Ring Sums, Bridges and Fundamental Sets

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1 Ring Sums

Definition 1 *Given two graphs* $G_1 = (V_1, E_1)$ *and* $G_2 = (V_2, E_2)$ *we define the* ring sum

$$G_1 \oplus G_2 = (V_1 \cup V_2, (E_1 \cup E_2) - (E_1 \cap E_2))$$

with isolated points dropped.

So an edge is in $G_1 \oplus G_2$ if and only if it is an edge of $G_!$, or an edge of G_2, but not both.

Theorem 2

- $\oplus$ *is commutative. i.e. For any graphs* G_1, G_2,

$$G_1 \oplus G_2 = G_2 \oplus G_1.$$

- $\oplus$ *is associative. i.e. For any graphs* G_1, G_2, G_3

$$G_1 \oplus (G_2 \oplus G_3) = (G_1 \oplus G_2) \oplus G_3.$$

- *For any fixed integer* n, $\overline{K_n}$ *is the identity of* $\oplus$. *i.e. For any graph* G *of order* n,

$$\overline{K_n} \oplus G = G.$$

- $\oplus$ *is idempotent, i.e. For any graph* G, G *is self inverse: under* $\oplus$, *so*

$$G \oplus G = \overline{K_n}.$$

Theorem 3 *For any graphs* G *and* H, $G \oplus H$ *is empty if and only if* $E(G) = E(H)$.

Proof: ($\Rightarrow$) Suppose that $G \oplus H$ is empty, this means that every edge of $E(G)$ is also in $E(H)$ and visa versa, so the two edge sets are equal.
($\Leftarrow$) Suppose that $E(G) = E(H)$, then very edge of G is also an edge of H and so there are no edges in $G \oplus H$. $\square$

2 Trees and Bridges

Definition 4 *An edge is a* bridge *if it is a cut-edge, i.e. $G - e$ is disconnected.*

Theorem 5 (Unique Tree Path Theorem) *For any tree T there is exactly one path between any pair of points of T.*

Proof: Let T be a tree. Since T is a tree, it is connected and so there is at least one path between any pair of vertices. It remains to show that there is not more than one.
Suppose not, so suppose that there are two vertices u and v in T such that there are two distinct paths between them. but then the two paths form a circuit and so T is not a tree. □

Theorem 6 (Bridge Theorem) *An edge of a graph G is a bridge if and only if it lies on no cycle of G.*

Proof: We prove the contrapositive. Namely, e is on a cycle if and only if it is not a bridge.
($\Rightarrow$) Suppose $e = u\,v$ lies on a cycle, then there is a $uv-$path P given by the rest of the cycle. So any $xy-$path requiring e can be replaced by an $xy-$path with P in place of e.
($\Leftarrow$) Let $e = u\,v$ be an edge in G which is not a bridge. Let G_1 be the component of G containing e, $G_1 - e$ is connected (otherwise e would be a bridge), so there is a $uv-$path P in $G_1 - e$. Adding e to P gives a cycle in G containing e. □

Corollary 7 (Tree Bridge Theorem) *Every edge of a tree is a bridge.*

Proof: Since T contains no cycles, every edge e of T is not on a cycle of T, and so by the Bridge Theorem (Theorem 6), e is a bridge of T. □

Corollary 8 *All trees are 1-edge connected.*

Theorem 9 (Tree Cycle Theorem) *Given a tree T the addition of any non-edge of T creates a cycle.*

Proof: Let T be a tree and $e \notin E(T)$, suppose $e = u\,v$. By the Unique Tree Path Theorem (Theorem 5) there is a unique $uv-$path P in T. Now, adding e to P gives a cycle. □

Corollary 10 *Any bridge of a graph G is in every spanning tree of G.*

Proof: (Contradiction) Suppose that e was a bridge in a graph G, and T a spanning tree which does not contain e. The addition of e to T creates a cycle containing e, by the Tree Cycle Theorem (Theorem 9). But this contradicts the Bridge Theorem (Theorem 6). □

3 Fundamental Circuits

Definition 11 *Given a graph $G = (V, E)$, with a specified spanning tree $T = (V, F)$ (so $F \subseteq E$):*

1. *The* co-tree *of G with respect to T is the set of edges in E which are not in F. i.e. co$-T = (V, E - F)$.*

   The edges in co$-T$ are called chords*.*

2. *By the Tree Cycle Theorem (Theorem 9) the addition of any chord to T creates a cycle, called the* Fundamental Circuits of G with respect to T*, one for each chord $e \in E($co$-T)$.*

Note that for each edge $e \in E(\text{co}-T)$ we have an associated cycle $C(e)$. Further, e is an edge of $C(e)$ and all other edges of $C(e)$ are edges of the tree T.

Lemma 12 (Circuit Closure) *The ring sum of a two edge disjoint collections of circuits is a collection of circuits.*

Proof: Note that a collection of circuits may be characterized by every vertex being of even degree.
Let C_1 and C_2 be two circuits and $C = C_1 \oplus C_2$. Let $v \in V(C)$, $d_1(v) =$ the degree of v in C_1, $d_2(v) =$ the degree of v in C_2 and $d(v) =$ the degree of v in C.
Since C_1 and C_2 are circuits we have that $d_1(v)$ and $d_2(v)$ are even, we wish to show that $d(v)$ is even.
If $v \in V(C_1)$, but $v \notin V(C_2)$, then every edge incident on v exists only in C_1 and so $d(v) = d_1(v)$, which is even. A similar argument holds if $v \notin V(C_1)$, but $v \in V(C_2)$.
Now suppose $v \in V(C_1) \cap V(C_2)$ and consider the edges, e_i, incident on v which are in both circuits, so $e_i \in E(C_1) \cap E(C_2)$, thus $e \notin E(C)$.
Suppose that there are k such edges, each counts once towards $d_1(v)$ and once towards $d_2(v)$ once, but is not counted in $d(v)$, thus $d(v) = d_1(v) - k + d_2(v) - k = d_1(v) + d_2(v) - 2k$, which is even. □

Theorem 13 (Fundamental Circuit Theorem) *Given a graph $G = (V, E)$, with a specified spanning tree $T = (V, F)$, any cycle in G can be expressed as the ring some of the fundamental circuits of G with respect to T. Further, no fundamental circuit is the ring sum of any of the others.*

Proof: Let $G = (V, E)$, with a specified spanning tree $T = (V, F)$, and let C be a cycle in G.
We can completely describe C be the set of edges in C. Order the edges $C = \{e_1, \dots, e_k, e_{k+1}, \dots, e_\ell\}$, so that $e_1, \dots e_k \in E(\text{co}-T)$ and $e_{k+1}, \dots e_\ell \in E(T)$.
For each e_i, $1 \le i \le k$, let $C(e_i)$, $1 \le i \le k$ be the fundamental circuit associated with e_i.
Consider $C' = C(e_1) \oplus C(e_2) \oplus \dots \oplus C(e_k)$, we claim that $C = C'$.
Since for $1 \le i, j \le k$, $e_i \in C(e_j)$ if and only if $i = j$ $e_1, \dots, e_k$ are edges of C' and these are the only edges of C' not in T.
Consider $C \oplus C'$, since the only chords in both are $e_1, \dots e_k$, $C \oplus C' \subseteq T$.
But C' is the ring sum of the fundamental circuits $C(e_i)$ and so is a collection of circuits by Circuit Closure (Lemma 12). C is a circuit by assumption and so $C \oplus C'$ is a collection of circuits, again by Circuit Closure (Lemma 12).
But T is a tree and hence contains no circuits, thus $C \oplus C' = \emptyset$. □

This says that the fundamental circuits form a basis for the set of circuits of a graph. Taking different spanning trees gives different bases.

4 Fundamental Edge-Cuts

For this section we consider that all initial graphs G are connected. If the initial graph G is not connected we consider the connected components individually.

Definition 14

1. *An* edge-cut C *is a set of edges which disconnect a graph. So* $G - C$ *is disconnected.*
2. *A* proper edge-cut *is an edge-cut which contains no edge-cut within it.*
3. *Given an edge-cut* C *we identify it with the partition of the vertices it induces,* $(V_1, V_2, \ldots V_k)$*, each being a connected component of* $G - C$*.*

If G is connected a proper edge-cut set will split G into exactly two pieces $(U, \overline{U})$ for some $U \subseteq V(G)$, here $\overline{U} = V(G) - U$. In particular, given a proper edge-cut C, producing partition $(U, \overline{U})$ every edge $e = u\,v$ of C has that $u \in U$ and $v \in \overline{U}$, otherwise any edge contained in the same connected component of $G - C$ could be removed from C.

Theorem 15 *If two proper edge-cut sets produce the same partition of the points then they are equal.*

Proof: (Contradiction) Let C_1 and C_2 be proper edge-cuts which both produce the connected components $(V_1, V_2, \ldots, V_k)$ and let $e = u\,v$ be an edge of C_1, but $e \notin C_2$.
Since C_1 is proper we must have that $u \in V_i$ and $v \in V_j$ for some $1 \leq i, j \leq k$ with $i \neq j$
But now e is a $uv-$path which connects V_i and V_j in $G - C_2$.
So C_2 cannot provide the same partition. □
This Theorem means that edge-cuts may be uniquely defined by the partition of the vertex set they induce.

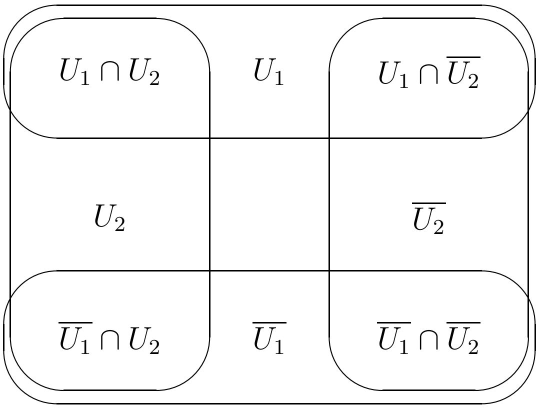
Theorem 16 *Given a graph* $G = (V, E)$*, with a specified spanning tree* $T = (V, F)$*, any cut-set must contain at least one edge of* T*.*

Proof: T is connected, and so provides a path between any pair of vertices. □

Lemma 17 (Edge-Cut Closure) *The ring sum of two distinct proper edge-cut sets is an edge-cut set.*

Proof: Let C_1 and C_2 be distinct edge-cut sets of a graph $G = (V, E)$ producing the partitions $(U_1, \overline{U_1})$ and $(U_2, \overline{U_2})$ respectively, where $U_1, U_2 \subseteq V$.
Every edge of C_1 has one endpoint in U_1 and the other in $\overline{U_1}$, and every edge of C_2 has one endpoint in U_2 and the other in $\overline{U_2}$.
Thus edges which are in both C_1 and C_2 must have one endpoints in either $U_1 \cap U_2$, $U_1 \cap \overline{U_2}$, $\overline{U_1} \cap U_2$

or $\overline{U_1} \cap \overline{U_2}$.
This means that the edges in both C_1 and C_2 must connect vertices in $U_1 \cap U_2$ to vertices in $\overline{U_1} \cap \overline{U_2}$ or vertices in $\overline{U_1} \cap U_2$ to vertices in $U_1 \cap \overline{U_2}$.
Thus if the edges of $C_1 \cap C_2$ are removed, the remaining edges provide the partition $(\overline{U_1} \cap \overline{U_2}) \cup \overline{U_1} \cap \overline{U_2}$ and $(\overline{U_1} \cap U_2) \cup (U_1 \cap \overline{U_2})$.



□

Given a connected graph $G = (V, E)$, with a specified spanning tree $T = (V, F)$ (so $F \subseteq E$) each edge e of $E(T)$ is a bridge of T. Thus each edge $e \in E(T)$ defines a partition of the vertex set V, $(U_e, \overline{U}_e)$, $U_e \subseteq V$. The *Fundamental Edge-Cuts* of G with respect to T, one for each edge e of T, is the proper edge-cut (containing e) which produces the partition $(U_e, \overline{U_e})$ of G.

Theorem 18 (Fundamental Edge-Cut Set Theorem) *Given a graph $G = (V, E)$, with a specified spanning tree $T = (V, F)$, any proper edge-cut set of G can be expressed as the ring some of the fundamental edge-cut sets of G with respect to T. Further, no fundamental edge-cut set is the ring sum of any of the others.*

Proof: Let $G = (V, E)$, with a specified spanning tree $T = (V, F)$, and let C be an edge-cut set in G.
Order the edges $C = \{e_1, \ldots, e_k, e_{k+1}, \ldots, e_\ell\}$, so that $e_1, \ldots e_k \in E(T)$ and $e_{k+1}, \ldots e_\ell \in E(\text{co}-T)$.
For each e_i, $1 \le i \le k$, let $C(e_i)$, $1 \le i \le k$ be the fundamental edge-cut of G associated with e_i.
Consider $C' = C(e_1) \oplus C(e_2) \oplus \ldots \oplus C(e_k)$, we claim that $C = C'$.
Since for $1 \le i, j \le k$, $e_i \in C(e_j)$ if and only if $i = j$ $e_1, \ldots, e_k$ are edges of C' and these are the only edges of C' not in co$-T$.
Consider $C \oplus C'$, since the only elements of T in both are $e_1, \ldots e_k$, $C \oplus C' \subseteq$co$-T$.
But C' is the ring sum of the fundamental edge-cuts $C(e_i)$ and so is an edge-cut by Edge-Cut Closure (Lemma 17). C is an edge-cut by assumption and so $C \oplus C'$ is an edge-cut, again by Edge-Cut Closure (Lemma 17).
But by Theorem 16 any edge-cut set must contain at least one edge of T, thus $C \oplus C' = \emptyset$. □

This says that the fundamental circuits form a basis for the set of circuits of a graph. Taking different spanning trees gives different bases.