Definitions and Review

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1 Introductory Graph Theory

Informally a *graph* is a set of points joined by lines.

Definition 1

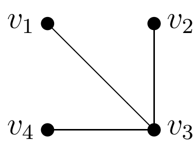
1. *A graph is a pair (V, E), where V is a set of vertices (also called points), and E is a set of edges (also called lines).*

   The number of vertices is called the order of a graph and the number of edges is called the size of the graph.

   Given a graph $G = (V, E)$ the vertex set *V is often denoted $V(G)$ and the* edge set *E by $E(G)$.*

2. *Each edge $e \in E$ is associated with a pair of points from V. If u and v are associated with the edge e they are called the endpoints of e, we often write $u\,v$ or $\{u, v\}$ to represent the edge e.*

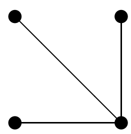
   Example 2

   $$V = \{v_1, v_2, v_3\},\ E = \{\{v_3, v_4\}, \{v_2, v_3\}, \{v_1, v_3\}\}$$

3. *In some cases we are interested in the structure of the graph. In these cases it is sometimes useful to drop the labels, the result is an unlabeled graph.*

   The graph abvoe could be represented as the unlabeled graph below.

4. *Two verticies are adjacent if they are both the endpoints of the same edge, they are also said to be neighbours.*

   An edge is said to be incident on each of its endpoints. Edges incident with a common vertex are called adjacent edges

   The set of all vertices adjacent to a given vertex, v, is called the neighbourhood of v.
   i.e. $N(v) = \{u \in V \mid u\,v \in E\}$

   A vertex in a graph which is incident with no edges is said to be isolated.

5. *A graph with no edges is called empty, otherwise it is called nonempty.*
   The graph with exactly one vertex and no edges is called the trivial graph

6. *A loop is an edge which joins a vertex to itself (i.e.* $e = \{v_1, v_1\}$*).*
   Two edges with the same set endpoints are said to be parallel. (i.e. $e_1 = \{v_1, v_2\}, e_2 = \{v_1, v_2\}$*).*
   A graph with no loops or parallel edges is called simple.
   A graph with parallel edges but no loops is called a multi-graph

7. *The degree of a vertex* v *in a graph* G *is the number of edges incident with* v*.*
   We denote the degree of v *by* $d(v)$*.*

8. *The total degree of a graph* G *is the sum of the degrees of all of its vertices.*

9. *A graph is called regular if every vertex has the same degree. If this degree is* k *then it is called* $k-$*regular.*

In this course we will mainly deal with simple graphs, so unless otherwise stated graphs will be assumed to be simple.

Definition 3 *A weighted graph is a graph in which each edge has an associated weight or cost.*

In a weighted graph we usually denote that weight of an edge e by $w(e)$, or if $e = u\,v$ we can write $w(u, v)$.

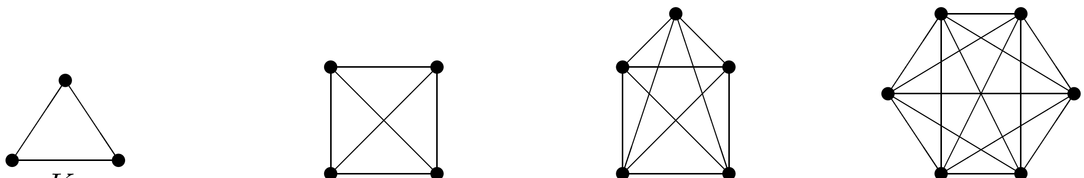
Definition 4 *A graph* $G = (V, E)$ *is bipartite if there is a partition of the vertex set* V *into two parts,* $U, W \subseteq V$ $(U \cap W = \emptyset$ *and* $U \cup W = V)$ *and every edge* $\{u, w\} \in E$ *has* $u \in U$ *and* $v \in V$*.*

That is in a bipartite graph all edges go from U to V, but no edges are wholly in U or in V.

1.1 The Complete Graphs

1. The complete graph on n verticies, K_n.
   K_n is the simple graph on n verticies, in which every vertex is adjacent to every other.

   Example 5



K_3 K_4 K_5 K_6

   Note that K_n is $(n-1)-$regular.

2. The Complete Bipartite graph on (n, m) verticies, $K_{n,m}$.

   The Complete Bipartite graph on nm verticies, $K_{n,m}$ is a bipartite graph with one part of n verticies and one part of m verticies.

   Example 6

   $K_{1,2}$ $K_{2,2}$ $K_{2,3}$ $K_{3,3}$

   Note that $K_{n,n}$ is $n-$regular. When $n \neq m$, $K_{n,m}$ is not regular; every vertex in the part of size m has degree n and every vertex in the part of size n has degree m.

1.2 Subgraphs

Definition 7 *Given two graphs $G = (V, E)$ and $H = (W, F)$:*

1. *A graph H is said to be a subgraph of G if every vertex of H is also a vertex of G and every edge of H is also an edge of G.*

   i.e. $W \subseteq V$ and $F \subseteq E$. We write $H \subseteq G$.

   In a similar vein G is said to be a supergraph of H.

2. *If either $V \neq W$ or $E \neq F$ we say that H is a proper subgraph of G and write $H \subset G$.*

   If $V = W$ we say the H is a spanning subgraph of G.

3. *Given a graph $G = (V, E)$ and a subset $W \subseteq V$, the subgraph of G induced by W is the graph $H = (W, F)$, whith point set W and edge set $F \subseteq E$ so that $e \in F \Leftrightarrow e \in E$ and both endpoints of e are in W.*

   If we remove a single vertex v from G (i.e. $W = V\{e\}$) we write $G - v$.

4. *In a similar manner, given a set of edges $F \subseteq E$ the edge induced subgraph with edge set F and vertex set obtained by deleting any resulting isolated vertices.*

   The subgraph of G obtained by removing a single edge e is denoted $G - e$.

5. *Any subgraph of a graph which is K_n is called a clique.*

Note that the removal of a vertex from a graph implies the removal of all edges incident with it, but the removal of an edge implies no further deletions. However, often after deleting a set of edges from a graph we also delete any verticies which have become isolated as a result.
If H is a subgraph of G we also say that G contains H.
Given a graph $G = (V, E)$ and an edge $e \notin E$ by $G + e$ we mean the graph obtained by adding the edge e to E. i.e. $G + e = (V, E \cup \{e\})$.

Theorem 8 *Every (simple) graph on n verticies is a subgraph of K_n.*

1.3 Joining Graphs

Definition 9

1. *Given a graph $G = (V, E)$ the* complement, $\overline{G} = (V, \overline{E})$, *where* $e \in \overline{E} \Leftrightarrow e \notin E$. *Thus $\overline{G}$ has the same vertex set as G and contains an edge between two vertices exactly when G doesn't.*

2. *Given two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ we define the* union $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$.

3. *Given two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ we define their* join $G_1 + G_2 = (V_1 \cup V_2, E_1 \cup E_2 \cup K)$, *where* $K = \{v\, v \mid u \in E_1,\ v \in E_2\}$, *the edges of* $K_{|V_1|,|V_2|}$.

4. *Given two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ we define their* Cartesian Product $G_1 \times G_2$ *by* $V(G_1 \times G_2) = V_1 \times V_2$, *the set of ordered pairs (u, v) with $u \in V_1$ and $v \in V_2$;* $E(G_1 \times G_2) = \{(u_1, u_2)\,(v_1, v_2) \mid u_1\, v_1 \in E_1 \text{ or } u_2\, v_2 \in E_1\}$.

5. *Given two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ we define the* ring sum $G_1 \oplus G_2$ *by* $V(G_1 \oplus G_2) = V_1 \cup V_2$, $E(G_1 \oplus G_2) = \{u\, v \mid u, v \notin E_1 \cup E_2\}$.

We observe that $G_1 \oplus G_2 = \overline{G_1 \cup G_2}$. However, usually for ring sums we have the same vertex set $V_1 = V_2$, but different edge sets $E_1 \neq E_2$, whereas for unions we often want disjoint unions, so $V_1 \cap V_2 = \emptyset$.
Also be aware that notations for these operations vary. In particular, some authors use $G_1 \vee G_2$, for the join and take $G_1 + G_2$ to be a disjoint union.

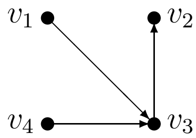
1.4 Directed Graphs

Definition 10

1. *A* directed graph *or* digraph *is a pair (V, E), where V is a set of* points *(also called* vertices*), and E is a set of* ordered *pairs of points from V called* arcs.

2. *Each arc $e \in E$ is associated with an ordered pair of points from V. If u and v are associated with the edge e they are called the* endpoints *of e, we often write $u\, v$ or (u, v) to represent the arc e.*

Example 11



$$V = \{v_1, v_2, v_3\},\ E = \{(v_4, v_3), (v_3, v_2), (v_1, v_3)\}$$

3. *The* indegree *of a point is the number of arcs pointing to it. We denote the indegree of v by $d^-(v)$.*

4. *The outdegree of a point is the number of arcs pointing out of it. We denote the outdegree of v by $d^+(v)$*

5. *A digraph is balanced if $d^+(v) = d^-(v)$ for every vertex $\in V$.*

In the example $V = \{v_1, v_2, v_3\}$, $E = \{(v_4, v_3), (v_3, v_2), (v_1, v_3)\}$

x	v_1	v_2	v_3	v_4
$d^-(x)$	0	1	2	0
$d^+(x)$	1	0	1	1

All of the definitions above hold for digraphs as well, replacing *graph* with *digraph* and *edge* with *arc*.
Given a digraph it naturally defines an (undirected simple) underlying graph by replacing each arc of the digraph with an edge and then deleting any parallel edges.

2 Connectivity, Paths and Circuits

Definition 12 *Let $G = (V, E)$ be a graph.*

1. *A walk from vertex v to vertex w is a finite sequence of adjacent verticies starting at v and ending at w: $v_1e_1v_2e_2 \ldots e_nv_n$ with $v_i \in V$, $e_i = \{v_i, v_{i+1}\} \in E$, $v_1 = v$ and $v_n = w$.*

   This is interpreted to mean that we start at v_1, take edge e_1 to vertex v_2 and so on.

   In a simple graph it is sufficient to list the vertices traversed, in a graph with parallel edges the edges must also be specified.

   The length of a walk is the number of edges in it.

   A closed walk is a walk that starts and ends at the same vertex. (So $v = w$.)

   If a walk is not closed it is open.

2. *A trail from v to w is a walk from v to w which does not contain any repeated edge.*

   A path from v to w is a trail which contains no repeated vertex. (So there are no repeated verticies or edges.)

   We use P_n to denote the simple path with n verticies.

3. *A circuit is a closed walk that contains no repeated edge. (So $v = w$ and there are no repeated edges.)*

   A simple circuit is a circuit which does not have any repeated veticies except the first and last. (So $v = w$ and there are no other repeated verticies or edges.)

   We use C_n to denote the simple circuit on n verticies, this is also referred to as an $n-$cycle. If n is even thsi is an even cycle, whereas if n is odd, it is an odd cycle.

4. *Two verticies $v, w \in V$ are connected if there is a walk from v to w. By convention every vertex is connected to itself.*

   A graph G is called connected if and only if every pair of verticies in G is connected.

5. *The distance between two vertices u and v is the length of the shortest path between them, and is denoted $d(u, v)$. An example of such a shortest path is called a geodesic.*

   The greatest distance between any two points in a graph is calle the diameter of the graph.

Theorem 13 *If a graph G contains a walk from u to v of length ℓ, then G contains a path of length k, where $k \leq \ell$.*

2.1 Connectivity

As noted above a graph G is *connected* if and only if there is a walk between any pair of vertices of G, and every vertex is connected to itself.

Theorem 14 *Given a graph $G = (V, E)$, connection between vertices is an equivalence relation on V.*

Proof: Let $G = (V, E)$ be a graph and $x, y, z \in V$. We must show that connection obeys the three equivalence rules: *reflexivity*, *symmetry* and *transitivity*

Reflexivity (For all $x \in V$ x is connected to itself.)
This is true by the definition of connection.

Symmetry (For any pair of vertices x and y, if x is connected to y, then y is connected to x.)
Let x be connected to y, thus there is a walk from x to y. Traverse this walk in reverse to obtain a walk from y to x.

Transitivity (For any vertices x, y and z if x is connected to y and y is connected to z then x is connected to z.)
Let x, y and z be vertices with x connected to y and y connected to z. Thus there is a walk from x top y, W_{xy} say, and a walk from y to z, W_{yz} say. The concatenation of these two walks is a walk from x to z, so x and z are connected. □

Recall that an equivalence relation divides the set into equivalence classes, sets of objects that are all relatesd to each other, but not to anything else. Thus connectivity divides the vertex set of a graph into connected components. Every pair of vertices in a connected component are connected (by a walk), but there is no walk between two vertices in different connected components.
Note that a connected graph has only one connected component.

Definition 15

1. *If a graph has more than one connected component it is called disconnected.*

2. *If v is a vertex in a connected graph $G = (V, E)$ such that $(G - v)$ is disconnected then v is called an articulation point.*

3. *If e is an edge in a connected graph $G = (V, E)$ such that $(G - e)$ is disconnected then e is called a cut-edge.*

4. *A connected component of a graph G that contains no articulation points is called a block.*

5. *A component with one or more articulation points is called seperable.*
6. *A set of vertices which disconnects a graph is called a vertex cut.*
7. *A connected component of a graph G is called k−vertex connected if there is a vertex cut of size k, but no set of $k-1$ vertices will disconnect the graph.*
8. *A set of edges which disconnects a graph is called an edge cut.*
9. *A connected component of a graph G is called k−edge connected if there is an edge cut of size k, but no set of $k-1$ edges will disconnect the graph.*
10. *A set of vertices or edges which disconnects a graph is referred to as a cut set.*

In the case of a digraph we can refine the notions of connectivity.

Definition 16

1. *A digraph is strongly connected if there is a path of arcs between any two vertices.*
2. *A digraph is weakly connected if the underlying graph is connected.*

Note that a strongly connected graph is always weakly connected, but a weakly connected graph may not be strongly connected.
In a digraph we may thus speak of stongly connected components and weakly connected components.