}+x\,y201\right)\ln\left(\frac{1}{144}+x\,y201\right)-\left(\frac{7}{72}+x\,y101\right)\ln\left(\frac{7}{72}+x\,y101\right)-\left(\frac{7}{48}-x\,y-x\,y201-x\,y101\right)\ln\left(\frac{7}{48}-x\,y-x\,y201-x\,y101\right)$$

```
> simplify(series(f_expanded, x, 3));
```

$$4\ln(3)-3\ln(2)+\left(-\frac{512\,y100^2}{63}+\frac{(488\,y-56\,y101+56\,y111-320\,y200-112\,y201-112\,y210+112\,y211)\,y100}{63}-\frac{512\,y101^2}{63}+\frac{(-488\,y-56\,y111-112\,y200-320\,y201+112\,y210-112\,y211)\,y101}{63}-\frac{512\,y111^2}{63}+\frac{(488\,y+112\,y200-112\,y201-112\,y210-320\,y211)\,y111}{63}-\frac{4640\,y200^2}{63}+\frac{(544\,y-224\,y201-224\,y210+224\,y211)\,y200}{63}-\frac{4640\,y201^2}{63}+\frac{(-544\,y+224\,y210-224\,y211)\,y201}{63}-\frac{4640\,y210^2}{63}+\frac{(-544\,y-224\,y211)\,y210}{63}-\frac{836\,y^2}{63}+\frac{544\,y211\,y}{63}-\frac{4640\,y211^2}{63}\right)x^2+\mathrm{O}(x^3) \quad \textbf{(76)}$$

```
> convert(%,polynom);
```

$$4\ln(3)-3\ln(2)+\left(-\frac{512\,y100^2}{63}\right. \quad \textbf{(77)}$$

$$+\frac{(488\,y-56\,y101+56\,y111-320\,y200-112\,y201-112\,y210+112\,y211)\,y100}{63}$$
$$-\frac{512\,y101^2}{63}$$
$$+\frac{(-488\,y-56\,y111-112\,y200-320\,y201+112\,y210-112\,y211)\,y101}{63}$$
$$-\frac{512\,y111^2}{63}+\frac{(488\,y+112\,y200-112\,y201-112\,y210-320\,y211)\,y111}{63}$$
$$-\frac{4640\,y200^2}{63}+\frac{(544\,y-224\,y201-224\,y210+224\,y211)\,y200}{63}-\frac{4640\,y201^2}{63}$$
$$+\frac{(-544\,y+224\,y210-224\,y211)\,y201}{63}-\frac{4640\,y210^2}{63}+\frac{(-544\,y-224\,y211)\,y210}{63}$$
$$\left.-\frac{836\,y^2}{63}+\frac{544\,y211\,y}{63}-\frac{4640\,y211^2}{63}\right)x^2$$

\$f\$ at \$\hat{\textbf{z}}\$ is:

The exponential parts of \$E\[Y^2]\$ and \$\E[Y]^2\$ cancel out, leaving the following divided by 9.

```
> simplify(subs(z = 1/4, z100 = 7/72, z101 = 7/72, z110 = 7/72,
  z111 = 7/72, z200 = 1/144, z201 = 1/144, z210 = 1/144, z211 =
  1/144,g))*sqrt((pi*n)^9/deter)
```

$$\frac{72\sqrt{2+\left(\frac{1}{4}\right)\left(\frac{9}{2}\right)}\,\sqrt{2}}{7} \tag{78}$$

```
> h:=proc(x); log(2*Pi*x*n)/2  end proc;
```

$h :=$ **proc**(x) (79)

VectorCalculus:-\`*\`(log(*VectorCalculus*:-\`*\`(*VectorCalculus*:-\`*\`(*VectorCalculus*:-\`*\`(2, Pi), *x*), *n*)), 1/2)

end proc

```
> pol:=h(1)+h(b)+h(9/2-b)+h(9/2)-h(9)-h(z211)-h(z111)-h(z-z211-
  z111)-h(z200)-h(z100)-h(z-z200-z100)-h(z210)-h(z110)-h(1/2-z-z210
  -z110)-h(z201)-h(z101)-h(1/2-z-z201-z101);
```

$$pol := \frac{\ln(2\,\pi\,n)}{2}+\frac{\ln(2\,\pi\,b\,n)}{2}+\frac{\ln\left(2\,\pi\left(\frac{9}{2}-b\right)n\right)}{2}+\frac{\ln(9\,\pi\,n)}{2}-\frac{\ln(18\,\pi\,n)}{2} \tag{80}$$
$$-\frac{\ln(2\,\pi\,z211\,n)}{2}-\frac{\ln(2\,\pi\,z111\,n)}{2}-\frac{\ln(2\,\pi\,(z-z211-z111)\,n)}{2}$$

$$-\frac{\ln(2\pi z200\, n)}{2} - \frac{\ln(2\pi z100\, n)}{2} - \frac{\ln(2\pi (z - z200 - z100)\, n)}{2}$$
$$-\frac{\ln(2\pi z210\, n)}{2} - \frac{\ln(2\pi z110\, n)}{2} - \frac{\ln\left(2\pi\left(\frac{1}{2} - z - z210 - z110\right) n\right)}{2}$$
$$-\frac{\ln(2\pi z201\, n)}{2} - \frac{\ln(2\pi z101\, n)}{2} - \frac{\ln\left(2\pi\left(\frac{1}{2} - z - z201 - z101\right) n\right)}{2}$$

```
> g := simplify(exp(pol), symbolic);
```

$$g := \left(\sqrt{2}\,\sqrt{b}\,\sqrt{9-2\,b}\right) \Big/ \left(32\,\sqrt{z211}\,\sqrt{z111}\,\sqrt{z-z211-z111}\,\sqrt{z200}\,\sqrt{z100}\,\sqrt{z-z200-z100}\,\sqrt{z210}\,\sqrt{z110}\,\sqrt{z201}\,\sqrt{z101}\,\pi^{9/2}\,\sqrt{1-2\,z-2\,z210-2\,z110}\,\sqrt{1-2\,z-2\,z201-2\,z101}\,n^{9/2}\right) \quad \textbf{(81)}$$

```
> g := subs(b = 2*z211 + 2*z200 + z111 + z100 - z110 - z101 + 2 + z
  - 2*z210 - 2*z201, g);
```

$$g := \left(\sqrt{2}\,\sqrt{2\,z211+2\,z200+z111+z100-z110-z101+2+z-2\,z210-2\,z201}\,\sqrt{5-4\,z211-4\,z200-2\,z111-2\,z100+2\,z110+2\,z101-2\,z+4\,z210+4\,z201}\right) \Big/ \left(32\,\sqrt{z211}\,\sqrt{z111}\,\sqrt{z-z211-z111}\,\sqrt{z200}\,\sqrt{z100}\,\sqrt{z-z200-z100}\,\sqrt{z210}\,\sqrt{z110}\,\sqrt{z201}\,\sqrt{z101}\,\pi^{9/2}\,\sqrt{1-2\,z-2\,z210-2\,z110}\,\sqrt{1-2\,z-2\,z201-2\,z101}\,n^{9/2}\right) \quad \textbf{(82)}$$

```
> simplify(subs(z = 1/4, z100 = 7/72, z101 = 7/72, z110 = 7/72,
  z111 = 7/72, z200 = 1/144, z201 = 1/144, z210 = 1/144, z211 =
  1/144, g))*sqrt((pi*n)^9/deter);
```

$$\frac{81\sqrt{\pi^9 n^9}}{7\,\pi^{9/2}\,n^{9/2}} \quad \textbf{(83)}$$

```
>
```