Row Space, Column Space and Nullspace

P. Danziger

1 Nullspace

Definition 1 *Given an* $m \times n$ *matrix* A *The* Nullspace *of* A *1s the set of solution to the equation* $A\mathbf{x} = \mathbf{0}$.

Notes

- The Nullspace of $A = \ker(A)$.
- When we are asked to give a subspace (such as the nullspace of a matrix) the easiest way to describe the subspace is to give a basis for the space.

Example 2

1. Find the Nullspace of the matrix

$$A = \begin{pmatrix} 1 & 0 & 1 & 2 \\ 1 & 1 & 0 & 1 \end{pmatrix}$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 1 & 2 & 0 \\ 1 & 1 & 0 & 1 & 0 \end{array}\right) \quad R_2 \to R_2 - R_1$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 1 & 2 & 0 \\ 0 & 1 & -1 & -1 & 0 \end{array}\right)$$

Let $s, t \in \mathbb{R}$, set

$$x_3 = s,\ x_4 = t,\ \text{so } x_2 = s + t \text{ and } x_1 = -s - 2t.$$

We can express the solution as

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = s \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -2 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

So every solution to $A\mathbf{x} = \mathbf{0}$ is a linear combination of $(-1, -1, 1, 0)$ and $(-2, -1, 0, 1)$. Thus the Nullspace of A is $\text{span}\{(-1, 1, 1, 0), (-2, 1, 0, 1)\}$.

2. Solve the system of equations $A\mathbf{x} = \mathbf{b}$, where A is as above and $\mathbf{b} = (1, 1)$.

   Row reducing gives

$$\left(\begin{array}{cccc|c} 1 & 0 & 1 & 2 & 1 \\ 1 & 1 & 0 & 1 & 1 \end{array}\right) \quad R_2 \to R_2 - R_1$$
$$\left(\begin{array}{cccc|c} 1 & 0 & 1 & 2 & 1 \\ 0 & 1 & -1 & -1 & 0 \end{array}\right)$$

   Let $s, t \in \mathbb{R}$, set $x_3 = s$, $x_4 = t$, so $x_2 = -s - t$ and $x_1 = 1 - s - 2t$.

   We can express the solution as

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -2 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

   Note that $\mathbf{x}_0 = (1, 0, 0, 0)$ is a solution to $A\mathbf{x} = \mathbf{b}$ and all solutions have the form $\mathbf{x}_0 + \mathbf{h}$, where $\mathbf{h} \in$Nullspace(A).

Theorem 3 *If $\mathbf{x}_0$ is any solution to the linear system $A\mathbf{x} = \mathbf{b}$ and $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is a basis for the nullspace of A, then $\mathbf{x}$ is a solution of $A\mathbf{x} = \mathbf{b}$ if and only if*

$$\mathbf{x} = \mathbf{x}_0 + c_1\mathbf{v}_1 + \ldots + c_k\mathbf{v}_k$$

Definition 4 *The dimension of the Nullspace of a matrix A is called the nullity of A.*

Example 5
Find nullity(A), where

$$A = \begin{pmatrix} 1 & 0 & 1 & 2 \\ 1 & 1 & 0 & 1 \end{pmatrix}$$

Since $\{(-1, 1, 1, 0), (-2, 1, 0, 1)\}$ is a basis for the nullspace of A, nullity$(A) = 2$.
Note that row operations do not change the solution set of a system of equations.

Theorem 6 *If A and B are row equivalent matrices, Nullspace(A) = Nullspace(B).*

2 Row Space and Column Space

Given an $m \times n$ matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{pmatrix}$$

We may consider the m rows of A as vectors in their own right.

$$\begin{aligned} \mathbf{r}_1 &= (a_{11}, a_{12}, \ldots, a_{1n}) \\ \mathbf{r}_2 &= (a_{21}, a_{22}, \ldots, a_{2n}) \\ &\vdots \\ \mathbf{r}_m &= (a_{m1}, a_{m2}, \ldots, a_{mn}) \end{aligned}$$

Similarly, we may consider the n columns of A as vectors in their own right.

$$\mathbf{c}_1 = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} \quad \mathbf{c}_2 = \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} \quad \cdots \quad \mathbf{c}_n = \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}$$

$\mathbf{r}_1, \ldots, \mathbf{r}_m$ are called the row vectors of A.
$\mathbf{c}_1, \ldots, \mathbf{c}_n$ are called the column vectors of A.

Definition 7 *Given an $m \times n$ matrix A*

1. *The* Row Space *of A is the space spanned by the row vectors of A. span$\{\mathbf{r}_1, \ldots, \mathbf{r}_m\}$.*
2. *The* Column Space *of A is the space spanned by the column vectors of A. span$\{\mathbf{c}_1, \ldots, \mathbf{c}_n\}$.*

Notes

1. The row space is a subspace of $\mathbb{R}^n$ and the column space is a subspace of $\mathbb{R}^m$. So $\dim(\text{RowSpace}(A)) \leq n$ and $\dim(\text{ColSpace}(A)) \leq m$.
2. When we are asked to give a subspace (such as the row space column space of a matrix) the easiest way to describe the subspace is to give a basis for the space.

We consider the effect of row operations on the row and column space.