Competition Highlights Canadian Mathematical Olympiad and Junior Olympiad (CMO/CJMO)

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The Canadian Mathematical Olympiad (CMO) is an annual, invitational, proof-based competition for Canadian students. It is considered to be Canada's premier national advanced mathematics competition. Students attempt to solve 5 problems in three hours, with each problem graded on a scale from 0 to 7. In 2020, the CMS introduced the Canadian Junior Mathematical Olympiad (CJMO), also by invitation only, a variant specifically for students in grade at most 10. These 3-hour competitions are held each March at a selected time and date (by default, the second Thursday of March). All official participants write at the same time and are proctored by their local school faculty or staff. For more information visit `https://cms.math.ca/competitions/cmo/`.

The CMO is an important contest for students with international aspirations, as a good performance leads to the Canadian Team Selection Test, and then onto the International Mathematical Olympiad itself. Qualification for the C(J)MO is primarily via the Canadian Open Mathematics Challenge (COMC), an open contest written in late October.

In total, the 2024 CMO was written by 91 students, with 87 official entrants. The CJMO was written by 21 students, all official entrants. Six Canadian provinces were represented, with the number of contestants as follows:

CMO: AB(7), BC (18), NS (1), ON (39), QC (7), SK (1)
CJMO: BC (7), NS (2), ON (12)

(note that Canadian citizens residing outside of Canada can also officially write the CMO, accounting for the discrepancy in numbers).

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Grading for both contests went relatively smoothly, with a team of 12 mathematicians, including professors, students, and former contestants, contributing their time. The top score on the CMO was 34, achieved by Warren Bei, and the mean score was 14.7. The Matthew Brennan Award for best solution went to Ming Yang for an excellent solution to problem 5. On the CJMO, a perfect score of 35 was achieved by Ryan Li, and the mean score was 18.3. A full breakdown of the marks assigned problem by problem is in Table 1.

Score	P1	P2	P3	P4	P5
7	41	68	22	3	3
6	9	4	12	1	0
5	7	3	4	4	0
4	5	0	2	1	0
3	4	1	3	3	0
2	5	4	4	11	1
1	11	0	0	4	9
0	9	11	44	64	78
Avg	4.71	5.78	2.98	0.95	0.35

(a) CMO

Score	P1	P2	P3	P4	P5
7	13	13	6	8	2
6	0	2	1	1	0
5	0	1	2	0	1
4	0	2	1	0	1
3	0	1	0	0	1
2	6	1	3	2	0
1	0	0	0	0	0
0	2	1	8	10	16
Avg	4.90	5.76	3.24	3.14	1.24

(b) CJMO

Table 1: C(J)MO score breakdown by problem.

An interesting problem that appeared this year was problem 2 on the CMO, which doubled as problem 4 on the CJMO.

Problem 1. *Jane writes down 2024 natural numbers around the perimeter of a circle. She wants the 2024 products of adjacent pairs of numbers to be exactly the set* $\{1!, 2!, \ldots, 2024!\}$*. Can she accomplish this?*

One can easily generalize this problem by replacing 2024 by $N \geq 2$ at each appearance, and a natural place to start is by looking at small values of N. For $N = 2$, both products are necessarily equal. This is not a very interesting observation, and clearly it does not generalize, so go on to $N = 3$. If the numbers are a, b, c in order, the products are ab, bc, ca, which must be $1! = 1, 2! = 2, 3! = 6$ in some order. One can list out all $3! = 6$ pairings of products and solve the algebra, but even if this works (it does!), it would not be easily generalizable to $N = 2024$. Instead, it makes sense to look at some invariants that are preserved, no matter what assignment of products is made.

A natural one is the sum: no matter what, we must have $ab + bc + ca = 1 + 2 + 6 = 9$. While this seems useful at first, it's not clear where to go from here. On the other hand, the *product* is also invariant, and this gives

$$(abc)^2 = (ab)(ac)(bc) = 1 \cdot 2 \cdot 6 = 12.$$

In fact, this allows us to solve for a, b, c! As they must be positive, $abc = \sqrt{12}$, and then from expressions analogous to

$$a = \frac{abc}{bc} = \frac{\sqrt{12}}{1, 2, \text{or } 6}$$

one can solve for a, b, c. In fact, this shows that if Jane wants to write down positive real numbers, she can always do so uniquely, no matter which order she wrote the products in!

However, there is a problem with this: the question requires that all numbers Jane writes are *natural*, and $\sqrt{12} \notin \mathbb{N}$. Indeed, we could have detected this without even solving for a, b, c: the expression $(abc)^2 = 12$ gives a contradiction, as abc is a positive integer and 12 is not a perfect square.

If you try a few more small cases you will quickly see that this fact is more general: if Jane can accomplish the task, then the product of all N numbers must be a perfect square. Indeed, if these numbers in order are $a_1, a_2, \ldots, a_N$, then the products formed are $a_1a_2, a_2a_3, \ldots, a_{N-1}a_N, a_Na_1$, which form the product $(a_1a_2 \cdots a_N)^2$. The following observation immediately follows.

Proposition 2. *If Jane can accomplish the task for some $N > 1$, then $1! \cdot 2! \cdots N!$ is a perfect square.*

If this product is a perfect square, then we do not have a contradiction, but we still do not yet know if Jane can accomplish the task! For example, if $N = 3$ and the products were $1, 3, 12$, then $abc = \sqrt{1 \cdot 3 \cdot 12} = 6$, giving the numbers $6, 2, \frac{1}{2}$, which do *not* work since $\frac{1}{2}$ is not integral. A further complicating observation is that if N is even, then there is no unique solution! For example, if $N = 4$, then replacing a_1, a_2, a_3, a_4 by $2a_1, 0.5a_2, 2a_3, 0.5a_4$ give the same sequence of adjacent products. At this point, we will hope that $1! \cdot 2! \cdots N!$ is never a perfect square, which would avoid this extra analysis.

It turns out that this is true for all $N > 1$. We will finish the problem by giving a few approaches for $N = 2024$, with the second being relatively straightforward to generalize to all $N > 1$.

Proposition 3. *The number $K = 1! \cdot 2! \cdots 2024!$ is not a perfect square.*

Solution 1. A basic way to show that a number x is not a square is to find a prime divisor p such that $p^2 \nmid x$. More generally, consider $v_p(x)$, the p-adic valuation of an integer x, which is the exponent of the highest power of the prime number p that divides x. If $v_p(x)$ is odd, then x is not a perfect square.

For our problem, we know that $p \mid x!$ if and only if $p \leq x$, so choosing a large p will limit the possible $x!$ it can divide, presumably making the analysis easier to consider. For example, $p = 2017$ will divide $2017!, 2018!, \ldots, 2024!$ each once, giving $v_p(K) = 8$, which is even! In fact, for any prime $1012 < p < 2024$, we end up with $v_p(K) = 2025 - p$, which is always even.

Since these large primes will never work, let's try going slightly lower. If we pick p such that $\frac{2024}{3} < p \leq \frac{2024}{2}$, then there is 1 contribution of p from $p!, (p+1)!, \ldots, (2p-1)!$, and 2 contributions from $(2p)!, (2p+1)!, \ldots, 2024!$. This totals to $p + 2(2025 - 2p)$, which is odd! Thus, proving that there is a prime in this range $[675, 1012]$ will suffice. Trial division shows that 677 is prime, and lies in the range required (see below for more on this step). $\square$

This solution contained a proof of special cases of Legendre's formula

$$v_p(x!) = \sum_{i=1}^{\infty} \left\lfloor \frac{x}{p^i} \right\rfloor,$$

which can be used to compute $v_p(K)$ in general.

Another observation is that this proof requires finding a prime p in a certain range. The prime number theorem states that $\pi(n)$, the count of prime numbers at most n, is asymptotic to $\frac{n}{\log n}$, where $\log n$ is the natural logarithm of n. Using this expression we can estimate the number of primes in the interval $[675, 1012]$ to be

$$\frac{1012}{\log 1012} - \frac{674}{\log 674} \approx 43,$$

reasonably close to $\pi(1012) - \pi(674) = 47$. If we are to attempt to find a prime by hand, we could pick sample numbers n from this range, and divide by the prime numbers up to $\sqrt{n} < 32$ to ensure primality.

How quickly can we expect this to work? First, by ensuring that the units digit of n is $1, 3, 7, 9$, we know n is not a multiple of 2 or 5. By ensuring the sum of the digits is not a multiple of 3, we know that n is also not a multiple of 3. This leaves 90 numbers, and more than half of them are prime. These are excellent odds, and we should expect to find a prime within a few guesses only, limiting the manual labour.

If we want to avoid prime computation all together, then we need a result that gives us primes in certain ranges. One of the most famous results in this area is *Bertrand's postulate*, which states that for all integers $n > 1$, there exists a prime p with $n < p < 2n$. Bertram conjectured this result in 1845 and proved it for all $n \leq 3,000,000$, but a full proof would have to wait for Chebyshev, who accomplished it in 1852.

Solution 2. Before trying the approach in Solution 1, we can factor out square factors from K to make our number smaller. One natural way to do this is note that $x!(x+1)! = (x+1) \cdot (x!)^2$. Doing this trick for all odd x up to 2023 yields

$$\begin{aligned} K &= 2 \cdot 4 \cdots 2024 \cdot (1! \cdot 3! \cdots 2023!)^2 \\ &= (2 \cdot 1)(2 \cdot 2) \cdots (2 \cdot 1012)(1! \cdot 3! \cdots 2023!)^2 \\ &= 2^{1012} \cdot 1012! \cdot (1! \cdot 3! \cdots 2023!)^2 \\ &= 1012!(2^{506} 1! \cdot 3! \cdots 2023!)^2. \end{aligned}$$

Thus, K is a perfect square if and only if $1012!$ is a perfect square. Bertrand's postulate implies that there is a prime p with $\frac{1012}{2} < p < 1012$, and such a p satisfies $v_p(1012!) = 1$, showing that $1012!$ is not a perfect square. $\square$