

Systems of Equations

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1 Systems of Equations

We can have more than one equation which we want to be simultaneously true.

1.0.1 2 Equations in 2 Unknowns

$$\begin{aligned}a_1x + b_1y &= c_1 \\ a_2x + b_2y &= c_2\end{aligned}$$

Here a_1, a_2, b_1, b_2, c_1 and c_2 are all constants.

x and y are variables.

We wish to find those values of x and y such that both equations are true.

There are 3 possibilities:

1. There lines meet in a single point and is a unique solution.
2. The lines are parallel and there is no solution.
3. The lines are linear multiples and there are an infinite number of solutions.

Example 1

1.

$$\begin{aligned}x + y &= 3 \\ 2x + 3y &= 7\end{aligned}$$

Has the **unique solution** $x = 2, y = 1$.

Here $a_1 = 1, a_2 = 2, b_1 = 1, b_2 = 3, c_1 = 3$ and $c_2 = 7$

2.

$$\begin{aligned}x + y &= 4 \\ 2x + 2y &= 6\end{aligned}$$

If $x + y = 4$, we would expect $2x + 2y = 8$, but this contradicts the second equation. So this system has **no solution**.

3.

$$\begin{aligned}x + y &= 4 \\ 2x + 2y &= 8\end{aligned}$$

In this case the equations are linear multiples of each other, and so represent the same line. We may represent the solution set by a

1 parameter solution:

Let $t \in \mathbb{R}, y = t, x = 4 - t$.

We can also write this as:

$$(x, y) = (4 - t, t) = (4, 0) + t(-1, 1)$$

1.1 2 equations in 3 unknowns:

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \end{aligned}$$

We wish to find those values of (x, y, z) such that both of these equations are simultaneously true. Geometrically we wish to find the intersection of two planes.

Two possibilities:

1. There are an infinite number of solutions.

In this case there are actually two sub-cases:

- The two planes meet in a line, so there is a 1-parameter solution.
- The two planes are the same, so there is a 2-parameter solution.

2. The two planes are parallel and do not meet, so there is no solution.

Example 2

- 1.

$$\begin{aligned} x + y + z &= 1 \\ x + 2y - z &= 0 \end{aligned}$$

Here $a_1 = 1$, $a_2 = 1$, $b_1 = 1$, $b_2 = 2$, $c_1 = 1$, $c_2 = -1$, $d_1 = 1$ and $d_2 = 0$.

This system has solution $2x + 3y = 1$, which has a **1-parameter solution**:

Let $t \in \mathbb{R}$, $y = t$, $x = \frac{1-3t}{2}$.

We also have (from the second equation)

$$z = x + 2y = \frac{1-3t}{2} + 2t = \frac{1+t}{2}$$

We can also write this as:

$$\begin{aligned} (x, y, z) &= \frac{1}{2}(1 - 3t, 2t, 1 + t) \\ &= \frac{1}{2}[(1, 0, 1) + t(-3, 2, 1)] \end{aligned}$$

- 2.

$$\begin{aligned} x + y + z &= 1 \\ 2x + 2y + 2z &= 2 \end{aligned}$$

Here $a_1 = 1$, $a_2 = 2$, $b_1 = 1$, $b_2 = 2$, $c_1 = 1$, $c_2 = 2$, $d_1 = 1$ and $d_2 = 2$.

In this case the equations are linear multiples and so represent the same plane.

Thus there is a **2-parameter solution**:

Let $s, t \in \mathbb{R}$, $y = s$, $z = t$, $x = 1 - s - t$.

We can also write this as

$$\begin{aligned} (x, y, z) &= (1 - s - t, s, t) \\ &= (1, 0, 0) + s(-1, 1, 0) + t(-1, 0, 1) \end{aligned}$$

- 3.

$$\begin{aligned} x + y + z &= 1 \\ x + y + z &= -2 \end{aligned}$$

In this case the planes are parallel and there is no solution.

1.2 3 Equations in 3 Unknowns

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}$$

We wish to find those values of (x, y, z) such that all three of these equations are simultaneously true.

Geometrically we wish to find the intersection of three planes.

Note that it is possible to have three non parallel planes that never meet.

Three Possibilities:

1. The three planes meet in a point, so there is a unique solution.
2. There are infinitely many solutions.

In this case there are actually two sub-cases:

- The planes meet in a line, so there is a 1-parameter solution.
 - The planes are the same, so there is a 2-parameter solution.
3. The three planes do not all meet simultaneously, so there is no solution.

Note that now this does **not** mean that the planes are parallel.

Example 3

1.

$$\begin{aligned} x + 2y - 3z &= 1 \\ &+ y + z = 1 \\ x + y + z &= 0 \end{aligned}$$

This has the unique solution $(x, y, z) = (-1, 1, 0)$.

2.

$$\begin{aligned} x + 2y - 3z &= 1 \\ &+ y + z = 1 \\ x + 3y - 2z &= 2 \end{aligned}$$

Has the 1 parameter solution: $t \in \mathbb{R}$

$$\begin{aligned} (x, y, z) &= (-1 + 5t, 1 - t, t), \text{ or} \\ (x, y, z) &= (-1, 1, 0) + t(5, -1, 1) \end{aligned}$$

3.

$$\begin{aligned} x + 2y - 3z &= 1 \\ &+ y + z = 1 \\ x + 3y - 2z &= 0 \end{aligned}$$

Has No solution.

Note that the planes are not parallel.