Definitions and Review

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1 Eulerian Graphs

The Seven bridges of Königsberg
Leonhard Euler (1707 - 1783) was one of the greatest and most prolific mathematicians of all time. At the beginning and and of his career he worked in St. Petersburg in Russia, but He was employed at the Berlin academy, court mathematician to Frederick the Great of Prussia from 1741 to 1766. Euler's output was prodigious, he published more than 500 books and papers during his lifetime, with another 400 appearing posthumously, his collected works span 74 volumes. Euler was also active in the popularisation of mathematics and science, he designed textbooks for Russian elementary schools, his lessons to Frederick the Great's niece Princess Anhalt-Dessau where written up in *Letters to a German Princess,* which was translated into seven languages and became a best seller.
One of Frederick the Great's holdings was the town of Königsberg, famed for its seven bridges across the river Pregel. One day a group of foreign dignitaries was visiting and he wished to show off the seven bridges. He asked Euler to devise a tour which would cross each of the bridges exactly once. Euler considered the problem and used a graph to model it - thus inventing graph theory. However, he was forced to report back that such a tour was impossible.

Definition 1

- *Given a graph G, an* <u>*Euler circuit*</u> *of G is a circuit which every edge of G exactly once. That is an Eulerian Circuit is a sequence of adjacent verticies which uses every vertex at least once, and every edge exactly once.*
- *A graph G is called* <u>*Eulerian*</u> *if and only if it has an Euler circuit.*

These sorts of problems are also known as <u>Postman Problems</u> A postman wishes to traverse a set of streets on his route, visiting each street exactly once. Modeling the route as a graph this asks for an Eulerian circuit of the corresponding graph.

Lemma 2 *If a graph is Eulerian then it is connected.*

Proof: If the graph is disconnected then there are two vertices with no path between them, these two vertices can never be on the same circuit.

Lemma 3 *If a graph is Eulerian then every vertex has even degree.*

Proof: Since we have a circuit, each time it visits a vertex, it does so twice, once in and once out.

Corollary 4 *If any vertex of a graph has odd degree then it is not Eulerian.*

Theorem 5 *If every vertex of a connected non empty graph G has even degree then G has an Eulerian circuit.*

Proof: We give an algorithm which always produces an Eulerian circuit.

1. Pick a vertex v to start.
   Set $C := v$.

2. Repeat the following:

   (a) Remove the edges of C from G and any vertices which become isolated.
   Call this graph G'. (G' may not be connected)

   (b) Pick a vertex u common to both C and G'.
   (There must be one since G is connected)

   (c) Pick a sequence of adjacent edges and points starting and ending at u.
   This is always possible since the degrees of verticies are all even. So whenever we reach a vertex there is an edge to leave by. G is finite, so we must eventually return.
   Call this circuit C'.

   (d) Set $C = C \cup C'$

   (e) If C contains all edges of G terminate and return C.

2 Hamiltonian Circuits

Definition 6 *A Hamiltonian Circuit of a graph G is a simple circuit of G that includes every vertex exactly once.*

Note We don't require that every edge be used.
This problem is in some sense complementary to the Euler circuit problem, here we require every vertex to be used exactly once, whereas an Eulerian circuit requires every edge to be used exactly once.
The following is a recursive algorithm to find Hamiltonian circuits in a graph G.

Routine Ham(v)
    Input a vertex v
    $C := C \cup \{v\}$ (Add v to the circuit C)
    $U := U \cup \{v\}$ (Add v to the visited bucket U)
    If $U = V$ and $vx \in E$ (every vertex has been visited - cycle complete)
        Return Success
    For each vertex $u \in N(v) - U$ (u is adjacent to v, but $u \notin U$)
        if Ham(u)
            Return Success
    Next u
    Return Fail
End Ham

Main routine
    Input a graph $G = (V, E)$ with n vertices
    Pick a vertex $x \in V$ to start
    $U := C := \phi$
    If Ham(x)
        Return C
    else
        Return Fail
End main

While this algorithm will eventually find a Hamiltonian circuit if one exists it may take some time. This is an example of a backtrack algorithm, it essentially tries every possible permutation exhaustively. If the graph does not contain a Hamiltonian circuit every possible combination of edges will be traversed.
For ease of calculation suppose that the input graph is $k-$regular (each vertex has degree k), but contains no Hamiltonian circuit. At each recursive level we must call Ham k times, each of these then in turn must call Ham k times and so on. Assuming that G is connected, the depth of is to traverse all n vertices, giving k^n total steps.
It is possible to improve this algorithm somewhat by ordering the vertices, but no polynomial time algorithm is known. In fact the problem of determining whether a graph has a Hamiltonian circuit is an example of a class of problems known as NP-complete.
Surprisingly, while the Euler circuit problem has an efficient algorithm for it's solution (running time of the Eulerian circuit algorithm is $O(|E|)$) there is no known efficient algorithm for finding a Hamiltonian circuit. This is despite the fact that the two problems seem superficially similar.
The best we can do efficiently is to say something about graphs which do **not** have Hamiltonian circuits.
If we think about trying to create a Hamiltonian circuit on a graph 3 rules become evident:

1. If a vertex has degree 2, then both of these edges must be used. (since we must visit the vertex on the circuit, and leave again)

2. No proper sub-circuit (a circuit which does not contain all of the vertices) can be part of a Hamiltonian circuit. (Otherwise the joining vertex would be visited twice.)

3. As we build a Hamiltonian circuit, after we have used a vertex all of the edges incident with that vertex may be deleted, since we may not use them again. (This may allow us to apply rule 1 again)

2.1 The Traveling Salesman problem

Problem A traveling salesman must travel to a number of cities, starting and ending at some given city. The links between cities have a cost associated with them. It is required to find a Hamiltonian circuit which minimizes the cost.

Definition 7 *A* <u>*weighted graph*</u> *is a graph in which each edge has an associated weight or cost.*