Linear Colouring of Binomial Random Graphs

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Abstract

We investigate the linear chromatic number $\chi_{\text{lin}}(\mathcal{G}(n,p))$ of the binomial random graph $\mathcal{G}(n,p)$ on n vertices in which each edge appears independently with probability $p = p(n)$. For a graph G, $\chi_{\text{lin}}(G)$ is defined as the smallest k such that G admits a k-colouring with the property that every path P in G receives a colour which appears on only one vertex of P. For dense random graphs ($np \to \infty$ as $n \to \infty$), we show that asymptotically almost surely $\chi_{\text{lin}}(\mathcal{G}(n,p)) \geq n(1 - O((np)^{-1/2})) = n(1 - o(1))$. Understanding the order of the linear chromatic number for subcritical random graphs ($np < 1$) and critical ones ($np = 1$) is relatively easy. However, supercritical sparse random graphs ($np = c$ for some constant $c > 1$) remain to be investigated.

1 Introduction

Let $G = (V, E)$ be a graph and let $\phi : V \to \{1, \dots, k\}$ be an assignment of k colours to the vertices of G. We say that ϕ is a *proper k-colouring* if for each $\{v, w\} \in E$, $\phi(v) \neq \phi(w)$. The *chromatic number* of G, denoted $\chi(G)$, is the smallest positive integer k such that a proper k-colouring of G exists.

Given a colouring ϕ and subset $S \subseteq V$, we say that a vertex $v \in S$ is a *centre* for S if $\phi(v)$ is distinct from $\phi(w)$ for all $w \neq v$ in S. A *centred k-colouring* of G is a k-colouring of G such that for every connected subgraph $H \subseteq G$, $V(H)$ has a centre. The *centred chromatic number* $\chi_{\text{cen}}(G)$ is the smallest k such that a centred k-colouring of G exists. Observe that a centred colouring is necessarily proper, since each edge $\{v, w\} \in E$ comprises a connected subgraph of G. Hence we have the inequality $\chi(G) \leq \chi_{\text{cen}}(G)$.

The centred chromatic number is an important and natural graph parameter that has been introduced under numerous names in the literature: rank function [21], vertex ranking number (or ordered colouring) [8], weak colouring number [12]. Its study was systematically

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undertaken by Nešetřil and Ossona de Mendez under the name of *tree-depth* [18]. The notion of tree-depth is related to the one of tree-width. The tree-width of a graph can be seen as a measure of closeness to a tree, while the tree-depth takes also into account the diameter of the tree. Both serve as important measures of sparsity of a graph [19, 20].

A *linear k-colouring* of G is a k-colouring such that every subgraph of G that is a path has a centre. The corresponding *linear chromatic number* $\chi_{\text{lin}}(G)$ is defined in the obvious way. A linear colouring is necessarily proper, as each edge $\{v,w\} \in E$ is a path of length one. On the other hand, a centred colouring is necessarily linear, since path subgraphs are connected. Therefore, we have $\chi(G) \le \chi_{\text{lin}}(G) \le \chi_{\text{cen}}(G)$. (In other works, e.g., [27], the term *linear colouring* has been used to refer to proper colourings with the property that the subgraph induced any pair of colour classes is a disjoint union of paths. This is distinct from the meaning here.)

The linear chromatic number was introduced by Kun, O'Brien, Pilipczuk, and Sulivan [15] who were motivated by finding efficiently-computable approximations of tree-depth in the class of bounded expansion graphs. The authors of [15] provide a family of graphs that contains, for every $\epsilon > 0$, a graph G with $\chi_{\text{cen}}(G) > (2-\epsilon)\chi_{\text{lin}}(G)$ and based on that they stated the following, quite bold, conjecture:

Conjecture 1.1 ([15])**.** *For all graphs G, $\chi_{cen}(G) \le 2\chi_{lin}(G)$.*

We are far away from proving this conjecture. The only class of graphs for which centred chromatic number is known to be bounded by a linear function of linear chromatic number is the class of bounded degree trees [15, Theorem 4]. Currently the best upper bound is proved by Bose, Dujmović, Houdrouge, Javarsineh, and Morin [5] who were able to prove that

$$\chi_{\text{cen}}(G) \le \chi_{\text{lin}}(G)^{10}\Big(\log(\chi_{\text{lin}}(G))\Big)^{O(1)}.$$

This result improved the bound proved by Czerwiński, Nadara, and Pilipczuk [6] (they reduced the exponent from 190 to 19) which, in turn, improved the original bound by Kun at al. [15] (with exponent 190). Bose at al. [5] provide further evidence in support of the conjecture by establishing that, if G is a $k \times k$ pseudo-grid, then $\chi_{\text{cen}}(G) = O(\chi_{\text{lin}}(G))$.

In this paper, we investigate the binomial random graph $\mathcal{G}(n,p)$ that is formally defined as a distribution over the class of graphs with the set of nodes $[n] := \{1,\ldots,n\}$ in which every pair $\{i,j\} \in \binom{[n]}{2}$ appears independently as an edge in G with probability p. Note that $p = p(n)$ may (and usually does) tend to zero as n tends to infinity. Most results in this area are asymptotic by nature. We say that $\mathcal{G}(n,p)$ has some property *asymptotically almost surely* (or *a.a.s.*) if the probability that $\mathcal{G}(n,p)$ has this property tends to 1 as n goes to infinity. For more about this model see, for example, [4, 11, 10].

The binomial random graph $\mathcal{G}(n,p)$ is notoriously a good candidate for constructing counterexamples to conjectures that seem to be false, including the seminal result of Erdős from 1959 [9] that "many consider [to be] one of the most pleasing uses of the probabilistic method, as the result is surprising and does not appear to call for nonconstructive techniques" (see [3]). The *girth* of a graph is the size of its shortest cycle. Erdős showed in [9] that for any k and ℓ there exists a graph G with girth more than ℓ and $\chi(G) > k$.

Alternatively, one can investigate random graphs to support various conjectures that seem to be true. In particular, $\mathcal{G}(n,p)$ with $p = 1/2$ yields a uniform distribution of (labeled) graphs on n vertices, so showing that a given conjecture holds a.a.s. for $\mathcal{G}(n, 1/2)$ is equivalent to proving that almost all graphs satisfy the conjecture. Many open problems are supported by such statements including the following, clearly biased, small sample of results of this flavour: Meyniel's conjecture [24, 25], Tutte's conjecture [26], and Jaeger's conjecture [7].

The results presented in this paper for dense binomial random graphs $\mathcal{G}(n,p)$ (that is, in the regime when $np \to \infty$) support Conjecture 1.1. Our main theorem is the following.

Theorem 1.2. *Let $\omega = \omega(n) \le n$ be any function that tends to infinity as $n \to \infty$, and let $p = \omega/n$. Then, the following holds a.a.s.:*

$$\chi_{lin}(\mathcal{G}(n,p)) \ge n - \frac{510n}{\sqrt{\omega}}.$$

In our proofs, we did not try to optimize the constants. Since $\chi_{\text{lin}}(G) \le \chi_{\text{cen}}(G)$ and, trivially, $\chi_{\text{cen}}(G) \le n$, Theorem 1.2 implies that a.a.s. $\chi_{\text{lin}}(G(n,p)) = (1+o(1))\chi_{\text{cen}}(G(n,p)) = (1+o(1))n$. In particular, we conclude that Conjecture 1.1 holds for almost all graphs.

Supporting Conjecture 1.1 is a nice implication but understanding the behaviour of the linear chromatic number for $\mathcal{G}(n,p)$ seems to be interesting on its own. In particular, our result implies the lower bound for the centred chromatic number of dense binomial random graphs proved in [22], where it was shown that $\chi_{\text{cen}}(G(n,\omega/n)) \ge n - O\left(n/\sqrt{\omega}\right)$ a.a.s.

Investigating the linear chromatic number for very sparse random graphs, before the giant component is formed, is relatively easy. We have the following theorem:

Theorem 1.3. *If $c \in (0,1)$ then, a.a.s.,*

$$\chi_{lin}(\mathcal{G}(n,c/n)) = \log_2 \log n + O(1).$$

If $c = 1$, then, a.a.s.,

$$\frac{1}{3}\log_2 n + O(1) \le \chi_{lin}(\mathcal{G}(n,c/n)) \le \frac{2}{3}\log_2 n + \omega$$

where $\omega = \omega(n)$ is an arbitrarily slowly growing function.

With very minor modifications, the proof of [22, Theorem 1.2] shows that the bounds in Theorem 1.3 above also hold a.a.s. for $\chi_{\text{cen}}(\mathcal{G}(n,c/n))$ (and in fact we use the result of [22] to prove Theorem 1.3.) Thus, Conjecture 1.1 holds a.a.s. in the subcritical regime, and holds "in the limit" a.a.s. in the critical regime, in the sense that a.a.s.

$$\limsup_{n\to\infty} \frac{\chi_{\text{cen}}(\mathcal{G}(n,c/n))}{\chi_{\text{lin}}(\mathcal{G}(n,c/n))} \le 2.$$

On the other hand, supercritical sparse random graphs (when $p = c/n$ for some constant $c > 1$) remain to be investigated. The proof of our main result, Theorem 1.2, can be adjusted to show that a.a.s. $\chi_{\rm lin}(\mathcal{G}(n, c/n)) = \Theta(n)$, provided that c is large enough. However, there seems to be no hope to apply the current argument to prove it for any $c > 1$. Maybe a.a.s. $\chi_{\rm lin}(\mathcal{G}(n, c/n)) = o(n)$ for some $c > 1$? That would show that Conjecture 1.1 is false, since a.a.s. $\chi_{\rm cen}(\mathcal{G}(n, c/n)) = \Theta(n)$ for any $c > 1$.

The paper is structured as follows. We first provide a high level sketch of the proof of the main theorem, Theorem 1.2 (see Subsection 1.1). Section 2 is devoted to the proof of Theorem 1.2. Observations that prove Theorem 1.3 can be found in Section 3.

1.1 Sketch of the Proof of Theorem 1.2

The starting point of the proof is an idea from the paper of Alon, McDiarmid, and Reed [2] on acyclic colourings. Let $x = x(n) \in (0, 1)$ and consider any colouring of the vertices of $\mathcal{G}(n, p)$ which uses as most $(1-x)n$ colour classes. By removing at most one vertex from each class, we can make the sizes of all classes even. Since we remove at most $(1 - x)n$ vertices, a set S of size at least xn remains. Vertices in S are necessarily in even classes and so of size at least 2. In particular, colours that were initially present only one time disappeared. Finally, we (arbitrarily) pair the vertices within each colour class, resulting in at least $xn/2$ pairs of vertices, where each pair is a subset of a single colour class. Let $\mathcal{P}$ be the set of pairs formed at this step.

We call a path in $\mathcal{G}(n, p)$ *bad* if it has no centre, and observe that any path on vertices from S which visits each pair of $\mathcal{P}$ either exactly twice or not at all is bad. To show that the coloring we started with is *not* linear, we seek a bad path for the pairing $\mathcal{P}$. Maybe in each pairing $\mathcal{P}$ there is always a short bad path? The answer is 'no'—it is relatively easy to construct a large set of pairs with no short bad paths a.a.s. Alternatively, one might simply look for a Hamilton path on the vertices in S. This also turns out to be too much to ask for as, in general, the subgraph of $\mathcal{G}(n, p)$ induced by the vertices in S may not even be connected. Indeed, there are many isolated vertices in $\mathcal{G}(n, p)$ for p below the threshold for connectivity $\bar{p} = \log n/n$ so this subgraph can have many isolated vertices.

However, something slightly weaker turns out to be true. By repeatedly removing pairs of vertices in S which contain a vertex of small degree until no such pairs remain, we reach a subset $S' \subseteq S$ and a sub-pairing $\mathcal{P}' \subseteq \mathcal{P}$. (This procedure is reminiscent of the construction of the *k-core* of the subgraph induced by S.) Provided that x is large enough, one can show that, a.a.s., not too many pairs are removed and that the resulting set S' induces a connected subgraph with good expansion. Using the now-standard rotation-extension technique of Pósa [23], it can then be shown that this subgraph has a Hamilton path a.a.s. Since $\mathcal{P}' \subseteq \mathcal{P}$, any such path is bad, and hence the colouring we started with is a.a.s. not linear.

2 Dense Case: $np \to \infty$ (Proof of Theorem 1.2)

We will use the following specific instances of Chernoff's bound. Let $X \in \text{Bin}(n, p)$ be a random variable distributed according to a Binomial distribution with parameters n and p. Then, a consequence of *Chernoff's bound* (see e.g. [11, Theorem 2.1]) is that for any $t \geq 0$ we have

$$\mathsf{Pr}(X \geq \mathsf{E}X + t) \quad \leq \quad \exp\left(-\frac{t^2}{2(\mathsf{E}X + t/3)}\right) \tag{1}$$

$$\mathsf{Pr}(X \leq \mathsf{E}X - t) \quad \leq \quad \exp\left(-\frac{t^2}{2\mathsf{E}X}\right). \tag{2}$$

We define a *set-pairing* to be a pair $(S, \mathcal{P})$ where $S \subseteq [n]$ is a set of even size and $\mathcal{P}$ is a set of the form

$$\{\{v_1, v_2\}, \{v_3, v_4\}, \ldots, \{v_{|S|-1}, v_{|S|}\}\}$$

where $v_1, v_2, \ldots, v_{|S|}$ is some ordering of the vertices of S. Given set-pairings $(S, \mathcal{P})$ and $(S', \mathcal{P}')$, we say that $(S', \mathcal{P}') \subseteq (S, \mathcal{P})$ if and only if $S' \subseteq S$ and $\mathcal{P}' \subseteq P$.

Throughout this section, $\omega = \omega(n)$ will denote a function of n which grows to infinity arbitrarily slowly and satisfies $\omega \leq n$ so that $p = \omega/n \leq 1$. Recall that for a graph G on vertex set $[n]$ and $S \subseteq [n]$, we let $G[S]$ denote the subgraph of G induced by the vertices in S.

Let $(S, \mathcal{P})$ be a set-pairing and let G be a graph on vertex set $[n]$. For a given $k \geq 0$ we define the *k-core of* $(S, \mathcal{P})$ in G, denoted $C_k^G(S, \mathcal{P})$ to be the maximal induced subgraph of $G[S]$ with minimum degree at least k and such that if $v \in V(C_k^G(S, \mathcal{P}))$ and $\{v, w\} \in \mathcal{P}$, then $w \in V(C_k^G(S, \mathcal{P}))$. Note that any maximal subgraph satisfying these conditions is necessarily unique, else a larger subgraph satisfying the same conditions could be constructed by taking a union. Thus the definition is unambiguous (though the k-core may be empty). Moreover, to find the k-core of $(S, \mathcal{P})$ one may repeatedly remove vertices of degree less than k (together with their partners in $\mathcal{P}$) until there is no vertex of degree less than k.

The key feature of $C_k^G(S, \mathcal{P})$ is that it respects the original pairing $\mathcal{P}$: for any $\{v, w\} \in \mathcal{P}$, either both v and w are in the k-core, or neither of them is. In this subsection, we establish some properties of k-cores of set pairings $(S, \mathcal{P})$ in the binomial random graph $\mathcal{G}(n, p)$. When the host graph is clear from context, we simply write $C_k(S, \mathcal{P})$.

We will first show that k-cores are large (Subsection 2.1) and have good expansion properties (Subsection 2.2). The results in these two subsections are adaptations of similar results in [14, Section 3] to the present application. These observations, combined via sprinkling with the rotation-extension technique of Pósa, imply that the corresponding k-cores have Hamilton paths (Subsection 2.3).

2.1 k-cores are Large

Our first lemma shows that, a.a.s., for every set-pairing $(S, \mathcal{P})$ with $|S| \geq cn/\sqrt{\omega}$, the core $C_{|S|p/3}(S, \mathcal{P})$ in $\mathcal{G}(n, p)$ has at least $|S|/2$ vertices.

Lemma 2.1. *Let $p = \omega/n$, $c > 0$, and let $G = \mathcal{G}(n,p)$. Then, a.a.s. for every set-pairing $(S, \mathcal{P})$ with $|S| \geq cn/\sqrt{\omega}$, there exists a set-pairing $(S', \mathcal{P}') \subseteq (S, \mathcal{P})$ such that $|S'| \geq |S|/2$ and the subgraph $G[S']$ has minimum degree at least $|S|p/3$.*

Proof. First, fix a set-pairing $(S, \mathcal{P})$ with $|S| \geq cn/\sqrt{\omega} = \Omega(\sqrt{n})$. We build a sub set-pairing $(S', \mathcal{P}')$ using the following simple algorithm. Set $S_0 = S$ and $\mathcal{P}_0 = \mathcal{P}$. For $i = 0, 1, 2, \ldots$, if $G[S_i]$ contains a vertex of degree less than $|S|p/3$, then let v_{i+1} be the smallest such vertex, and let w_{i+1} be its partner such that $\{v_{i+1}, w_{i+1}\} \in \mathcal{P}_i$; set $S_{i+1} = S_i \setminus \{v_{i+1}, w_{i+1}\}$ and $\mathcal{P}_{i+1} = P_i \setminus \{\{v_{i+1}, w_{i+1}\}\}$. If no vertex of degree less than $|S|p/3$ exists in $G[S_i]$ (which is trivially true if $S_i = \varnothing$), then the algorithm terminates after i steps.

Let $T = T(S, \mathcal{P})$ be the termination time of the algorithm and let $S' = S_T$. If $T < |S|/2$, then clearly $G[S']$ is a subgraph of $\mathcal{G}(n,p)$ on $|S| - 2T$ vertices of minimum degree at least $|S|p/3$. To prove the lemma, we will show that a.a.s. $T(S, \mathcal{P}) \leq |S|/4$ for all set-pairings $(S, \mathcal{P})$ with $|S| \geq cn/\sqrt{\omega}$.

For each $i \geq 1$, let $B_i = \{v_1, v_2, \ldots, v_i\}$, where the v_j's are as defined in the algorithm. By construction, we have $|E(B_i, S_i)| < i \cdot |S|p/3$. Suppose that $T > |S|/4$. Then, at step $t = \lfloor |S|/4 \rfloor$ of the algorithm, we find disjoint sets B_t and S_t of sizes $\lfloor |S|/4 \rfloor = (1+o(1))|S|/4$ and $|S| - 2\lfloor |S|/4 \rfloor = (1 + o(1))|S|/2$, respectively, such that

$$|E(B_t, S_t)| < \left\lfloor \frac{|S|}{4} \right\rfloor \cdot \frac{|S|p}{3} \leq \frac{|S|^2\omega}{12n}.$$

The preceding shows that for any set-pairing $(S, \mathcal{P})$ with $|S| \geq cn/\sqrt{\omega}$, the event $\{T(S, \mathcal{P}) > |S|/4\}$ implies the existence of a pair of disjoint sets $P, Q \subseteq [n]$ such that $|P| = \lfloor |S|/4 \rfloor$, $|Q| = |S| - 2\lfloor |S|/4 \rfloor$, and $|E(P,Q)| < \frac{|S|^2\omega}{12n}$. Thus,

$$\mathsf{Pr}\left(\bigcup_{(S,\mathcal{P})} \left\{ T(S, \mathcal{P}) > \frac{|S|}{4} \right\} \right) \leq \mathsf{Pr}\left(\bigcup_{P,Q} \left\{ |E(P,Q)| < \frac{s^2\omega}{12n} \right\} \right), \tag{3}$$

where the union on the left is taken over all set-pairings with $|S| \geq cn/\sqrt{\omega}$ and the union on the right is over all disjoint P, Q with $|P| = \lfloor s/4 \rfloor$, $|Q| = s - 2\lfloor s/4 \rfloor$, and $s \geq cn/\sqrt{\omega}$.

Consider a fixed pair of disjoint sets P, Q of sizes $\lfloor s/4 \rfloor$ and $s - 2\lfloor s/4 \rfloor$, respectively, for some $s \geq cn/\sqrt{\omega}$. The cut size $|E(P,Q)|$ is the binomial random variable $X \sim \mathrm{Bin}(|P||Q|, p)$ with mean

$$\mathsf{E}[X] = |P||Q|p = (1 + o(1))\frac{s^2\omega}{8n}.$$

From Chernoff's bound (2) applied with $t = \mathsf{E}[X] - \frac{s^2\omega}{12n} = (1 + o(1))\mathsf{E}[X]/3$, we then get

$$\begin{aligned} \mathsf{Pr}\left(|E(P,Q)| < \frac{s^2\omega}{12n} \right) &\leq \exp\left\{ -(1 + o(1))\frac{\mathsf{E}[X]}{18} \right\} = \exp\left\{ -(1 + o(1))\frac{s^2\omega}{144n} \right\} \\ &\leq \exp\left\{ -\frac{s^2\omega}{150n} \right\} \leq \exp\left\{ -\frac{cs\sqrt{\omega}}{150} \right\}, \end{aligned}$$

where the second equality holds for n sufficiently large, and in the final equality we use the fact that $s \geq cn/\sqrt{\omega}$. For a given s, the number of choices for the sets P and Q is at most

$$\binom{n}{\lfloor s/4 \rfloor}\binom{n}{s-2\lfloor s/4 \rfloor} \leq n^{O(1)}\left(\frac{4ne}{s}\right)^{s/4}\left(\frac{2ne}{s}\right)^{s/2} = n^{O(1)}\left(\frac{2^{\frac{4}{3}}ne}{s}\right)^{\frac{3s}{4}}.$$

(The $n^{O(1)}$ factor is the price paid for getting rid of ceilings; the constant implied in the $O(\cdot)$ notation does not depend on s.) Using the fact that $s \geq cn/\sqrt{\omega}$, the right-hand side above is at most

$$n^{O(1)}\left(\frac{2^{\frac{4}{3}}\sqrt{\omega}e}{c}\right)^{\frac{3s}{4}} = \exp\left\{(1+o(1))\frac{3s}{8}\log\omega\right\} \leq \exp\left\{\frac{s\log\omega}{2}\right\}.$$

(The inequality holds for n sufficiently large.) Thus, the probability that there exist disjoint sets P, Q of sizes $\lfloor s/4 \rfloor$ and $s - 2\lfloor s/4 \rfloor$, respectively, such that $|E(P,Q)| \leq \frac{s^2\omega}{12n}$ is at most

$$\exp\left\{-\frac{cs\sqrt{\omega}}{150}+\frac{s\log\omega}{2}\right\} = \exp\left\{-\frac{cs\sqrt{\omega}}{150}\left(1-\frac{75\log\omega}{c\sqrt{\omega}}\right)\right\} \leq \exp\left\{-\frac{cs\sqrt{\omega}}{200}\right\}.$$

(As always, the inequality holds for n sufficiently large.) It follows that

$$\begin{aligned} \mathsf{Pr}\left(\bigcup_{P,Q}\left\{|E(P,Q)| \leq \frac{s^2\omega}{12n}\right\}\right) &\leq \sum_{s=\lceil cn/\sqrt{\omega}\rceil}^{n} \exp\left\{-\frac{cs\sqrt{\omega}}{200}\right\} \\ &\leq ne^{-\Omega(n)} = o(1). \end{aligned}$$

Based on (3), we conclude that a.a.s. $T(S,\mathcal{P}) \leq |S|/4$ for all set-pairings $(S,\mathcal{P})$ with $|S| \geq cn/\sqrt{\omega}$. This completes the proof of the lemma. □

2.2 k-cores are Good Expanders

The previous lemma, Lemma 2.1, shows that, a.a.s., for every set-pairing $(S,\mathcal{P})$ with $|S| \geq 2cn/\sqrt{\omega}$, the core $C_{|S|p/3}(S,\mathcal{P})$ in $\mathcal{G}(n,p)$ has at least $|S|/2 \geq cn/\sqrt{\omega}$ vertices. By definition, the minimum degree of $C_{|S|p/3}(S,\mathcal{P})$ is at least $|S|p/3 \geq |V(C_{|S|p/3}(S,\mathcal{P}))|p/3$. Our next lemma implies that a.a.s. for every set-pairing $(S,\mathcal{P})$ with $|S| \geq 2cn/\sqrt{\omega}$, the core $C_{|S|p/3}(S,\mathcal{P})$ in $\mathcal{G}(n,p)$ is a good expander.

Lemma 2.2. *Let $p = \omega/n$ and $c > 0$. The following properties hold a.a.s.:*

i) For any subgraph H of $\mathcal{G}(n,p)$ on at least $cn/\sqrt{\omega}$ vertices with $\delta(H) \geq |V(H)|p/3$, we have $|N_H(X) \setminus X| > 2|X|$ for every $X \subseteq V(H)$ with $|X| \leq |V(H)|/45$.

ii) Every induced subgraph H of $\mathcal{G}(n,p)$ with $\delta(H) \geq |V(H)|p/3$ on at least $cn/\sqrt{\omega}$ vertices is connected.

Proof. We begin with i). Suppose that there is a subgraph H of $\mathcal{G}(n,p)$ on $s \geq cn/\sqrt{\omega}$ vertices with minimum degree at least $sp/3$ that fails the expansion condition in the statement. Let $X \subseteq V(H)$ be a subset of vertices with $|X| \leq s/45$ and such that $|N_H(X) \setminus X| \leq 2|X|$. Then, $N_H(X) \setminus X$ is contained in some $Y \subseteq V(H)$, disjoint from X, with $|Y| = 2|X|$. In H, there are at most $\binom{|X|}{2} + |X||Y| \leq \frac{5}{2}|X|^2$ possible edges incident with X. At least

$$\frac{\delta(H)|X|}{2} \geq \frac{sp|X|}{6} = \frac{s\omega|X|}{6n}$$

of these edges must be present in H, and hence also in $\mathcal{G}(n,p)$. Writing $|X| = j \leq s/45$, the probability that this occurs for a given pair of sets X and Y is at most

$$\binom{\lfloor \frac{5}{2}j^2 \rfloor}{\lceil \frac{s\omega j}{6n} \rceil} p^{\lceil s\omega j/6n \rceil} \leq \left(\frac{15ejn}{s\omega} p \right)^{\lceil s\omega j/6n \rceil} \leq \left(\frac{15ej}{s} \right)^{s\omega j/6n}.$$

For $s \geq cn/\sqrt{\omega}$, let $\mathcal{B}_s$ be the event that there exists a subgraph H of $\mathcal{G}(n,p)$ with s vertices and minimum degree at least $sp/3$ such that the expansion condition in the statement of the lemma fails. We have

$$\begin{aligned}
\mathsf{Pr}(\mathcal{B}_s) &\leq \sum_{j=1}^{\lfloor s/45 \rfloor} \binom{n}{j} \binom{n}{2j} \left(\frac{15ej}{s} \right)^{s\omega j/6n} \\
&\leq \sum_{j=1}^{\lfloor s/45 \rfloor} \left[\frac{1}{4} \left(\frac{ne}{j} \right)^3 \left(\frac{15ej}{s} \right)^{s\omega/6n} \right]^j \\
&\leq \sum_{j=1}^{\lfloor s/45 \rfloor} \left[\frac{1}{4} \left(\frac{ne}{j} \right)^3 \left(\frac{15ej}{s} \right)^{c\sqrt{\omega}/6} \right]^j, \qquad (4)
\end{aligned}$$

where in the final inequality we use that $15ej/s < 1$ for $j \leq \lfloor s/45 \rfloor$ and that $s \geq cn/\sqrt{\omega}$. We will show that the last sum above is $o(1/n)$ uniformly in s. This will suffice to finish the proof of part i), since it implies that

$$\mathsf{Pr}\left(\bigcup_{s=\lceil cn/\sqrt{\omega} \rceil}^{n} \mathcal{B}_s \right) \leq \sum_{s=\lceil cn/\sqrt{\omega} \rceil}^{n} \mathsf{Pr}(\mathcal{B}_s) = n \cdot o(1/n) = o(1).$$

Now, we bound the sum (4). We remark first that, since $\omega \leq n$, we have $s \geq cn/\sqrt{\omega} = c\sqrt{n}$ and, in particular, $s \gg \log n$. We will split the sum (4) into two parts corresponding to $j \leq \lfloor \log n \rfloor$ and, respectively, $j > \lfloor \log n \rfloor$.

For $1 \leq j \leq \lfloor \log n \rfloor$, we have

$$\frac{1}{4} \left(\frac{ne}{j} \right)^3 \left(\frac{15ej}{s} \right)^{c\sqrt{\omega}/6} \leq \frac{(ne)^3}{4} \left(\frac{15e \log n}{c\sqrt{n}} \right)^{c\sqrt{\omega}/6} =: g(n).$$

where in the inequality we use that $1 \le j \le \log n$ and $s \ge c\sqrt{n}$. It is easy to see that $g(n) = o(1/n)$, and hence

$$\sum_{j=1}^{\lfloor \log n \rfloor} \left[\frac{1}{4} \left(\frac{ne}{j}\right)^3 \left(\frac{15ej}{s}\right)^{c\sqrt{\omega}/6} \right]^j \le \sum_{j=1}^{\lfloor \log n \rfloor} (g(n))^j = O(g(n)) = o(1/n).$$

For $\lfloor \log n \rfloor + 1 \le j \le \left\lfloor \frac{s}{45} \right\rfloor$, observe that

$$\begin{aligned} \frac{1}{4} \left(\frac{ne}{j}\right)^3 \left(\frac{15ej}{s}\right)^{c\sqrt{\omega}/6} &= \frac{(15e^2)^3}{4} \left(\frac{n}{s}\right)^3 \left(\frac{15ej}{s}\right)^{c\sqrt{\omega}/6-3} \\ &\le \frac{(15e^2)^3}{4} \left(\frac{\sqrt{\omega}}{c}\right)^3 \left(\frac{15e}{45}\right)^{c\sqrt{\omega}/6-3} \quad := h(n). \end{aligned}$$

Now, $h(n) = \exp\{O(\log \omega) - \Omega(\sqrt{\omega})\} = \exp\{-\Omega(\sqrt{\omega})\} = o(1)$, and hence for n large enough, $h(n) < 1/3 < 1/e$ and so we have

$$\begin{aligned} \sum_{j=\lfloor \log n \rfloor + 1}^{\lfloor s/45 \rfloor} \left[\frac{1}{4} \left(\frac{ne}{j}\right)^3 \left(\frac{15ej}{s}\right)^{c\sqrt{\omega}/6} \right]^j &\le \sum_{j=\lfloor \log n \rfloor + 1}^{\infty} (h(n))^j = O\left((h(n))^{\log n}\right) \\ &= O((1/3)^{\log n}) = o(1/n). \end{aligned}$$

Thus, we conclude that for $s \ge cn/\sqrt{\omega}$, the sum (4) is $o(1/n)$, uniformly in s. This completes the proof of part i).

For ii), let H be an induced subgraph on $s \ge cn/\sqrt{\omega}$ vertices with $\delta(H) \ge |V(H)|p/3$. By part i), we may assume that H does not have a component with $\frac{s}{45}$ or fewer vertices. If H has more than one component of size greater than $s/45$, then we find a pair of disjoint sets of $\lceil s/45 \rceil$ vertices each which induce no edges between them in $\mathcal{G}(n,p)$. (The assumption that H is induced is necessary here.) The probability of finding such sets is at most

$$\begin{aligned} \binom{n}{\lceil s/45 \rceil}^2 (1-p)^{\lceil s/45 \rceil^2} &\le O(\omega) \left(\frac{45ne}{s}\right)^{2s/45} e^{-s^2\omega/45^2 n} \\ &\le O(\omega) \left(\frac{45e\sqrt{\omega}}{c}\right)^{2s/45} e^{-cs\sqrt{\omega}/45^2} \\ &= \exp\left\{ -\frac{cs\sqrt{\omega}}{45^2} \left(1 - O\left(\frac{\log \omega}{\sqrt{\omega}}\right)\right) \right\} \\ &\le \exp\left\{ -\frac{cs\sqrt{\omega}}{50^2} \right\}, \end{aligned}$$

with the final inequality holding for n large enough. Thus, the probability that there exists an induced subgraph H on $s \ge cn/\sqrt{\omega}$ vertices with multiple components of size greater than $s/45$ is at most

$$\sum_{s=\lceil cn/\sqrt{\omega} \rceil}^{n} \exp\left\{ -\frac{cs\sqrt{\omega}}{50^2} \right\} \le ne^{-\Omega(n)} = o(1).$$

We conclude that a.a.s., every induced subgraph H of $\mathcal{G}(n,p)$ on at least $cn/\sqrt{\omega}$ vertices with $\delta(H) \geq |V(H)|p/3$ is connected. This finishes part ii) of the proof and so the proof of the lemma is finished. □

2.3 Sprinkling and Pósa Rotations

The main result in this section is the following.

Theorem 2.3. *Let $p = \omega/n$ and $G = \mathcal{G}(n,p)$. Then, a.a.s., for all set-pairings $(S,\mathcal{P})$ with $|S| \geq 510n/\sqrt{\omega}$, there is a nonempty sub-pairing $(S',\mathcal{P}') \subseteq (S,\mathcal{P})$ such that $G[S']$ has a Hamilton path.*

Before we prove Theorem 2.3, let us show that Theorem 1.2 follows from it.

Proof of Theorem 1.2. Let $p = \omega/n$. Let $\phi : [n] \to \{1,2,\ldots,c\}$ be a colouring of the vertices of $\mathcal{G}(n,p)$ with $c \leq n - 510n/\sqrt{\omega}$ colour classes. We construct a set-pairing $(S(\phi),\mathcal{P}(\phi))$ associated to ϕ as follows. For each $j \in \{1,2,\ldots,c\}$, if $|\phi^{-1}(j)|$ is odd, let v be the smallest vertex in $\phi^{-1}(j)$ and set $S_j := \phi^{-1}(j) \setminus \{v\}$; otherwise, let $S_j := \phi^{-1}(j)$.

For each j such that $s_j := S_j > 0$, let $v_{j_1}, v_{j_2}, \ldots, v_{j_{s_j}}$ be the vertices of S_j in increasing order, and define the pairing $\mathcal{P}_j := \{\{v_{j_1}, v_{j_2}\}, \{v_{j_3}, v_{j_4}\}, \ldots, \{v_{j_{s_j-1}}, v_{j_{s_j}}\}\}$. Finally, define the set-pairing $(S(\phi),\mathcal{P}(\phi))$ by

$$S(\phi) := \bigcup_{j=1}^{c} S_j \quad \text{and} \quad \mathcal{P}(\phi) := \bigcup_{j=1}^{c} \mathcal{P}_j.$$

Note that

$$|S(\phi)| = \left|\bigcup_{j=1}^{c} S_j\right| = \sum_{j=1}^{c} |S_j| \geq \sum_{j=1}^{c} (|\phi^{-1}(j)| - 1) = n - c \geq \frac{510n}{\sqrt{\omega}}.$$

If ϕ is a linear colouring of $\mathcal{G}(n,p)$, then $(S(\phi),\mathcal{P}(\phi))$ cannot contain any nonempty sub-pairing $(S'(\phi),\mathcal{P}'(\phi))$ with a Hamilton path. But, by Theorem 2.3 and the fact that $|S(\phi)| \geq 510n/\sqrt{\omega}$, a.a.s. $(S(\phi),\mathcal{P}(\phi))$ contains such a sub-pairing, regardless which colouring ϕ with at most $n - 510n/\sqrt{\omega}$ colour classes is considered. Thus, we conclude that a.a.s. no linear colouring of $\mathcal{G}(n,p)$ with at most $n - 510n/\sqrt{\omega}$ colour classes exists, and hence a.a.s.

$$\chi_{\text{lin}}(\mathcal{G}(n,p)) > n - \frac{510n}{\sqrt{\omega}},$$

which finishes the proof of Theorem 1.2. □

It remains to prove Theorem 2.3. To that end, we will use the rotation-extension technique of Pósa [23]. This procedure requires a two-round exposure of the edges of $\mathcal{G}(n,p)$. That is, to generate the random graph for a given p, we choose two values $0 \leq p_1, p_2 \leq p$ such that $p = p_1 + p_2 - p_1 p_2$, then generate independent random graphs $\mathcal{G}(n,p_1)$ and $\mathcal{G}(n,p_2)$.

It is easy to see that the graph obtained by taking the union of $\mathcal{G}(n,p_1)$ and $\mathcal{G}(n,p_2)$, and collapsing any double edges into single edges is distributed as $\mathcal{G}(n,p)$. In our case, $p=\omega/n$, and we can take $p_1=\frac{\omega}{2n}$ and $p_2=\frac{\omega}{2n}+\epsilon\geq\frac{\omega}{2n}$, where $\epsilon=O((\omega/n)^2)$.

Rather than giving a full explanation of the technique here, we refer instead to the treatment in [10, Chapter 6]. The crucial lemma is the following, which is a straightforward consequence of [13, Corollary 2.10]:

Lemma 2.4. *Let r be a positive integer, and let $G=(V,E)$ be a connected graph in which every subset $X\subseteq V$ of size $|X|\leq r$ satisfies $|N(X)\setminus X|>2|X|$. Suppose that the longest path in G has $h\leq|V|-2$ edges. Then there are at least $r^2/2$ non-edges of G such that the addition of any one of them results in a graph G' whose longest path has at least $h+1$ edges.*

In the light of the above lemma, we will call a graph H *good* if H is connected and satisfies $|N(X)\setminus X|>2|X|$ for every $X\subseteq V(H)$ with $|X|\leq|V(H)|/45$.

We will use the following observation.

Lemma 2.5. *Let G_1 be any simple graph on vertex set $[n]$. Sample $\mathcal{G}(n,p_2)$ and consider $G=G_1\cup\mathcal{G}(n,p_2)$, collapsing double edges if needed. Then, the following property holds a.a.s.: for all subsets $S\subseteq[n]$ with $|S|\geq 255n/\sqrt{\omega}$ such that $G_1[S]$ is good, $G[S]$ contains a Hamilton path.*

Proof. Consider a set $S\subseteq[n]$ of size $|S|\geq 255n/\sqrt{\omega}$ that induces a good subgraph $H=G_1[S]$. Using Lemma 2.4, we will greedily build a Hamilton path on the vertices in S as the edges of $\mathcal{G}(n,p_2)$ are exposed one-by-one. For a graph G, define $\lambda(G)$ to be the number of edges in a longest path in G.

Let $\{e_1,e_2,\ldots,e_r\}$ be the edges in $\mathcal{G}(n,p_2)$ which join pairs of vertices in S, listed in a random order. Note that r is distributed as $\mathrm{Bin}\left(\binom{|S|}{2},p_2\right)$, which has mean asymptotic to $|S|^2p_2/2\geq 255^2n/4$. Using Chernoff's bound (2), it is easy to show that $r\geq|S|^2p_2/4$ with probability $1-o(2^{-n})$. We condition on this outcome, and henceforth assume $r\geq|S|^2p_2/4$. Note that we only exposed the number of edges in $\mathcal{G}(n,p_2)$ that fall into the set S; the locations of these edges are still unexposed.

For $1\leq j\leq r$, inductively define $H_j=H_{j-1}\cup\{e_j\}$, where we take $H_0=H$. Since we assume H is good, and adding edges to a good graph preserves the property of being good, H_j is good for all j.

Now, fix $j\geq 0$ and condition on the outcome of H_j (there is no conditioning necessary for $j=0$, when we simply have $H_0=H$). Suppose that $\lambda(H_j)<|S|-1$, that is, H_j does not have a Hamilton path. By Lemma 2.4, there exists a set B_j of at least $\frac{|S|^2}{2\cdot(45)^2}=\frac{|S|^2}{4050}$ non-edges of H_j such that if $e_{j+1}\in B_j$, then we have $\lambda(H_{j+1})\geq\lambda(H_j)+1$. The edge e_{j+1} is uniformly distributed over pairs of vertices in S which are not in the set $\{e_1,e_2,\ldots,e_j\}$. Crudely, there are at most $\binom{|S|}{2}$ such pairs. Since none of the pairs in B_j are in $\{e_1,e_2,\ldots,e_j\}$ by definition, we therefore have

$$\mathsf{Pr}(e_{j+1}\in B_j\,|\,e_1,e_2,\ldots,e_j)\geq\binom{|S|}{2}^{-1}\frac{|S|^2}{4050}\geq\frac{1}{2025}\,,$$

given that $\lambda(H_j) < |S|-1$. Clearly, if $\lambda(H_j) = |S|-1$, then $\lambda(H_{j+1}) = |S|-1$ as well. So for any $0 \le j \le r-1$, either H_j has a Hamilton path, or the length of a longest path increases by at least 1 from H_j to H_{j+1} with probability at least $1/2025$, independently of the history up to time j. Thus, for as long as H_j has no Hamilton path, $\lambda(H_j)$ stochastically dominates a Binomial random variable with mean $j/2025$. In particular,

$$\begin{aligned} \mathsf{Pr}(H_r \text{ has no Hamilton path}) &\le \mathsf{Pr}(\lambda(H_r) < |S|-1) \\ &\le \mathsf{Pr}\left(\mathrm{Bin}\left(r, \frac{1}{2025}\right) < |S|-1\right). \end{aligned} \tag{5}$$

Conditioned on $r \ge |S|^2 p_2/4$, the $\mathrm{Bin}(r, \frac{1}{2025})$ random variable has mean

$$\frac{r}{2025} \ge \frac{1}{2025}\frac{|S|^2 p_2}{4} \ge \frac{1}{16200}\left(\frac{255n}{\sqrt{\omega}}\right)^2 \frac{\omega}{n} > 4n,$$

where in the last inequality we use the fact that $255 > 2\sqrt{16200} \approx 254.558$. By Chernoff's bound (2) with $t = \frac{r}{2025} - n > \frac{3}{4}\cdot\frac{r}{2025}$,

$$\begin{aligned} \mathsf{Pr}\left(\mathrm{Bin}\left(r, \frac{1}{2025}\right) < |S|-1\right) &\le \mathsf{Pr}\left(\mathrm{Bin}\left(r, \frac{1}{2025}\right) < n\right) \\ &\le \exp\left\{-\frac{9}{32}\cdot\frac{r}{2025}\right\} \\ &\le \exp\left\{-\frac{9n}{8}\right\} \\ &= o(2^{-n}), \end{aligned}$$

and so $\mathsf{Pr}(H_r \text{ has no Hamilton path}) = o(2^{-n})$ as well by (5).

In summary, we have shown that, conditioned on $r \ge |S|^2 p_2/4$, the subgraph $H_r = G[S]$ contains a Hamilton path with probability $1-o(2^{-n})$. Since $r \ge |S|^2 p_2/4$ also with probability $1-o(2^{-n})$, it follows that $\mathsf{Pr}(G[S] \text{ has no Hamilton path}) = o(2^{-n})$. A union bound over the at most 2^n choices for the set S completes the proof of the lemma. □

Now, we can finish the proof of Theorem 2.3.

Proof of Theorem 2.3. By Lemma 2.1, applied to $\mathcal{G}(n, p_1)$ with $p_1 = p/2 = (\omega/2)/n$, a.a.s., for every set pairing $(S, \mathcal{P})$ with

$$|S| \ge \frac{(255\sqrt{2})n}{\sqrt{\omega/2}} = \frac{510n}{\sqrt{\omega}},$$

there exists a set-pairing $(S', \mathcal{P}') \subseteq (S, \mathcal{P})$ such that $|S'| \ge |S|/2 \ge 255n/\sqrt{\omega}$ and the subgraph $G[S']$ has minimum degree at least $|S|p_1/3 \ge |S'|p_1/3$. By Lemma 2.2, applied again to $\mathcal{G}(n, p_1)$, a.a.s. every induced subgraph H of $\mathcal{G}(n, p_1)$ on at least

$$\frac{(255/\sqrt{2})n}{\sqrt{\omega/2}} = \frac{255n}{\sqrt{\omega}}$$

vertices with $\delta(H) \geq |V(H)|p_1/3$ is good.

Combining the two above observations together, we establish that a.a.s. in $\mathcal{G}(n, p_1)$, for every set-pairing $(S, \mathcal{P})$ with $|S| \geq 510n/\sqrt{\omega}$, there exists an induced subgraph that is good and has at least $255n/\sqrt{\omega}$ vertices. Then, by Lemma 2.5, a.a.s. each of these subgraphs becomes Hamiltonian after adding the edges from $\mathcal{G}(n, p_2)$. This finishes the proof of the theorem. □

3 Sparse Case: $np \leq 1$ (Proof of Theorem 1.3)

In this section, we give some results about $\chi_{\text{lin}}(\mathcal{G}(n, p))$ in the regime $p = c/n$, $c \leq 1$. These results are implied directly by the arguments of Perarnau and Serra from [22], where the centred chromatic number of $\mathcal{G}(n, p)$ is studied under the name of *tree-depth*. The relevant result therein is the following.

Theorem 3.1 ([22] Theorem 1.2)**.** *The following hold a.a.s.:*

$$\chi_{cen}(\mathcal{G}(n, c/n)) = \begin{cases} \Theta(\log\log n) & c \in (0,1) \\ \Theta(\log n) & c = 1. \end{cases}$$

Though not stated explicitly in the paper, it is clear upon closer examination that the arguments of [22] also give the constants implied by the Θ-notation in Theorem 3.1. In particular, their arguments directly imply

$$\frac{1}{2}\log_2 \log n + O(1) \leq \chi_{\text{cen}}(\mathcal{G}(n, c/n)) \leq \log_2 \log n + O(1) \tag{6}$$

a.a.s. when $c \in (0, 1)$ and

$$\frac{1}{3}\log_2 n + O(1) \leq \chi_{\text{cen}}(\mathcal{G}(n, c/n)) \leq \frac{2}{3}\log_2 n + \omega \tag{7}$$

when $c = 1$, where $\omega = \omega(n)$ is an arbitrarily slowly growing function of n.

Since $\chi_{\text{lin}}(G) \leq \chi_{\text{cen}}(G)$ for any graph G, the upper bounds in (6) and (7) also hold for $\chi_{\text{lin}}(\mathcal{G}(n, p))$. As we will see, the techniques used in [22] to prove the lower bounds in (6) and (7) apply equally well to linear colourings, and thus these bounds hold as well for $\chi_{\text{lin}}(\mathcal{G}(n, p))$. Moreover, a result from [16] actually gives an improvement to the lower bound in the subcritical case which removes the $1/2$ factor from the leading term. With these considerations we are able to deduce Theorem 1.3.

Let us make a few observations before beginning the proof. First, for any graph G, since any linear colouring of G is necessarily a linear colouring of every subgraph of G, we have

$$\chi_{\text{lin}}(G) \geq \max_{H \subseteq G} \chi_{\text{lin}}(H), \tag{8}$$

where the maximum is taken over all subgraphs H of G. Next, observe that any connected subgraph of the path on k vertices P_k is necessarily a path, and hence linear and the centred colourings are equivalent on a path:

$$\chi_{\text{cen}}(P_k) = \chi_{\text{lin}}(P_k). \tag{9}$$

Finally, it is well known, and easy to show, that

$$\chi_{\text{cen}}(P_k) = \lfloor \log_2 k \rfloor + 1. \tag{10}$$

Together, (8), (9), and (10) imply that for any graph G and any component C of G, we have

$$\chi_{\text{lin}}(G) \geq \log_2(\text{diam}(C)). \tag{11}$$

Observation (11) is all we need to complete the proof of Theorem 1.3.

Proof of Theorem 1.3. Recall that we only need to show the lower bounds on $\chi_{\text{lin}}(\mathcal{G}(n, c/n))$; the upper bounds are implied by (6) and (7).

For $c < 1$, the diameter of the largest component in $\mathcal{G}(n, c/n)$ is typically of order $\sqrt{\log n}$, but there are components of smaller cardinality with diameter of order $\log n$ (see [16]). We conclude that a.a.s. $\chi_{\text{lin}}(\mathcal{G}(n, c/n)) \geq \log_2 \log n + O(1)$ by (11). (Note that this also improves the lower bound in (6) proved in [22].)

Similarly, for $c = 1$, the diameter of the largest component in $\mathcal{G}(n, 1/n)$ is known to be typically of order $n^{1/3}$ (see [17]) implying that a.a.s. $\chi_{\text{lin}}(\mathcal{G}(n, c/n)) \geq \frac{1}{3} \log_2 n + O(1)$. □

Let us mention that for $c > 1$, a.a.s. $\mathcal{G}(n, c/n)$ contains a path of length $\Omega(n)$ (see, for example, [1]) and so a.a.s. $\chi_{\text{lin}}(\mathcal{G}(n, c/n)) = \Omega(\log n)$. In fact, the non-existence of linear colouring is clearly a monotonic property so the same bound is implied by the fact that a.a.s. $\chi_{\text{lin}}(\mathcal{G}(n, 1/n)) = \Omega(\log n)$.

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