Matchings

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1 Matchings

Definition 1

1. *Given a graph $G = (V, E)$, a matching in G is a set of non-loop edges $M \subseteq E$ such that no two elements of M are adjacent. ie No two edges from M share the same endpoints.*

2. *The size of a matching is the number of edges in it.*

3. *A maximal matching is a matching to which no edges can be added without violating the adjacency conditions.*

4. *A maximum matching is a matching is a matching of maximum size among all possible matchings in a graph.*

   Note that it is possible to have a maching wich is maximal, but not maximum (eg P_3).

5. *In a weighted graph G a maximum weight matching is a matching whose edges collectively have maximum weight over all possible matchings in G.*

6. *Given a matching M on a graph G, the vertices of G which are incident on some edge of M are said to be saturated by M, those which are not incident on any vertex of M are said to be unsaturated by M. We also say $M-$saturated and $M-$unsaturated respectively. A vertex which is unsaturated by M is also called free.*

7. *A perfect matching of a graph G is a matching in which every vertex of G is incident on some edge of G. That is every vertex is saturated by M.*

8. *Given a matching M an $M-$alternating path is a path whose edges are alternately in and not in M.*

9. *Given a matching M an $M-$augmenting path is an $M-$alternating path whose first and last vertices are free (unsaturated).*

Matchings arise naturally when we are trying to pair different things, possibly subject to some constraints.

Theorem 2 *Given a matching M and an $M-$augmenting path P, then there is a larger matching M'.*

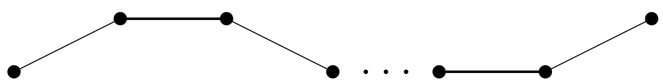
Proof: Define M' as follows:

- For each edge e of P,

  - if $e \in M$ then $e \notin M'$,
  - if $e \notin M$ then $e \in M'$.

- For each edge e not in P,
  - if $e \in M$ then $e \in M'$,
  - if $e \notin M$ then $e \notin M'$.

So M' agrees with M off P and is the opposite on P.



$|M'| = |M| + 1$

2 Covering and Independence

In this section we assume that G has no isolated vertices.

Definition 3 *Given a graph G*

1. *A vertex cover (of the edge set) is a set of vertices $S \subseteq V(G)$ such that S contains at least one endpoint of every edge of G.*
2. *An edge cover (of the vertex set) is a set of edges $F \subseteq E(G)$ such that every point of G is incident to some edge of F.*
3. *An independent set is a set of vertices $S \subseteq V(G)$ which contain no edge of G. That is*

$$\forall x, y \in S, x\,y \notin E(G).$$

We define maximum values for each of these. Note that a maximum set of independent (non intersecting) edges is exactly a maximum matching.

Maximum size of an independent set	$\alpha(G)$
Maximum Matching	$\alpha'(G)$
Minimum size of a vertex Cover	$\beta(G)$
Minimum size of an edge Cover	$\beta'(G)$

Matchings, covers and independent sets are closely related to each other. We can now investigate some relationships between these values.

Theorem 4 *For any graph G, $\alpha'(G) \leq \beta(G)$.*

Proof: No vertex can cover two edges of a matching, so the size of any vertex cover is at least the size of a maximum matching.

Theorem 5 *For any graph G, $\alpha(G) \leq \beta'(G)$.*

Proof: No edge can cover two vertices from an independent set.

Theorem 6 *Given a graph G on n vertices, S is an independent set of G, if and only if $\overline{S}$ is a vertex cover and hence $\alpha(G) + \beta(G) = n$*

Proof: ($\Rightarrow$) If S is an independent set of G, then every edge is incident on some vertex in $\overline{S}$, otherwise there would be an edge in S.
($\Leftarrow$) Let $\overline{S}$ be a vertex cover, hence every edge has an endpoint in $\overline{S}$, and no edge can be contained in S. So S is an independent set.

Lemma 7 *A minimal edge cover of a graph G is a disjoint union of $k-$stars ($K_{1,k}$).*

Proof: Let F be an edge cover.
If any edge e of F has both of its endpoints incident on edges of F then $F - \{e\}$ is also a vertex cover.
So F is not minimal and any minimal edge cover cannot contain such an edge.
Hence each component of a minimal edge cover has at most one vertex of degree > 1.

Theorem 8 (Gallai (1959)) *Given a graph G on n vertices, $\alpha'(G) + \beta'(G) = n$*

Proof: ($\beta'(G) \leq n - \alpha'(G)$) Let M be a maximum matching.
We obtain a vertex cover F by adding an edge to M for each free vertex.
We have one edge in F for each $M-$unsaturated vertex, and one edge of F for every two vertices incident on an edge of M.
Thus $|F| = n - |M|$.
So $\beta'(G) \leq n - \alpha'(G)$.
($\alpha'(G) \leq n - \beta'(G)$) Let F be a minimal edge cover and let ℓ be the number of components in F. Each of which is a $k-$star for some k by the Lemma.
Now, since F has an edge for each non cenrtal vertex in the star $|F| = n - \ell$.
Obtain a matching M by taking an edge from each of the ℓ $k-$stars of F.
$|M| = \ell = n - |F|$.
In the case where G is bipartite we have some stronger results.

Theorem 9 (König (1931)) *If G is a bipartite graph then the maximum size of a matching is the minimum size of a vertex cover. ie If G is bipartite $\alpha'(G) = \beta(G)$.*

Note that the condition of bipartite is necessary, C_5 has $\alpha'(C_5) = 2$, but $\beta(C_5) = 3$.

Theorem 10 (König (1916)) *If G is a bipartite graph then the maximum size of an edge cover is the minimum size of an independent set. ie If G is bipartite $\alpha(G) = \beta'(G)$.*

Proof: We have that $n = \alpha(G) + \beta(G) = \alpha'(G) + \beta'(G)$, and since G is bipartite, $\alpha'(G) = \beta(G)$.