Matrices

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1 Matrices

1.1 Definitions

Definition 1

1. A Matrix is an $m \times n$ (m by n) array of numbers.

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

2. The entries in a matrix are called the *components* of the matrix and can be written as a_{ij}, where i indicates the row number and runs from 1 to m, and j indicates the column number and runs from 1 to n.

3. A Vector is a $1 \times n$ or $n \times 1$ matrix. That is an ordered set of n numbers.

   We say that such a vector is of dimension n.

4. A scalar is a number (usually either real or complex).

Notation 2

- We generally use uppercase letters from the beginning of the alphabet $(A, B, C \dots)$ to denote matrices.

- A matrix is identified with its components, given a matrix A with components a_{ij}, $1 \leq i \leq m$, $1 \leq j \leq n$, we may write
$$A = [a_{ij}]$$

Example 3

Find the 4×4 matrix A with compoents given by $a_{ij} = i + j$.

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{pmatrix}$$

- We generally use lowercase boldface letters from the end of the alphabet ($\mathbf{u}, \mathbf{v}, \mathbf{w} \ldots$) to denote vectors.
- We use the convention that $\mathbf{u} = (u_1, u_2, \ldots, u_n)$, $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, etc.
- If $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ then the scalars $x_1, x_2, \ldots, x_n$ are called the components of $\mathbf{x}$.
- We denote the set of all vectors of dimension n whose components are real numbers by $\mathbb{R}^n$.
- We denote the set of all vectors of dimension n whose components are complex numbers by $\mathbb{C}^n$.

Note This definition of vector differs from the usual 'High School' definition involving magnitude and direction.

1.2 Special Matrices and Vectors

1. **The Identity matrix**
   The identity matrix is a square matrix with 1's down the diagonal, and zeros elsewhere. The $n \times n$ identity matrix is denoted I_n.

$$I_n = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \end{pmatrix}$$

2. **The Zero Matrix**
   The zero matrix is an $m \times n$ matrix, all of whose entries are 0.

$$\begin{pmatrix} 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 \end{pmatrix}$$

2 Operations on Matrices

1. **Transpose**
   Given an $m \times n$ matrix, A, the transpose of A is obtained by interchanging the rows and columns of A. We denote the transpose of A by A^t, or A^T.

   Notes:

   - If A is $m \times n$ then A^T will be $n \times m$.
   - If $A = [a_{ij}]$, then $[a_{ij}]^T = [a_{ji}]$.

Example 4

(a)

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}^t = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$$

(b)

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}^t = \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix}$$

(c)

$$(1, 2, 3)^t = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

2. **Matrix Addition**

Given two $m \times n$ matrices

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \dots & b_{mn} \end{pmatrix}$$

We may define the sum of A and B, $A + B$, to be the sum componentwise, i.e.

$$A + B = \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \dots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \dots & a_{2n} + b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & \dots & a_{mn} + b_{mn} \end{pmatrix}$$

Componentwise: $[a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}]$.

This works for vectors as well.

$$\begin{aligned} \mathbf{u} + \mathbf{v} &= (u_1, u_2, \dots, u_n) + (v_1, v_2, \dots, v_n) \\ &= (u_1 + v_1, u_2 + v_2, \dots, u_n + v_n) \end{aligned}$$

Note that matrix addition is only defined if A and B have the same size.

Example 5

(a)

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} + \begin{pmatrix} 10 & 11 & 12 \\ 13 & 14 & 15 \\ 16 & 17 & 18 \end{pmatrix}$$
$$= \begin{pmatrix} 1+10 & 2+11 & 3+12 \\ 4+13 & 5+14 & 6+15 \\ 7+16 & 8+17 & 9+18 \end{pmatrix}$$
$$= \begin{pmatrix} 11 & 13 & 15 \\ 17 & 19 & 21 \\ 23 & 25 & 27 \end{pmatrix}$$

(b)

$$\begin{aligned} (1,2,3) + (4,5,6) &= (1+4, 2+5, 3+6) \\ &= (5,7,9) \end{aligned}$$

3. **Matrix Multiplication**

   (a) **Scalar Multiplication**
   Given a matrix A, and a scalar k, we define the <u>scalar product</u> of k with A, kA by multiplying each entry of A by k.

$$kA = k \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} = \begin{pmatrix} ka_{11} & ka_{12} & \dots & ka_{1n} \\ ka_{21} & ka_{22} & \dots & ka_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ ka_{m1} & ka_{m2} & \dots & ka_{mn} \end{pmatrix}$$

   Componentwise: $k\,[a_{ij}] = [ka_{ij}]$.
   Note that this works for vectors as well.

$$k\mathbf{u} = k(u_1, u_2, \dots, u_n) = (ku_1, ku_2, \dots, ku_n)$$

   Example 6

$$10 \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = \begin{pmatrix} 10 & 20 & 30 \\ 40 & 50 & 60 \\ 70 & 80 & 90 \end{pmatrix}$$

(b) **Matrix Multiplication**

If A and B are two matrices where A has the same number of columns as B has rows (i.e. A is $m \times n$ and B is $n \times r$) we define the matrix product, AB to be the matrix in which the i, j^{th} entry is made up of the dot product of the i^{th} row of A with the j^{th} column of B.

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix},$$

$$B = \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1r} \\ b_{21} & b_{22} & \dots & b_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \dots & b_{nr} \end{pmatrix}$$

$$AB = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1r} \\ b_{21} & b_{22} & \dots & b_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \dots & b_{nr} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} + \ldots + a_{1n}b_{n1} & \dots & a_{11}b_{1r} + a_{12}b_{2r} + \ldots + a_{1n}b_{nr} \\ a_{21}b_{11} + a_{22}b_{21} + \ldots + a_{2n}b_{n1} & \dots & a_{21}b_{1r} + a_{22}b_{2r} + \ldots + a_{2n}b_{nr} \\ \vdots & \ddots & \vdots \\ a_{m1}b_{11} + a_{m2}b_{21} + \ldots + a_{mn}b_{n1} & \dots & a_{m1}b_{1r} + a_{m2}b_{2r} + \ldots + a_{mn}b_{nr} \end{pmatrix}$$

Example 7

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \\ 3 & 2 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} 1\times 9 + 2\times 6 + 3\times 3 & 1\times 8 + 2\times 5 + 3\times 2 & 1\times 7 + 2\times 4 + 3\times 1 \\ 4\times 9 + 5\times 6 + 6\times 3 & 4\times 8 + 5\times 5 + 6\times 2 & 4\times 7 + 5\times 4 + 6\times 1 \\ 7\times 9 + 8\times 6 + 9\times 3 & 7\times 8 + 8\times 5 + 9\times 2 & 7\times 7 + 8\times 4 + 9\times 1 \end{pmatrix}$$

$$= \begin{pmatrix} 9 + 12 + 9 & 8 + 10 + 6 & 7 + 8 + 3 \\ 36 + 30 + 18 & 32 + 25 + 12 & 28 + 20 + 6 \\ 63 + 48 + 27 & 56 + 40 + 18 & 49 + 32 + 9 \end{pmatrix}$$

$$= \begin{pmatrix} 30 & 24 & 18 \\ 84 & 69 & 54 \\ 138 & 114 & 90 \end{pmatrix}$$

Note that Matrix multiplication is only defined if A has the same number of columns as B has rows.

BIG Note

Matrix multiplication is **NOT** commutative. i.e. It is **NOT** true that $AB = BA$ (where defined).

Example 8

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$