Quantum Mechanics in a Two Dimensional Spacetime: What is a Wavefunction?

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Abstract

Conventional quantum mechanics specifies the mathematical properties of wavefunctions and relates them to physical experiments by invoking the Born postulate. There is no known direct connection between wavefunctions and any external physical object. However, in the case of a two dimensional spacetime there is a completely classical context for wavefunctions where the connection with an external reality is transparent and unambiguous. By examining this case, we show how a classical stochastic process assembles a Dirac wavefunction based solely on the detailed counting of reversible paths. A direct comparison of how a related process assembles a Probability Density Function reveals both how and why PDFs and wavefunctions differ, including the ubiquitous implication of complex numbers for the latter. The appearance of wavefunctions in a context that is free of the complexities of quantum mechanics suggests the study of such models may shed some light on interpretive issues.

Key words: quantum mechanics, special relativity, stochastic processes, Feynman chessboard

1. Introduction

The empirical accuracy of quantum mechanics makes the theory unsurpassed in the history of science. Despite this, there continue to be aspects of the theory that many scientists find controversial[1, 2]. The practically universal agreement that *quantum mechanics provides a superb probability*

	Classical	**Quantum**
Kinetic 'picture'	Kac (Poisson)	Chessboard
Telegraph/Dirac	$\frac{\partial \mathbf{U}}{\partial t} = \boldsymbol{\sigma}_z \frac{\partial U}{\partial z} + a\,\boldsymbol{\sigma}_x U$	$\frac{\partial \Psi}{\partial t} = \boldsymbol{\sigma}_z \frac{\partial \Psi}{\partial z} + i\,m\,\boldsymbol{\sigma}_x \Psi$
Telegraph/KG	$\frac{\partial^2 U}{\partial t^2} = \frac{\partial^2 U}{\partial z^2} + a^2 U$	$\frac{\partial^2 \psi}{\partial t^2} = \frac{\partial^2 \psi}{\partial z^2} + (i\,m)^2 \psi$
Heat/Schrödinger	$\frac{\partial U}{\partial t} = D\,\frac{\partial^2 U}{\partial x^2}$	$\frac{\partial \psi}{\partial t} = i\,D\,\frac{\partial^2 \psi}{\partial x^2}$
Characteristic	Bernoulli	Anti-Bernoulli
Random Variable	$X \in \{1, 0\}$	$Y \in \{1, 0, -1\}$

Table 1: Three sets of partial differential equations are compared. The left column contains phenomenological equations that have a basis in Kinetic theory. The PDF solutions are expected values of sums of the Bernoulli random variable. The right column contains 'quantum' equations obtained from the classical equations through a formal analytic continuation. We show that these equations also have a kinetic theory basis in which the solutions are expected values of sums of the Anti-Bernoulli random variable.

calculus does not extend to questions involving the theory's interpretation. Opinions on interpretive issues cover a large spectrum.

Compare this situation with that of classical statistical mechanics and diffusion. In Table(1) six Partial Differential Equations are listed. In the centre column are three classical PDEs that describe diffusive processes. Their solutions are typically probability density functions that are obtained by counting paths with a random variable that detects the presence or absence of a path. The random variable X in this case is Bernoulli:

$$X = \begin{cases} 1 & \text{path link present} \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

the stochastic version of an indicator function. Ultimately the PDF solutions are continuum limits of the expected values of sums of the Bernoulli random variable. That is:

$$U(x,t) = E[\sum_{\text{Path Ensemble}} X]. \tag{2}$$

The solutions can be treated as probability densities since the sums of the Bernoulli random variable are non-negative and continuity of the paths allow U to be normalizable as a PDF in the continuum limit. Indeed, (2) is an expression of the frequency-based picture of probability.

On the right of the table we see respectively the Dirac, Klein-Gordon and Schrödinger equations. Each of these may be obtained from the classi-

cal equation in the same row by the conversion of a single real constant to an imaginary constant. However the solutions of these equations are wavefunctions, not PDFs. The presence of the imaginary constant removes the solutions from the domain of functions that would satisfy the properties of a PDF. Although the classical equations on the left are ultimately phenomenological with a basis in kinetic theory, the equations on the right are regarded as fundamental with no prior basis in an underlying microscopic model.

Comparing the two sets of equations, we understand the classical equations well enough to see how they arise from elementary properties of small classical particles in random motion. If we ask the question "What is a probability density function?" in the context of the solutions of these equations, we get a precise answer that is transparent with little need for 'interpretation'.

The purpose of this article is to show that we can do the same for the quantum equations and associated wavefunctions in a two dimensional spacetime provided particle paths treat both dimensions as spacelike in a manner that we shall describe shortly. The result is interesting in a number of ways. The model we discuss provides a simple classical model that can, in principle, be used to quantitatively simulate single-particle quantum mechanics in one dimension. Hints that this may be extended to three dimensions exist[3, 4] and will be confirmed in a future work. In addition to this, wavefunctions appear here as natural generalizations of PDFs to include counting processes for reversing paths. As such they may be studied as stochastic processes independently of their context in quantum mechanics. Finally, by comparing the classical and quantum contexts we anticipate that interpretative issues about quantum mechanics[1] may be brought into sharp focus.

2. Kac's model and PDFs

We begin by reviewing a version of Kac's model for the Telegraph equations[5]. Consider walks taking place on a lattice in the $x-y$ plane. Particles traverse diagonal links on the lattice moving in the $+y$-direction at each step. At the end of each step they decide to continue in the same direction, or occasionally to switch x-direction with a small probability $m\epsilon < 1$ where ϵ is the lattice spacing and m is a positive constant. We can imagine that at each diagonal lattice link there is a bin that collects the contribution of each path to the link. Thus at any fixed value of y on the lattice, any given path contributes a 1 to one particular bin, and a 0 to all the rest. The bin counts are then just sums of Bernoulli random variables as in (1). Since there are two types

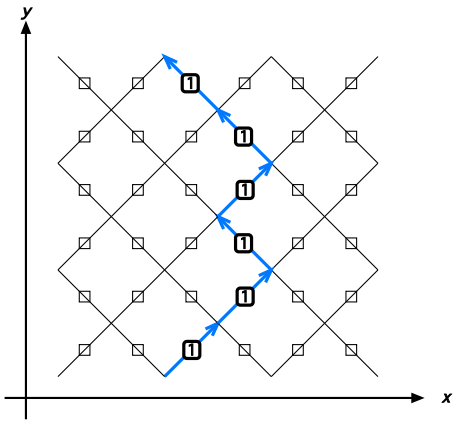

Figure 1: A walk in the $x-y$ plane for Kac's model. Walks are along diagonal links that are traversed in the $+y$ direction. Each link is counted by a Bernoulli random variable that is 1 if the bond is present or 0 if it is not.

of links with orientation parallel (+) and anti-parallel (-) to the x-axis, we consider two densities, $u_{\pm}$. These will be functions of both x and y. For simplicity let us first ignore the x-dependence and define:

$$W_{\pm}(y) = \sum_{\forall x} u_{\pm}(x,y) \tag{3}$$

At each step, the expected fraction of walkers that change state is just $m\epsilon$ so $W_{\pm}$ must obey the difference equations:

$$\begin{aligned} W_+(y+\epsilon) &= (1-\epsilon m)\, W_+(y) + \epsilon m\, W_-(y) \\ W_-(y+\epsilon) &= (1-\epsilon m)\, W_-(y) + \epsilon m\, W_+(y) \end{aligned} \tag{4}$$

These equations just express the conservation of particle number in the ensemble. They may be solved exactly, although our interest is primarily in the continuum limit, $\epsilon \to 0$. In that limit we may replace (4) by the differential equations:

$$\begin{aligned} \frac{dW_+}{dy} &= -m\, W_+ + m\, W_- \\ \frac{dW_-}{dy} &= -m\, W_- + m\, W_+ \end{aligned} \tag{5}$$

or

$$\frac{d\mathbf{W}}{dy} = -m\,\mathbf{W} + m\,\sigma_{\mathbf{x}}\mathbf{W} \tag{6}$$

where $\sigma_{\mathbf{x}}$ is the first Pauli matrix. If we start the system with the initial condition $W_+(0) = 1, \quad W_-(0) = 0$ we find that the result is the two component density:

$$\mathbf{W}(y) = e^{-my}\begin{pmatrix}\cosh(my)\\ \sinh(my)\end{pmatrix}. \tag{7}$$

Note that the sum of the two components is unity and the initial distribution of all paths oriented in the $+x$ direction rapidly decays to the equilibrium situation of equal occupancy. Equation(7) represents the expected value of normalized sums of the path-counting Bernoulli random variable (1). We see that the result is a y-dependent probability mass function between the two states.

The number of paths to each point on the lattice from the origin may be calculated exactly but, for our purposes, it will suffice to write down the conservation of particle number taking into account dependence on x and then take the continuum limit. Examination of paths on the lattice readily leads to:

$$\begin{aligned} u_+(x, y+\epsilon) &= (1-\epsilon m)\,u_+(x-\epsilon, y) + \epsilon m\,u_-(x, y) \\ u_-(x-\epsilon, y+\epsilon) &= (1-\epsilon m)\,u_-(x, y) + \epsilon m\,u_+(x-\epsilon, y) \end{aligned} \tag{8}$$

To lowest order in ϵ we see that:

$$\begin{aligned} \frac{\partial u_+}{\partial y} &= \frac{\partial u_+}{\partial x} - m\,u_+ + m\,u_- \\ \frac{\partial u_-}{\partial y} &= -\frac{\partial u_-}{\partial x} - m\,u_- + m\,u_+ \end{aligned} \tag{9}$$

For suitable initial conditions the solutions are normalized densities that satisfy the requirements of a two component probability density function. To make contact with the first PDE in Table(1) let

$$\mathbf{U}(x, y) = e^{my}\begin{pmatrix} u_+ \\ u_- \end{pmatrix} \tag{10}$$

then(9) is

$$\frac{\partial \mathbf{U}}{\partial y} = \sigma_{\mathbf{z}}\frac{\partial \mathbf{U}}{\partial x} + m\,\sigma_{\mathbf{x}}\mathbf{U} \tag{11}$$

If we associate y with a time variable, (11) is a form of the Telegraph equations. As in the simpler probability mass function case(5), the density arises from the expected value of sums of the Bernoulli random variable.

3. The Entwined Path Model

The probability density functions that arise in the Kac model are expected values of the sums of Bernoulli random variables. The role of the Bernoulli random variable is simply to identify whether the path passes through a link or not. This leads to a correct counting of paths provided all paths are continuous and intersect lines of constant y at a single point. Paths that double back and intersect lines of constant y multiple times are not correctly counted by simply using a Bernoulli random variable. However, as illustrated in Figure(2), paths with reversed links may be counted using a modified path-recognition random variable.

Consider the anti-symmetric random variable:

$$Y = \begin{cases} 1 & \text{path link traversed in the } +y \text{ direction.} \\ 0 & \text{link not traversed} \\ -1 & \text{path link traversed in the } -y \text{ direction.} \end{cases} \tag{12}$$

The 'Anti-Bernoulli' random variable extends path recognition to take into account paths that can be traversed in either direction with respect to the y-axis. Provided paths are continuous and eventually traverse the region of interest, the Anti-Bernoulli random variable will correctly count the number of continuous paths that cross a planar region.

We now consider a version of the Kac model that employs a reversing walk over the $x - y$ plane[6]. The walk starts at the origin, proceeds in the $+y$ direction past some maximum value of y and then returns to the origin along a route predetermined by the walk in the $+y$ direction. The walker follows the same instructions as for the Kac walk except at every other call for a direction change, the walker drops a marker but keeps going in the same direction. The return path then follows the markers back to the origin. This is illustrated on the left side of Figure(3). Each path out to the outer boundary and back forms a chain of oriented rectangles, the orientation changing at each crossing point. If we consider the right-hand boundary of this chain of rectangles, the Anti-Bernoulli weight for each link records the orientation as either positive or negative. We call the right-hand

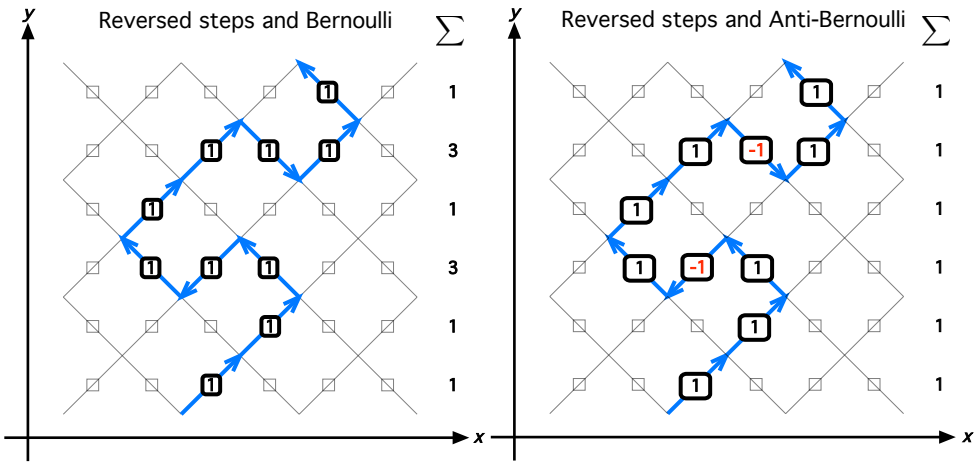

Figure 2: Walks in the $x-y$ plane that have links that may be traversed in any of the four possible directions are not correctly counted by the Bernoulli random variable. Note the variable sum over x of the Bernoulli random variable on the left. The sum sometimes counts one path and sometimes counts three. The Anti-Bernouli random variable on the right allows for reversed traversal and the sum over x correctly counts the single path.

boundary the 'enumerative path'. We shall use this to count contributions to the 'vorticity' of the areas enclosed by the paired paths. The enumerative paths have exactly the same statistics as the Kac paths of the previous model. The inter-corner intervals are determined in the same way; only the counting is different, allowing as it does for both forward and return path segments.

Let us proceed as we did for the original Kac walks. We initially neglect the x-dependence of the density we wish to find. That is we write

$$\Phi_{\pm}(y) = \sum_{\forall x} \phi_{\pm}(x,y) \tag{13}$$

where $\phi_{\pm}(x,y)$ is the expected value of the sums of the Anti-Bernoulli random variable at (x,y). At each step the expected fraction of walkers that change state is just $m\epsilon$ so $\Phi_{\pm}$ must obey the difference equations:

$$\begin{aligned} \Phi_{+}(y+\epsilon) &= (1-\epsilon m)\,\Phi_{+}(y) - \epsilon m\,\Phi_{-}(y) \\ \Phi_{-}(y+\epsilon) &= (1-\epsilon m)\,\Phi_{-}(y) + \epsilon m\,\Phi_{+}(y) \end{aligned} \tag{14}$$

Note the minus sign in the scattering term of the first equation. This is because Φ_{+} paths result only from a continuation of Φ_{+} paths or from the

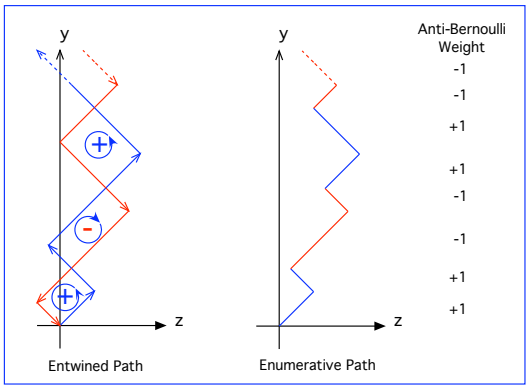

Figure 3: Entwined walks in the $x-y$ plane. The walk on the left goes out to the outer boundary at some maximum value of y and then returns to the origin via a series of markers dropped on the way out. The sequence of rectangles formed by the forward and return paths alternate orientation and the right hand boundary, called the enumerative path, alternates between the two non-zero value of the Anti-Bernoulli random variable every second 'corner'.

scattering of Φ_- paths with the opposite value of the Anti-Bernoulli random variable. The scattering term in the second equation has a positive sign because this scattering does not change the sign of the Anti-Bernoulli random variable.

Equation(14) finds the expected value of the Anti-Bernoulli random variable over the ensemble. Notice that since the Entwined path on the left of Fig.(3) returns to the origin, the whole ensemble of paths can be traversed by a single particle. The ensemble of paths in the $x-y$ plane can be concatenated into a single path! The continuum limit, $\epsilon \to 0$ of equation(14) is easily taken. In that limit we may replace (4) by the differential equations:

$$\begin{aligned} \frac{d\Phi_+}{dy} &= -m\,\Phi_+ - m\,\Phi_- \\ \frac{d\Phi_-}{dy} &= -m\,\Phi_- + m\,\Phi_+ \end{aligned} \tag{15}$$

or

$$\frac{d\mathbf{\Phi}}{dy} = -m\,\mathbf{\Phi} - m\,\mathbf{i}\,\sigma_{\mathbf{y}}\mathbf{\Phi} \tag{16}$$

where $-\mathbf{i}\,\sigma_{\mathbf{y}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is a real anti-symmetric matrix. If we start the

system with the initial condition $\mathbf{\Phi}_+(0) = 1, \quad \mathbf{\Phi}_-(0) = 0$ we find that the result is the two component state:

$$\mathbf{\Phi}(y) = e^{-my} \begin{pmatrix} \cos(my) \\ \sin(my) \end{pmatrix}. \tag{17}$$

Comparing this result with (7) we see that we no longer have the two components of a probability mass function. On the other hand, apart from the exponential decay, the two components can be thought of as two components of a vector that rotates with constant angular speed as a function of y! The effect of counting with the Anti-Bernoulli random variable has been, in part, to allow the stochastic process to interpolate between the four directions giving a vector that rotates with a continuous phase in the continuum limit.

As in the previous case, the expected number of paths to each point on the lattice from the origin may be calculated exactly with the appropriate Anti-Bernoulli weight, but for our purposes it is easiest to write down the expected net particle number, taking into account dependence on x and then take the continuum limit. Examination of paths on the lattice readily leads to:

$$\begin{aligned} \phi_+(x, y+\epsilon) &= (1-\epsilon m)\,\phi_+(x-\epsilon, y) - \epsilon m\,\phi_-(x, y) \\ \phi_-(x-\epsilon, y+\epsilon) &= (1-\epsilon m)\,\phi_-(x, y) + \epsilon m\,\phi_+(x-\epsilon, y) \end{aligned} \tag{18}$$

To lowest order in ϵ we see that:

$$\begin{aligned} \frac{\partial \phi_+}{\partial y} &= \frac{\partial \phi_+}{\partial x} - m\,\phi_+ - m\,\phi_- \\ \frac{\partial \phi_-}{\partial y} &= -\frac{\partial \phi_-}{\partial x} - m\,\phi_- + m\,\phi_+ \end{aligned} \tag{19}$$

If we write $\phi_\pm = e^{-mt}\psi_\pm$, $\Psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}$, (19) becomes:

$$\frac{\partial \Psi}{\partial y} = \sigma_{\mathbf{z}} \frac{\partial \Psi}{\partial x} - im\sigma_{\mathbf{y}}\Psi \tag{20}$$

Notice this is a form of the Dirac equation (where $\hbar = c = 1$) where we regard y as the time variable. The derivation of equation(20) did not require a formal analytic continuation or an appeal to anything other than the use of

a characteristic random variable that was required to correctly count paths that could double back along the y-direction. Although the components of Ψ are real in equation (20), standard non-relativistic approximations take us through the Klein-Gordon equation to the Schrödinger equation where the phase of the wavefunction is essentially a frequency modulation of the high-frequency oscillation of (20) at $\nu = m$[7].

From our derivation of (20) it is clear that the relation to kinetic theory is very direct. The ensemble of paths that gives rise to the equation from the single source at the origin may be concatenated to form a single path. As a result, the ergodic hypothesis is not necessary here. In principle we can simulate any solution of (20) on a lattice by constructing a long path and counting contributions using the Anti-Bernoulli random variable. In practice the number of walk configurations goes up exponentially in y and makes such simulations impractical for anything but small values of y. Although the Telegraph equations have the same problem with the configuration space becoming large, in practice, simulations are much less sensitive to sparse coverage of configuration space. However, as has been shown elsewhere[8, 9], (20) has the advantage that large-y simulations of solutions may be accomplished using stochastic initial conditions but deterministic propagation. This allows one to circumvent the problem of simulation times that are of exponential order.

4. Discussion

In the Introduction we noted that the wavefunction in quantum mechanics is both fundamental and unrelated to any known underlying kinetic theory. This is in contrast to the PDF solutions of similar equations in classical statistical mechanics. By examining how classical densities are constructed through counting paths, we see that PDFs are *expected values of sums of Bernoulli random variables.* Noting that the Bernoulli random variable is unable to correctly count reversing paths we replace it with the 'Anti-Bernoulli' random variable Y of Eqn.(12). The relevant density is then:

$$\psi(x,t) = E[\sum_{\text{Path Ensemble}} Y]. \tag{21}$$

the generalization of equation (2) to counting reversible paths.

Using the new random variable to count Entwined Paths in the $x-y$ plane gives us densities that satisfy the Dirac equation. *No formal analytic*

continuation or interpretational ambiguities are encountered in this context for wavefunctions. Wavefunctions of the Entwined Path model have a kinetic theory basis that is as transparent and complete as the Kac-model basis for the Telegraph equations. Features like phase and unitary propagation that are often considered harbingers of quantum mechanics, are here features of ensemble averages that count reversible paths.

The fixed ontology of the Entwined Path model illustrates why the use of wavefunctions in the *quantum context* can produce paradoxical results. The derivation of Dirac wavefunctions assumes that the paths themselves have a free large-y boundary (see Fig. 3), *thus large-y information is encoded into the wavefunction itself.* Wavefunctions then evolve the initial conditions at $y = 0$ subject to a stochastic process that must be undisturbed out to values of y outside of the region of interest. In the quantum context, y is 'observer time' and the the implication is that wavefunctions encode some 'future' information and should evolve from the initial conditions according to the Dirac equation *provided there is no external interference in the future.*

This suggest a form of retro-causality reminiscent of the 'Absorber Theory of Radiation' program of Wheeler and Feynman[10, 11]. Conventional quantum mechanics does not assume that wavefunctions encode a 'free future' boundary condition as they do automatically in the Entwined Path model. Instead, unitary propagation ends at observation with a collapse phenomenon. However, it is precisely here that interpretations of quantum mechanics differ greatly and comparison with stochastic models may be beneficial.

The association of quantum mechanics with 'reversible diffusion' was originally suggested by Schrödinger in 1931[13] and has appeared in many different forms ever since[1] so a qualitative connection between Dirac wavefunctions and reversible paths not unexpected. The surprise is the simplicity of the underlying stochastic model. Essentially no physics and little mathematics beyond elementary counting arguments are needed to arrive at the Dirac equation from Entwined paths. Wavefunctions arise as a natural consequence of extending path counting to include reversible paths. How far the simplicity of this picture holds into higher dimension and multi-particle cases remains an interesting question for further investigation.

[1]The deBroglie-Bohm picture[14] is an example as are Nelson related views [15]-[20]

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