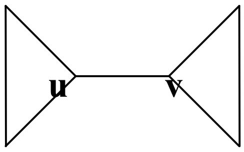
Section 5.1

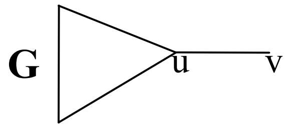
5.1
a) P_n has n − 2 cut vertices and n − 1 bridges.

b) From theorem 5.1 we know that both end points of a bridge are cut vertices if they have degrees ≥ 2. Since we want a graph with more cut vertices than bridges, we may try to draw one using this property. Here is a very simple one with one bridge (the edge uv) and two cut vertices (u and v) In this case both u and v have degree 3.



5.2
a) A k-star

b) G –u has two components while G – u – v has only one.



5.6
This is an if and only if statement so it has two directions:

1. A 3-regular graph G has a cut vertex if it has a bridge.

Proof: This is easy. Both end points of a bridge of a 3 regular graph have degree 3 so they must be cut vertices from theorem 5.1

2. A 3-regular graph G has a bridge if it has a cut vertex

The solution consists of two steps. In the first step we prove a more general observation which is also of independent interest.

Claim: If G be a connected graph, v is a cut vertex of G, then the vertices adjacent to v in G cannot all belong to the same component of G − v.

Proof of claim: Let $a_{1,}\, a_2 \ldots a_n$ be all the (distinct) vertices adjacent to the cut vertex v.

Suppose the claim is not true (so $a_{1,}\, a_2 \ldots a_n \ldots$ all belong to the same component of G − v)

Let x, y be two vertices of $G - v$. Since G is connected, there is a path α in G joining x to y. If α passes through v, it must enter v via one of the edges incident with it, say va_i, and exit v through a different edge va_j (i and j are different because α is a path so it cannot visit the same vertex twice) The vertices v, a_i, a_j appear in α in the order a_i, v, a_j. Since a_i and a_j lie in the same component of $G - v$, there is a path γ in $G - v$ connecting them. We can replace the vertex sequence a_i, v, a_j in α with a_i, γ, a_j to get a path α' connecting x and y. Note that γ does not pass through v. Since α is a path, it can only visit v once, there is no other occurrence of v in α. It follows that α' is a path connecting x and y without passing through v, *hence α' is a path in $G - v$.*

We showed that any two points x and y in $G - v$ can be connected by a path in $G - v$. Therefore $G - v$ is connected. This contradicts the assumption that v is a cut vertex.

We conclude that $a_1, a_2 \ldots a_n$ cannot all lie in the same component of $G - v$, proving the claim.

Now we prove that if a connected, 3-regular graph has a cut vertex it must have a bridge.

Proof: Let v be a cut vertex of G. Since G is 3 regular v has exactly 3 neighbours (i.e. vertices adjacent to v) a, b and c.

From step 1, the vertices a, b and c must belong to more than one components of $G - v$. There are only two possibilities: 1) two vertices belong to same component and the remaining vertex belongs to a different component or 2) the three vertices belong to three different components. In either case there is a vertex which is “singled out” in that it does not belong in the same component of $G - v$ with either of the remaining two vertices. Without loss of generality, let a be a “singled out” vertex, *so there is no path in $G - v$ connecting it to either b or c.*

We claim the edge $e = va$ is a bridge for G. We prove this by contradiction. If e was not a bridge, then $G - e$ would be connected. It follows that there is a path γ in $G - e$ connecting a and v. Since b and c are the only vertices adjacent to v in $G - e$, γ must reach v via either vb or vc. So the path γ' obtained from γ by removing its last edge (either vb or vc) is a path in $G - e$ connecting a to b or c. Since γ is a path it can visit v only once, it follows that γ' does not pass through v. In other words, *γ' is a path in $G - v$.* We showed that a can be connected to b or c via a path in $G - e$. This is a contradiction because a is a “singled out” vertex.

We conclude that e is a bridge.

Finally, if G is not connected, we can restrict our attention to the connected component where the cut vertex belongs and apply the above result.

Remarks:

1. Since the only connected 2-regular graphs are C_n (you can prove this by induction on the order of the graph) , they have neither cut vertices nor bridges, the assertion that a 2 regular graph has a cut vertex if and only if it has a bridge holds trivially.

2. In the proof above, the assumption that G is 3 regular was only used to assert that the cut vertex has exactly three neighbours. So we have actually proved something more general: if a cut vertex of a connected graph has degree 2 or 3, then at least one of the edges that incident the cut vertex is a bridge (if it has degree 2, then both edges incident to it are bridges)

3. To see why the proof fails if the cut vertex has degree > 3, try to carry out the same argument if v has four neighbours a, b, c, d. It won't work because these four vertices can belong to two components of $G - v$ in such a way that two vertices belong to each component so that no vertex is "singled out". For example the vertices a, d belong to one component and b, c belong to another. If we try to show that va is a bridge by arguing as above, we wouldn't be able to assert that there is a path leading from a to b or c to obtain a contradiction because we may find a path connecting a to v via the edge vd. Since a and d belong to the same component of $G - v$, there is no contradiction that we can find a path in $G - v$ connecting them.

Section 5.3

5.21

b) No such graph because every k connected graph is j connected for $j < k$.

d) No such graph because every k edge connected graph is j edge connected for $j < k$.