

Colouring

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1 Vertex Colourings

In this section we assume that all graphs are loopless.

Definition 1 Given a graph $G = (V, E)$

- A k -coloring of G is a map $f : V \rightarrow S$, where $|S| = k$. Despite the colour terminology usually $S = \{1, \dots, k\}$.
- The elements of S are referred to as colours. The colour of a vertex v is the image $f(v)$.
- A colour class in G is the set of vertices which all have the same colour: $C_i = \{v \in V \mid f(v) = i\}$
- A k -colouring is proper if no two adjacent vertices have the same colour.

$$\text{i.e. } uv \in E \Rightarrow f(u) \neq f(v).$$

- A graph is k -colourable if it has a proper k -colouring.
- The chromatic number of G , denoted $\chi(G)$, is the minimum k such that G is k -colourable.
- The clique number of G is the size of the largest clique in G . i.e. The largest subgraph of G isomorphic to K_ℓ for some ℓ . The clique number of G is denoted $\omega(G)$.

Note Colouring can be thought of as a generalization of the notion of bipartition. In a bipartite (2-partite) graph $\chi(G) = 2$ the parts are the colour classes.

Theorem 2 A graph is bipartite if and only if it is 2-colourable.

Proof: (\Rightarrow) Let G be a bipartite graph with bipartition X, Y and let $S = \{1, 2\}$.

$$\text{Define } f \text{ by } f(v) = \begin{cases} 1 & \text{if } v \in X \\ 2 & \text{if } v \in Y \end{cases}$$

(\Leftarrow) Suppose that G is a 2-colourable graph and let f be a proper two colouring with color classes X and Y .

Since f is proper there are no edges within X or within Y , so G is bipartite with bipartition X, Y . In general a graph G is k -partite if and only if $\chi(G) = k$.

Theorem 3 For any graph G $\chi(G) \geq \omega(G)$.

Proof: Every vertex of a k -clique must receive a different colour, since they are all adjacent.

Note This bound is not tight.

1.1 Greedy Colouring

Consider the following **Greedy algorithm**:

Input: A graph G

Order the vertices in some order (like by BFS or DFS but it could just be random) $\{v_1, v_2, \dots, v_n\}$
 Go through the list assigning a colour to each vertex in turn so that it doesn't get the same colour as any of its previously labeled neighbors.

Note that we need at most $\Delta(G) + 1$ colors, where $\Delta(G)$ is the largest vertex degree in G . Thus we have the following Theorem:

Theorem 4 For any graph G , $\chi(G) \leq \Delta(G)$.

In fact Brookes improved this result:

Theorem 5 (Brooks (1941)) If G is a connected graph other than a complete graph or an odd cycle then $\chi(G) \leq \Delta(G)$

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We also have a characterisation theorem using subgraphs in a similar spirit to Hall's or Tutte's theorems for matchings.

Theorem 6 (Szekeres, Wilf (1968)) For any graph G , $\chi(G) \leq 1 + \max_{H \subseteq G}(\delta(H))$, where $\delta(H)$ is the minimum degree of a vertex in H .

Proof: Let G be a graph with n vertices and let $k = \max_{H \subseteq G}(\delta(H))$.

We will define an ordering of the vertices so that the greedy algorithm needs at most $k + 1$ colours.

We use $d_H(v)$ to denote the degree of v in the induced subgraph H of G .

Since $G \subseteq G$ there is a vertex v of G such that $d_G(v) \leq k$.

Put $v_n = v$ and $G_n = G$.

Now let $G_{n-1} = G - v_n$.

Since $G_{n-1} \subseteq G$ there is a vertex v' of G_{n-1} such that $d_{G_{n-1}}(v') \leq k$.

Put $v_{n-1} = v'$ and $G_{n-2} = G_{n-1} - v_{n-1}$.

Proceed in this way to produce an ordering on the vertices of G : v_1, v_2, \dots, v_n , and a sequence of subgraphs G_1, G_2, \dots, G_n with $V(G_i) = \{v_1, \dots, v_i\}$ such that $d_{G_i}(v_i) \leq k$.

Now greedy colour using this ordering.

At any stage, since G_i consists of those vertices which have been labeled so far and $d_{G_i}(v_i) \leq k$, we may label v_i with a colour distinct from all its neighbours if we have $k + 1$ colours.

2 Edge Colourings

Definition 7 Given a graph $G = (V, E)$

- A k -edge coloring of G is a map $f : E \rightarrow S$, where $|S| = k$. Despite the colour terminology usually $S = \{1, \dots, k\}$.
- The elements of S are referred to as colours. The colour of an edge e is the image $f(e)$.

- A colour class in G is the set of edges which all have the same colour: $C_i = \{e \in E \mid f(e) = i\}$
- A k -edge colouring is proper if no two adjacent edges have the same colour.
- A graph is k -edge colourable if it has a proper k -edge colouring.
- The edge chromatic number of G , denoted $\chi'(G)$, is the minimum k such that G is k -edge colourable.

Note Edge colourings may be thought of as a generalisation of factorizations.

A factorization can be thought of as an edge colouring of a decomposition.

If we can find a colouring of the edges of a graph so that every colour class has size $n/2$ we have produced a 1-factorization.

Theorem 8 For any simple graph G , $\chi'(G) \geq \Delta(G)$

Proof: Let v be a vertex of maximum degree $\Delta(G)$. Every edge incident on v must get a different colour, so we need at least $\Delta(G)$ colours to colour these edges.

Theorem 9 If G is bipartite then $\chi'(G) = \Delta(G)$

Proof: Let G be a bipartite graph, with bipartition X, Y .

We know that every regular bipartite graph has a perfect matching.

Thus we first embed G in a larger regular bipartite graph H .

To construct H , first add vertices to the smaller part until both parts are of equal size.

If the resulting graph is not regular there must be at least one vertex in each part which has degree less than $\Delta(G)$, add an edge between them.

Continue in this manner until every vertex has degree $\Delta(G)$.

Now since H is regular bipartite it has a perfect matching.

Remove this perfect matching and give all its edges the same colour.

Continue in this manner, to produce a 1-factorisation (colouring) of H in which each 1-factor gets a different colour and uses $\Delta(G)$ colours.

To obtain a colouring of G we simply remove any edges not in G from the colour classes.

Theorem 10 (Vizing (1965), Gupta (1966)) For any graph G either $\chi'(G) = \Delta(G)$ or $\chi'(G) = \Delta(G) + 1$.

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Note that Given an arbitrary graph G has $\chi'(G) = \Delta$ or $\chi'(G) = \Delta(G) + 1$ is an NP complete problem.

The proof of Vizing's Theorem gives a polynomial time algorithm for $\Delta(G) + 1$ colouring G .