RYERSON UNIVERSITY
MTH 714
ASSIGNMENT #2 - SOLUTIONS

1. (a) The formula is not valid. We need to show that its negation, which is equivalent to

$$\forall x \exists y[p(x,y) \wedge \neg p(y,x) \wedge \neg(p(x,x) \leftrightarrow p(y,y))]$$

is satisfiable.

If we attempt to construct a semantic tableau for this negation and label this negation as (1), we get the following node after applying a γ-rule:

$$(2) \qquad \forall x \exists y[p(x,y) \wedge \neg p(y,x) \wedge \neg(p(x,x) \leftrightarrow p(y,y))], \quad \exists y[p(a,y) \wedge \neg p(y,a) \wedge \neg(p(a,a) \leftrightarrow p(y,y))]$$

and, after applying a δ-rule, we get the node

$$(3) \qquad \forall x \exists y[p(x,y) \wedge \neg p(y,x) \wedge \neg(p(x,x) \leftrightarrow p(y,y))], \quad p(a,b) \wedge \neg p(b,a) \wedge \neg(p(a,a) \leftrightarrow p(b,b))$$

After this, the only rule that can be applied is γ which would introduce new constant symbols in place of x. In any case, the branch is going to be infinite and we see that if the second formula in the node (3) can be made true in some interpretation, that would provide us with a model in which the formula is satisfied. In fact, the second formula in (3) will give us the smallest model in which the original formula (without $\neg$) is false:

$$I = (\{a,b\}, p)$$

where the binary relation is given as follows:

$$p(a,a) = T, \qquad p(a,b) = T, \qquad p(b,a) = F, \qquad p(b,b) = F$$

[It is also possible to use the binary relation p where $p(a,a) = F$, $p(a,b) = T$, $p(b,a) = F$, $p(b,b) = T$.]

Remark: It is possible to find a more "natural" model without using a semantic tableaux. One such model is

$$I_2 = (\mathbb{N}, p)$$

where the binary relation p is defined in the following way:

$$p(x,y) \text{ is true if and only if } \lfloor \sqrt{x} \rfloor^2 \leq y$$

and $\lfloor x \rfloor$ denotes the largest integer less than x. (This is known as the "floor" function on integers). The verification that this interpretation makes the negation true is more complicated and that solution will be posted later.

(b) This formula is not valid. To see that, use the interpretation

$$I = (\mathbb{Z}, \leq).$$

2. (a) The initial node is the negation of the original formula:

$$(1) \qquad \neg[\exists x(B(x) \rightarrow C(x)) \rightarrow (\forall x B(x) \rightarrow \exists x C(x))]$$

Using an α-rule, we get a single descendant:

$$(2) \qquad \exists x(B(x) \rightarrow C(x)), \neg(\forall x B(x) \rightarrow \exists x C(x))$$

Applying an α-rule again, we get a single descendant of (2):

$$(3) \qquad \exists x(B(x) \rightarrow C(x)), \forall x B(x), \neg\exists x C(x)$$

Next, we apply a δ-rule, which eliminates the first existential quantifier in (3):

$$(4) \qquad B(a) \rightarrow C(a), \forall x B(x), \neg\exists x C(x)$$

Using the γ-rule on the second formula in (4) and replacing x with a, we get:

$$(5) \qquad B(a) \rightarrow C(a), B(a), \forall x B(x), \neg\exists x C(x)$$

Now, apply the γ-rule again, this time on $\neg\exists x C(x)$ to get

$$(6) \qquad B(a) \rightarrow C(a), B(a), \neg C(a), \forall x B(x), \neg\exists x C(x)$$

Using a β-rule, we see that the previous node has two descendants:

$$(7) \qquad \neg B(a), B(a), \neg C(a), \forall x B(x), \neg\exists x C(x)$$

and

$$(8) \qquad C(a), B(a), \neg C(a), \forall x B(x), \neg\exists x C(x)$$

Both leaves (7) and (8) are marked as closed, so the formula must be valid, since its negation is unsatisfiable.

(b)

1.	$\{\exists x(B(x) \to C(x)), \forall x B(x)\} \vdash \exists x(B(x) \to C(x))$	assumption
2.	$\{\exists x(B(x) \to C(x)), \forall x B(x)\} \vdash \forall x B(x)$	assumption
3.	$\{\exists x(B(x) \to C(x)), \forall x B(x)\} \vdash B(a) \to C(a)$	C-Rule on 1
4.	$\{\exists x(B(x) \to C(x)), \forall x B(x)\} \vdash B(a)$	Axiom 4 on 2 (Specification)
5.	$\{\exists x(B(x) \to C(x)), \forall x B(x)\} \vdash C(a)$	MP 4,3
6.	$\{\exists x(B(x) \to C(x)), \forall x B(x)\} \vdash \exists x C(x)$	Theorem 1 on 5
7.	$\{\exists x(B(x) \to C(x))\} \vdash \forall x B(x) \to \exists x C(x)$	Ded. Rule on 6
8.	$\vdash \exists x(B(x) \to C(x)) \to (\forall x B(x) \to \exists x C(x))$	Ded. Rule on 7

3.

$$\begin{aligned}
&\forall z \exists y(P(y, g(y), z) \vee \neg \forall x Q(x)) \wedge \neg \forall z \exists x \neg R(f(x, z), z) \\
&\equiv \forall z \exists y(P(y, g(y), z) \vee \neg \forall x Q(x)) \wedge \neg \forall w \exists u \neg R(f(u, w), w) \\
&\equiv \forall z \exists y \exists x(P(y, g(y), z) \vee \neg Q(x)) \wedge \exists w \forall u R(f(u, w), w) \\
&\equiv \forall z \exists y \exists x \exists w \forall u[(P(y, g(y), z) \vee \neg Q(x)) \wedge R(f(u, w), w)] \\
&\approx \forall z \forall u[(P(i(z), g(i(z)), z) \vee \neg Q(j(z))) \wedge R(f(u, k(z)), k(z))]
\end{aligned}$$

where i, j, and k are new unary function symbols such that

$$y = i(z), \quad x = j(z), \quad w = k(z)$$

The clausal form is

$$\{P(i(z), g(i(z)), z) \neg Q(j(z)), R(f(u, k(z)), k(z))\}$$

4. We will solve the problem by using the equational approach (the problem can also be solved using Robinson's Unification Algorithm):

$$\begin{aligned}
f(z, g(a, y)) &= f(f(u, v), w) \\
h(z) &= h(f(a, b))
\end{aligned}$$

By equating the components inside each equation, we get:

$$\begin{aligned}
z &= f(u, v) \\
g(a, y) &= w \\
z &= f(a, b)
\end{aligned}$$

We reverse the order in the second equation:

$$
\begin{aligned}
z &= f(u, v) \\
w &= g(a, y) \\
z &= f(a, b)
\end{aligned}
$$

From the first and the third equation, we get:

$$
\begin{aligned}
f(u, v) &= f(a, b) \\
w &= g(a, y) \\
z &= f(a, b)
\end{aligned}
$$

Again, if we equate the corresponding terms in the first equation, we get:

$$
\begin{aligned}
u &= a \\
v &= b \\
w &= g(a, y) \\
z &= f(a, b)
\end{aligned}
$$

The system is now in a solved form, so an m.g.u. is

$$\mu = \{u \leftarrow a, v \leftarrow b, w \leftarrow g(a, y), z \leftarrow f(a, b)\}$$

5. Label the clauses in the set as

$$\{(1)\{\neg P(x), \neg P(f(a)), Q(y)\}, (2)\{P(y)\}, (3)\{\neg P(g(b, x)), \neg Q(b)\}\}$$

First, we unify literals $Q(y)$ and $Q(b)$ using the substitution $\sigma_1 = \{y \leftarrow b\}$ and apply resolution on clauses (1) and (3) to get:

$$(4) \qquad \{\neg P(x), \neg P(f(a)), \neg P(g(b, x))\}$$

Next, unify the literals $P(y)$ and $P(g(b, x))$ using $\sigma_2 = \{y \leftarrow g(b, x)\}$ and resolve the clauses (2) and (4):

$$(5) \qquad \{\neg P(x), \neg P(f(a))\}$$

Then, we unify $P(y)$ and $P(f(a))$ using $\sigma_3 = \{y \leftarrow f(a)\}$ after which we resolve (2) and (5):

$$(6) \qquad \{\neg P(x)\}$$

Finally, using the unifying substitution $\sigma_4 = \{y \leftarrow x\}$, we unify (2) and (6) to get the empty clause. Therefore, the given set of clauses is not satisfiable.