MTH 607 Midterm Solutions

1.

(a) Order: 8, Size: 10, Total Degree: 20, Degree Sequence: 4, 4, 2, 2, 2, 2, 2, 2.

(b) $\kappa(G) = 1$, $\lambda(G) = 2$, $\delta(G) = 2$, $\Delta(G) = 4$.

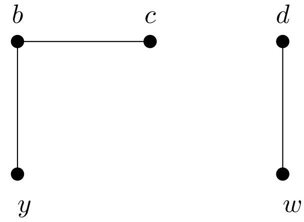
(c) **Adjacency table**

a	b, x
b	a, c, y, z
c	b, z
d	w, z
x	a, y
y	b, x
z	b, c, d, w
w	d, z

(d) Spanning tree not shown, there are many.

(e) z is an articulation point.
There are no bridges as G is 2−connected.

(f) Induced Subgraph

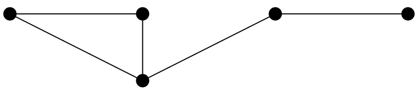


(g) $\{ab, xy, dz\}$

(h) Not bipartite - contains a C_3, bcz for example

(i) Not shown, there are many.

2. For each of the following either explain why the specified graph cannot exist (quote any theorems you use), or draw a graph with the given property.

(a) 

(b) DNE. There are an odd number of vertices of odd degree.

(c) DNE. Whitney's 1st theorem says $\kappa \le \lambda$.

(d) P_4

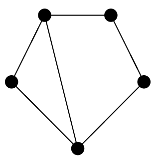
(e) DNE. Every non trivial tree has at least one leaf, height 2 implies at least 3 vertices.

(f) DNE. $C_3 \subseteq K_4 \subseteq G$, but no bipartite graph may contain an odd cycle.

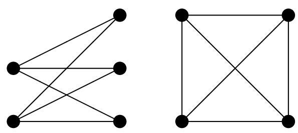
(g) DNE. 2-connected means $\kappa = 2$, if there is a bridge $\lambda = 1$. Whitney's 1st theorem says $\kappa \le \lambda$.

(h) DNE. For any pair of vertices, u and v of such a graph $d(u) + d(v) = n - 2 + n - 2 = 2n - 4$. But for $n \ge 4$ $2n - 4 \ge n - 1$, so $d(u) + d(v) \ge n - 1$ and so the graph is connected.

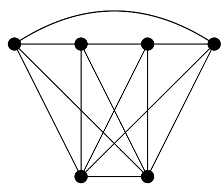
3.



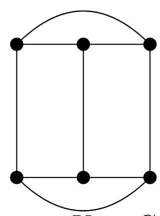
$\overline{P_5}$



$K_{2,3} \cup K_4$

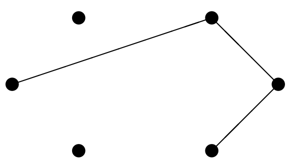


$C_4 + K_2$



$K_2 \times C_3$

4. (a)

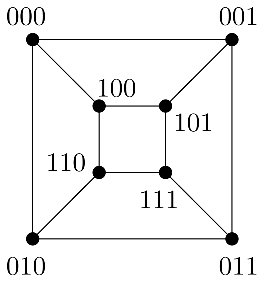


co−T

(b) $\{e_1, e_8\}$, $\{e_2, e_4, e_8\}$, $\{e_3, e_7\}$ $\{e_5, e_7, e_8\}$ $\{e_6, e_8\}$

(c) Take the ring sum of any two of the above. eg. $\{e_1, e_8\} \oplus \{e_6, e_8\} = \{e_1, e_6\}$

5. (a) H contains a cycle of length 5 (C_5), but G does not. OR G is bipartite, but G is not.

(b) 

Other solutions are possible, using the symmetries of the graph.

6. Show that if a *connected* graph G has two vertices u and v both of odd degree then G has a $uv-$path.

   Since G is connected, there is a $uv-$path between any pair of vertices.

   Note that this question should have read:

   Show that if a graph G has and odd degree vertex, then there are two vertices u and v both of odd degree which have a $uv-$path between them.

7. Show that a tree with a vertex of degree k contains at least k leaves.

   Let T be a tree and v a vertex of degree k in T.
   For each edge incident with v, follow a random path.
   Such a path must terminate, since T is finite and it must terminate in a leaf.
   Call these leaves $v_1, \ldots, v_k$.
   Each of the leaves thus visited must be unique, since there is a unique path from v to each v_i.

   Alternate proof (By Contradiction)
   Let T be a tree with less than k leaves and v a vertex of degree k in T.
   By the UTPT there is a unique path from v to each of the leaves of T. Each of these paths uses exactly one of the edges incident on v.
   Pick an edge which lies on no such path, and follow a path starting with this edge.
   Such a path must terminate, since T is finite and it must terminate in a leaf.
   But this is a leaf that was not counted previously.