# Properties of Transformations

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## 1 Transformations from $\mathbb{R}^n \longrightarrow \mathbb{R}^m$

### 1.1 General Transformations

A general transformation maps vectors in $\mathbb{R}^n$ to vectors in $\mathbb{R}^m$. We write $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ to indicate this.

**Example 1**

1. Given $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $T(x, y) = (x + 1, y)$, find $T(0, 1)$.
   $T(0, 1) = (1, 1)$.
2. Given $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $T(x, y) = (|x|, |y|)$, find $T(-1, 2)$
   $T(-1, 2) = (1, 2)$.
3. Given $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$, $T(x, y) = (x, y, x + y)$, find $T(-1, 2)$.
   $T(-1, 2) = (-1, 2, 1)$.
4. Given $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$, $T(x, y, z) = (x + z, y + z)$, find $T(1, 2, -1)$.
   $T(1, 2, -1) = (0, 1)$.

**Definition 2** *Given a transformation* $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$

1. The domain is all those values $\mathbf{x} \in \mathbb{R}^n$ where $T(\mathbf{x})$ is defined.
$$\mathrm{dom}(T) = \{\mathbf{x} \in \mathbb{R}^n \mid T(\mathbf{x}) \text{ is defined }\}.$$
2. The range of $T$ is the set of values in $\mathbf{y} \in \mathbb{R}^m$ for which there is an $\mathbf{x} \in \mathbb{R}^n$ such that $\mathbf{y} = T(\mathbf{x})$.
   i.e. $\mathbf{y} \in \mathbb{R}^n$ for which there exists $\mathbf{x} \in \mathbb{R}^m$ such that $\mathbf{y} = T(\mathbf{x})$.
$$\mathrm{ran}(T) = \{\mathbf{y} \in \mathbb{R}^n \mid \text{there exists } \mathbf{x} \in \mathbb{R}^m \text{ such that } \mathbf{y} = T(\mathbf{x})\}.$$
3. The kernal of a transformation $T$ is the set of $\mathbf{x} \in \mathbb{R}^n$ such that $T(\mathbf{x}) = \mathbf{0}$ and is denoted $\ker(T)$.

## 1.2 Matrix Transformations

We can interpret a matrix as a map. Given an $m \times n$ matrix $A$, for each $\mathbf{x} \in \mathbb{R}^n$ define $\mathbf{y} \in \mathbb{R}^m$ by

$$\mathbf{y} = A\mathbf{x}.$$

Thus each $m \times n$ matrix $A$ is associated with a transformation $T_A$ where $T_A(\mathbf{x}) = A\mathbf{x}$.
A transformation which comes from an associated matrix is called a *matrix transformation*
Given a matrix transformation $T_A$, we write

$$A = [T_A].$$

**Notes:**

- If $T$ is a matrix transformation associated with an $m \times n$ matrix $A$ then $\text{dom}(T) = \mathbb{R}^n$.
- If $T$ is a matrix transformation then finding the kernal is equivalent to solving the homogeneous system $A\mathbf{x} = \mathbf{0}$.

**Example 3**

1.

$$\text{Let } A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

(a) Find the image of the vector $(1, 2, 1)$.

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 4 \end{pmatrix}$$

(b) Find $T_A(1, 1, 0)$.

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$$

So $T_A(1, 1, 0) = (2, 1, 2)$

(c) Find $\text{dom}(T_A)$.
$T_A$ is a matrix transformation, so $\text{dom}(T_A) = \mathbb{R}^3$.

(d) Find $\ker(T_A)$.
Find $\mathbf{x} \in \mathbb{R}^3$ such that $T(\mathbf{x}) = \mathbf{0}$, i.e. Solve $A\mathbf{x} = \mathbf{0}$.

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right) \quad R_3 \longrightarrow R_3 - R_1$$
$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

Only solution is the trivial solution $x_1 = x_2 = x_3 = 0$. So $\ker(T_A) = \{\mathbf{0}\}$.

(e) Find $\mathrm{ran}(T_A)$.

We must find all $\mathbf{y} \in \mathbb{R}^3$ so that $\mathbf{y} = A\mathbf{x}$ for some $\mathbf{x} \in \mathbb{R}^3$. i.e. Solve $A\mathbf{x} = \mathbf{y}$.

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & y_1 \\ 0 & 1 & 1 & y_2 \\ 1 & 1 & 1 & y_3 \end{array}\right) \qquad R_3 \longrightarrow R_3 - R_1$$
$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & y_1 \\ 0 & 1 & 1 & y_2 \\ 0 & 0 & 1 & y_3 - y_1 \end{array}\right)$$

Which has a unique solution for every $\mathbf{y} \in \mathbb{R}^3$. So $\mathrm{ran}(T_A) = \mathbb{R}^3$.

2.

$$\text{Let } A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix}$$

(a) Find $\ker(T_A)$.

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 1 & 0 \end{array}\right) \qquad R_3 \longrightarrow R_3 - R_1$$
$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right) \qquad R_3 \longrightarrow R_3 - R_2$$
$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Solution: Let $t \in \mathbb{R}$, $x_3 = t, y = -t, x = t$, or $\mathbf{x} = t(1, -1, 1)$, which is a line parallel to $(1, -1, 1)$ through the origin.

$$\ker(T_A) = \{\mathbf{x} \in \mathbb{R}^3 | \mathbf{x} = t(1, -1, 1), t \in \mathbb{R}^3\}$$

(b) Find $\mathrm{ran}(T_A)$.

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & y_1 \\ 0 & 1 & 1 & y_2 \\ 1 & 2 & 1 & y_3 \end{array}\right) \qquad R_3 \longrightarrow R_3 - R_1$$
$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & y_1 \\ 0 & 1 & 1 & y_2 \\ 0 & 1 & 1 & y_3 - y_1 \end{array}\right) \qquad R_3 \longrightarrow R_3 - R_2$$
$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & y_1 \\ 0 & 1 & 1 & y_2 \\ 0 & 0 & 0 & y_3 - y_1 - y_2 \end{array}\right)$$

Which will have solution only when $y_1 + y_2 - y_3 = 0$.

$$\mathrm{ran}(T_A) = \{(x, y, z) \in \mathbb{R}^3 \mid x + y - z = 0\}$$

So the range of $T_A$ is the plane $x + y - z = 0$ and $T_A$ maps $\mathbb{R}^3$ to this plane.

(c) Is $(1,1,0)$ in the range of $T_A$?

From the above equation the vector $(x,y,z)$ is in the range of $T_A$ if and only if $x+y-z=0$. In this case, $(x,y,z)=(1,1,0)$, we have $1+1+0=2\neq 0$. So $(1,1,0)\notin \text{ran}(T_A)$.

If we did not have the work above we would be asking to solve $A\mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$.

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 2 & 1 & 0 \end{array}\right) \qquad R_3 \longrightarrow R_3 - R_1$$
$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{array}\right) \qquad R_3 \longrightarrow R_3 - R_2$$
$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{array}\right)$$

Which has no solution, so the original vector $(1,1,0)$ is not in the range.

(d) Is $(1,1,2)$ in the range of $T_A$?

Once again given the answer to part 2b we can see that $1+1-2=0$, so $(1,1,2)\in \text{ran}(T_A)$.

If we did not have the answer to part 2b we would proceed as follows.

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 2 & 1 & 2 \end{array}\right) \qquad R_3 \longrightarrow R_3 - R_1$$
$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{array}\right) \qquad R_3 \longrightarrow R_3 - R_2$$
$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Which has solution set $(t, 1-t, t), t\in\mathbb{R}$. So $(1,1,2)\in \text{ran}(T_A)$.

(e) Describe the transformation $T_A$.

This transformation maps $\mathbb{R}^3$ to the plane $x+y-z=0$ along lines parallel to $t(1,-1,1)$.

**Theorem 4** *Given an $m\times n$ matrix $A$, the range of $T_A$ is a subspace of $\mathbb{R}^m$ and the kernel of $T_A$ is a subspace of $R^n$.*

## 1.3 Linear Transformations

**Definition 5** *Given a transformation $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$, $T$ is called a* Linear Transformation *if for every $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ and every scalar $c$ the following two properties hold:*

1. $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$.
2. $T(c\mathbf{u}) = cT(\mathbf{u})$.

**Theorem 6** *A Transformation $T$ is a linear transformation if and only if for every $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ and every pair of scalars $c$ and $d$*

$$T(c\mathbf{u} + d\mathbf{v}) = cT(\mathbf{u}) + dT(\mathbf{v}).$$

**Proof:**($\Rightarrow$) If $T$ is a linear transformation then the result follows from properties 1 and 2.

$$T(c\mathbf{u} + d\mathbf{v}) = T(c\mathbf{u}) + T(c\mathbf{v}) = cT(\mathbf{u}) + dT(\mathbf{v}).$$

($\Leftarrow$) Suppose that $T$ satisfies $T(c\mathbf{u} + d\mathbf{v}) = cT(\mathbf{u}) + dT(\mathbf{v})$. for every $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ and every pair of scalars $c$ and $d$ In particular when $c = d = 1$ we get $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$. Which is property 1. When $\mathbf{v} = \mathbf{0}$ we get property 2, $T(c\mathbf{u}) = cT(\mathbf{u})$.

**Theorem 7** *A transformation is linear if and only if it is a matrix transformation*

**Proof:**($\Leftarrow$) Suppose $T = T_A$ for some matrix $A$, we will show that $T$ is linear. Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ and $c$ be a scalar.

$$\begin{array}{lrcl} \text{Consider} & T_A(c\mathbf{u} + d\mathbf{v}) & = & A(c\mathbf{u} + d\mathbf{v}) \\ & & = & Ac\mathbf{u} + Ad\mathbf{v} \\ & & = & cT_A(\mathbf{u}) + dT_A(\mathbf{v}) \end{array}$$

($\Rightarrow$) We will show this in due course.

**Theorem 8** *If a transformation is linear then $T(\mathbf{0}) = \mathbf{0}$.*

**Proof:**We must have $T(\mathbf{u}) = T(\mathbf{u} + \mathbf{0}) = T(\mathbf{0}) + T(\mathbf{u})$. Which implies $T(\mathbf{0}) = \mathbf{0}$. $\square$

**Notes**

- This Theorem can only be used to show that a transformation is **not** Linear, by showing that $T(0) \neq 0$.
- It is possible to have a transformation for which $T(\mathbf{0}) = \mathbf{0}$, but which is not linear. Thus, it is not possible to use this theorem to show that a transformation is linear, only that it is not linear.
- To show that a transformation is linear we must show that the rules 1 and 2 hold, or that $T(c\mathbf{u} + d\mathbf{v}) = cT(\mathbf{u}) + dT(\mathbf{v})$.

**Example 9**

1. Show that $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $T(x, y) = (x + 1, y)$ is not linear.

   $T(0, 0) = (1, 0) \neq (0, 0)$. So this transformation is not linear.

2. Show that $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $T(x, y) = (|x|, |y|)$ is not linear.

   $T(0,0) = (0,0)$ so Theorem 8 does not apply.
   Now consider $T(1,1) + T(-1,1) = (1,1) + (1,1) = (2,2)$.
   But $T((1,1) + (-1,1)) = T(0,2) = (0,2)$.
   So rule 1 does not apply to this transformation.

3. Show that the transformation $T : \mathbb{R}^4 \longrightarrow \mathbb{R}^2$, given by $T(x_1, x_2, x_3, x_4) = (x_1 + x_2, x_3 + x_4)$ is linear.

   Let $\mathbf{u} = (u_1, u_2, u_3, u_4)$ and $\mathbf{v} = (v_1, v_2, v_3, v_4)$ be vectors in $\mathbb{R}^4$ and $c$ and $d$ be scalars.
   Consider

$$\begin{aligned}
&T(c\mathbf{u} + d\mathbf{v}) \\
&= T(cu_1 + dv_1, cu_2 + dv_2, cu_3 + dv_3, cu_4 + dv_4) \\
&= ((cu_1 + dv_1) + (cu_2 + dv_2), (cu_3 + dv_3) + (cu_4 + dv_4)) \\
&= (c(u_1 + u_2) + d(v_1 + v_2), c(u_3 + u_4) + d(v_3 + v_4)) \\
&= c(u_1 + u_2, u_3 + u_4) + d(v_1 + v_2, v_3 + v_4) \\
&= cT(u_1, u_2, u_3, u_4) + dT(v_1, v_2, v_3, v_4) \\
&= cT(\mathbf{u}) + dT(\mathbf{v})
\end{aligned}$$

**Theorem 10** *Translations are not linear.*

Translations are maps of the form $T(\mathbf{x}) = \mathbf{x} + \mathbf{a}$, for some fixed non zero vector $\mathbf{a} \in \mathbb{R}^{\mathbf{n}}$.
Consider $T(\mathbf{0}) = \mathbf{0} + \mathbf{a} = \mathbf{a} \neq \mathbf{0}$. $\square$

**Theorem 11** *Linear transformations map subspaces to subspaces.*

## 1.4 Finding the Matrix Associated with a Linear Transformation

Given any linear transformation $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ The matrix $A$ such that $T = T_A$ is the matrix whose columns are the vectors

$$\mathbf{w}_1 = T(\mathbf{e}_1),\ \mathbf{w}_2 = T(\mathbf{e}_2),\ \ldots,\ \mathbf{w}_n = T(\mathbf{e}_n),$$

$T$ acting on each of the elementary vectors.

$$\text{So} \quad A = \begin{pmatrix} \mathbf{w}_1\ \mathbf{w}_2\ \ldots\ \mathbf{w}_n \end{pmatrix} = \begin{pmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) & \ldots & T(\mathbf{e}_n) \end{pmatrix}.$$

Given a vector $\mathbf{u} = (u_1, u_2, \ldots, u_n) \in \mathbb{R}^n$, we can write

$$\mathbf{u} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + \ldots + u_n\mathbf{e}_n$$

Now

$$A\mathbf{u} = \begin{pmatrix} \mathbf{w}_1\ \mathbf{w}_2\ \ldots\ \mathbf{w}_n \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} = u_1\mathbf{w}_1 + u_2\mathbf{w}_2 + \ldots + u_n\mathbf{w}_n$$

To see that $T(\mathbf{u}) = A\mathbf{u}$ consider

$$\begin{aligned} T(\mathbf{u}) &= T(u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + \ldots + u_n\mathbf{e}_n) \\ &= T(u_1\mathbf{e}_1) + T(u_2\mathbf{e}_2) + \ldots + T(u_n\mathbf{e}_n) \\ &= u_1T(\mathbf{e}_1) + u_2T(\mathbf{e}_2) + \ldots + u_nT(\mathbf{e}_n) \\ &= u_1\mathbf{w}_1 + u_2\mathbf{w}_2 + \ldots + u_n\mathbf{w}_n \\ &= A\mathbf{u} \end{aligned}$$

**Example 12**
Find a matrix for the transformation $T : \mathbb{R}^4 \longrightarrow \mathbb{R}^2$, given by

$$T(x_1, x_2, x_3, x_4) = (x_1 + x_2, x_3 + x_4).$$

$$T(\mathbf{e}_1) = T(1,0,0,0) = T(\mathbf{e}_2) = T(0,1,0,0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

$$T(\mathbf{e}_3) = T(0,0,1,0) = T(\mathbf{e}_4) = T(0,0,0,1) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

$$\text{So} \quad A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

For example

$$T(1,2,3,4) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ 7 \end{pmatrix}$$

### 1.4.1 Differentiation as a linear Operator

In calculus we have the following Theorem:

**Theorem 13** *Given any two differentiable functions $f$ and $g$ and a constant $c \in \mathbb{R}$,*

$$\frac{d(f+g)}{dx} = \frac{df}{dx} + \frac{dg}{dx}$$

$$\frac{d(cf)}{dx} = c\left(\frac{df}{dx}\right)$$

Which says that differentiation is a linear operator. For somplicity we restrict ourselves to polynomials, but note that by Taylors Theorem, any differentiable function can be approximated by a polynomial of sufficiently high degree.
Let

$$\begin{aligned} \mathbb{P}_n &= \{\text{Polynomials of degree at most } n\} \\ &= \{a_0 + a_1x + a_2x^2 + \ldots + a_nx^n \mid a_0, a_1, \ldots, a_n \in \mathbb{R}\} \end{aligned}$$

Now, any polynomial $a_0+a_1x+a_2x^2+\ldots+a_nx^n$ is uniquely represented by the vector $(a_0, a_1, \ldots, a_n) \in \mathbb{R}^{n+1}$, so there is a correspondence between the set of polynomials of degree at most $n$, $\mathbb{P}_n$ and $\mathbb{R}^{n+1}$, by

$$a_0 + a_1x + a_2x^2 + \ldots + a_nx^n \in \mathbb{P}_n \longleftrightarrow (a_0, a_1, \ldots, a_n) \in \mathbb{R}^{n+1}.$$

Given a polynomial $p(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$, $\dfrac{dp}{dx} = a_1 + 2a_2x + \ldots + na_nx^{n-1} \in P_{n-1}$. We can consider the corresponding map $D : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^n$ given by

$$D(a_0, a_1, a_2 \ldots, a_n) = (a_1, 2a_2, \ldots, na_n).$$

Now to find the matrix associated with this map $D$ we consider the effect of $D$ on the standard basis vectors $\mathbf{e}_i$.

$$\begin{array}{rcll} D(\mathbf{e}_1) & = & \mathbf{0} & \\ D(\mathbf{e}_i) & = & i\mathbf{e}_{i-1} & i > 1 \end{array}$$

So the matrix associated with the differentiation operator $D$, is the $(n-1) \times n$ matrix

$$[D] = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 2 & \ldots & 0 \\ & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & n \end{pmatrix}$$

# 2 A Library of Transformations

We want to build up a library of simple transformations, just as we have a library of simple functions. There are two special transformations:

- **The Identity Transformation**

$$T_I : \mathbb{R}^n \longrightarrow \mathbb{R}^n, \;\; T_I(\mathbf{x}) = \mathbf{x} \text{ for every } \mathbf{x} \in \mathbb{R}^n.$$

- **The Zero Transformation**

$$T_0 : \mathbb{R}^n \longrightarrow \mathbb{R}^n, \;\; T_0(\mathbf{x}) = \mathbf{0} \text{ for every } \mathbf{x} \in \mathbb{R}^n.$$

Any linear (matrix) transformation can be made up by taking compositions of the following basic types of transformation:

1. Dilations
2. Rotations
3. Reflections
4. Projections
5. Shears

We have already seen that composition of matrix maps is equivalent to matrix multiplication. that is

$$T_A \circ T_B = T_{AB} \text{ or equivalently } T_A(T_B(\mathbf{x})) = T_{AB}(\mathbf{x}).$$

## 2.1 Dilations

A dilation makes every vector bigger or smaller by a constant factor $k$. Thus a dilation of a vector $\mathbf{x} \in \mathbb{R}^n$ looks like $D_k(\mathbf{x}) = k\mathbf{x} = kI\mathbf{x}$. Thus the associated matrix $[D_k] = kI$.

**Example 14**

1. Find a matrix for the transformation in $\mathbb{R}^2$ which shrinks every vector to half its original size.

$$\frac{1}{2}I_2 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

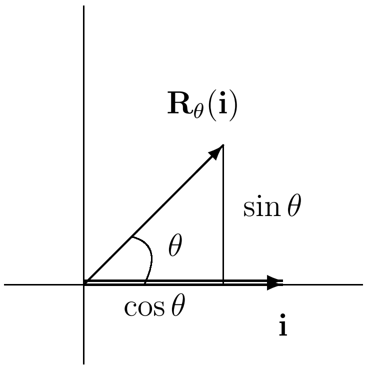
2. Find a dilation by a factor of 2 in $\mathbb{R}^3$

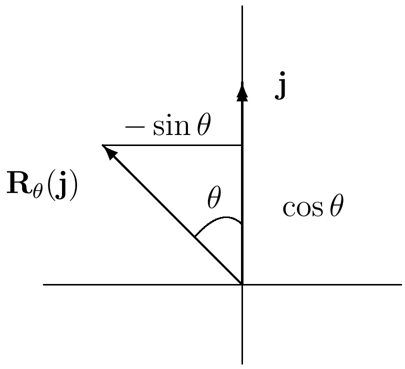
$$2I_3 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

## 2.2 Standard Transformations in $\mathbb{R}^2$

We now consider some basic standard transformations in $\mathbb{R}^2$.

1. **Rotations in $\mathbb{R}^2$** Consider a counter-clockwise rotation about the origin in $\mathbb{R}^2$.

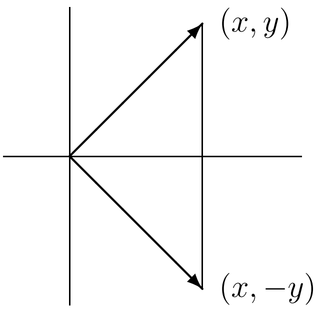


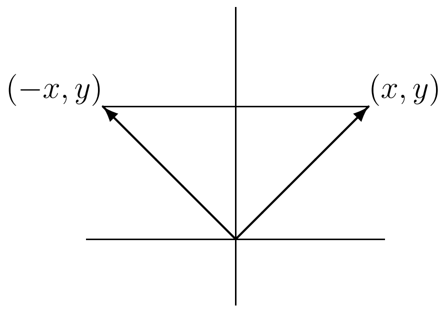

So $R_\theta(\mathbf{i}) = (\cos\theta, \sin\theta)$ and $R_\theta(\mathbf{j}) = (-\sin\theta, \cos\theta)$, and so

$$\boxed{[R_\theta] = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}}$$

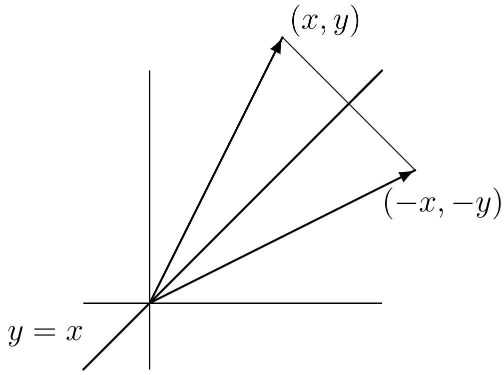
2. **Reflections in $\mathbb{R}^2$**




$F_x(x, y) = (x, -y)$
Reflection about $x-$axis




$F_y(x, y) = (-x, y)$
Reflection about $y-$axis




$F_{(1,1)}(x, y) = (-x, -y)$
Reflection about the line $y = x$

- Reflection about the $x-$axis.

$$F_x(x, y) = (x, -y)$$

$F_x(1, 0) = (1, 0)$ and $F_x(0, 1) = (0, -1)$, so

$$\boxed{[F_x(x, y)] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}$$

- Reflection about the $y-$axis.

$$F_y(x, y) = (-x, y)$$

$F_y(1, 0) = (-1, 0)$ and $F_y(0, 1) = (0, 1)$, so

$$\boxed{[F_y(x, y)] = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}}$$

- Reflection about the line $y = x$.

$$F_{(1,1)}(x, y) = (y, x)$$

$F_{(1,1)}(1, 0) = (0, 1)$ and $F_{(1,1)}(0, 1) = (1, 0)$, so

$$\boxed{\left[F_{(1,1)}(x, y)\right] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}$$

Suppose that we wish to find the matrix for the reflection about an arbitrary line parallel to a vector $\mathbf{u}$, which makes an angle of $\theta$ with the $x-$axis.

Now $R_{-\theta}(\mathbf{u})$ maps $\mathbf{u}$ parallel to the $x-$axis and so in this transformed coordinate system reflection about $\mathbf{u}$ is reflection about the $x-$axis.

Finally we transform back using $R_\theta$.

Thus $F_{\mathbf{u}} = R_\theta \circ F_x \circ R_{-\theta}$, or $R(x,y) = R_\theta(F_x(R_{-\theta}(x,y)))$

But composition of maps is matrix multiplication, so

$$\begin{aligned}
[R_{\mathbf{u}}] &= [R_\theta][F_x][R_{-\theta}] \\
&= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{pmatrix} \\
&= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \\
&= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} \\
&= \begin{pmatrix} \cos^2\theta - \sin^2\theta & 2\cos\theta\sin\theta \\ 2\cos\theta\sin\theta & -(\cos^2\theta - \sin^2\theta) \end{pmatrix} \\
&= \begin{pmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{pmatrix}
\end{aligned}$$

Where we have used the double angle formulas:

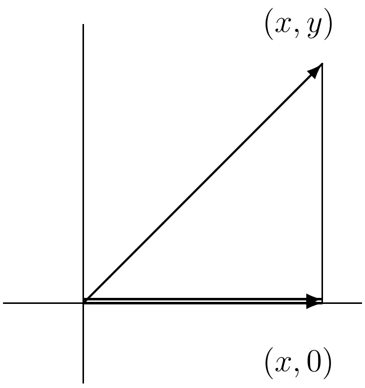
$$\begin{aligned}
\cos(2\theta) &= \cos^2\theta - \sin^2\theta, \\
\sin(2\theta) &= 2\cos\theta\sin\theta
\end{aligned}$$

**Example 15** *Find the matrix for a reflection about the line $t\mathbf{v}$, where $\mathbf{v} = (\sqrt{3}, -1)$.*

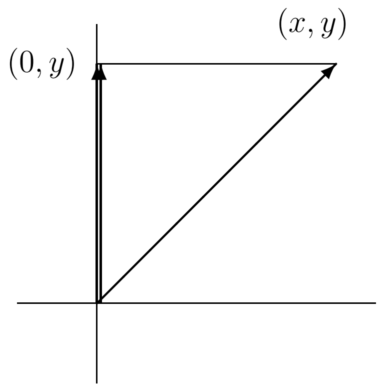
$\mathbf{v}$ makes an angle of $\theta = \arctan\left(\frac{-1}{\sqrt{3}}\right)$ with the $x-$axis. Further, since the second coordinate is negative, $\mathbf{v}$ lies in the fourth quadrant. So $\theta = \frac{11\pi}{6} = -\frac{\pi}{6}$.

$$\begin{aligned}
F_{\mathbf{v}} &= \begin{pmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{pmatrix} = \begin{pmatrix} \cos\left(-2\frac{\pi}{6}\right) & \sin\left(-2\frac{\pi}{6}\right) \\ \sin\left(-2\frac{\pi}{6}\right) & -\cos\left(-2\frac{\pi}{6}\right) \end{pmatrix} \\
&= \begin{pmatrix} \cos\left(\frac{\pi}{3}\right) & -\sin\left(\frac{\pi}{3}\right) \\ -\sin\left(\frac{\pi}{3}\right) & -\cos\left(\frac{\pi}{3}\right) \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -\sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix}
\end{aligned}$$

3. **Projections in $\mathbb{R}^2$**




$P_x(x,y) = (x,0)$
Projection onto the $x-$axis




$P_y(x,y) = (0,y)$
Projection onto the $y-$axis

- Projection onto the $x-$axis
$$P_x(x,y) = (x,0)$$
$P_x(1,0) = (1,0)$ and $P_x(0,1) = (0,0)$, so
$$\boxed{[P_x(x,y)] = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}$$

- Projection onto the $y-$axis
$$P_y(x,y) = (0,y)$$
$P_x(1,0) = (0,0)$ and $P_x(0,1) = (0,1)$, so
$$\boxed{[P_x(x,y)] = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}$$

Suppose that we wish to find the matrix for the projection onto an arbitrary line parallel to a vector $\mathbf{u}$, which makes an angle of $\theta$ with the $x-$axis.

As above $R_{-\theta}(\mathbf{u})$ maps $\mathbf{u}$ parallel to the $x-$axis and so in this transformed coordinate system reflection about $\mathbf{u}$ is projection onto the $x-$axis.

Finally we transform back using $R_\theta$.

Thus $P_\mathbf{u} = R_\theta \circ P_x \circ R_{-\theta}$, or $R(x,y) = R_\theta(P_x(R_{-\theta}(x,y)))$

But composition of maps is matrix multiplication, so
$$\begin{aligned}
[P_\mathbf{u}] &= [R_\theta][P_x][R_{-\theta}] \\
&= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{pmatrix} \\
&= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \\
&= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{pmatrix}
\end{aligned}$$

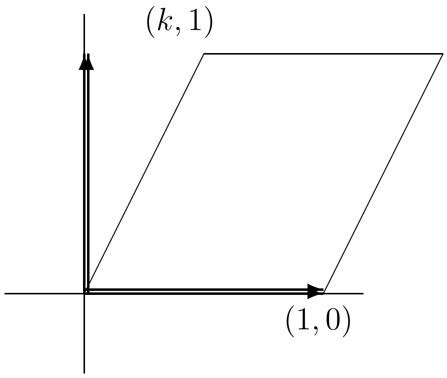
**Example 16** *Find the matrix for a projection onto the line $t\mathbf{v}$, where $\mathbf{v} = (\sqrt{3}, -1)$.*

As above $\mathbf{v}$ makes an angle of $\theta = \arctan\left(\frac{-1}{\sqrt{3}}\right)$ with the $x-$axis. Further, since the second coordinate is negative, $\mathbf{v}$ lies in the fourth quadrant. So $\theta = \frac{11\pi}{6} = -\frac{\pi}{6}$.

$$\begin{aligned}
P_\mathbf{v} &= \begin{pmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{pmatrix} = \begin{pmatrix} \cos^2\left(-\frac{\pi}{6}\right) & \cos\left(-\frac{\pi}{6}\right)\sin\left(-\frac{\pi}{6}\right) \\ \cos\left(-\frac{\pi}{6}\right)\sin\left(-\frac{\pi}{6}\right) & \sin^2\left(-\frac{\pi}{6}\right) \end{pmatrix} \\
&= \begin{pmatrix} \left(\frac{\sqrt{3}}{2}\right)^2 & \left(\frac{\sqrt{3}}{2}\right)\left(-\frac{1}{2}\right) \\ \left(\frac{\sqrt{3}}{2}\right)\left(-\frac{1}{2}\right) & \left(-\frac{1}{2}\right)^2 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 3 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}
\end{aligned}$$

4. **Shears in $\mathbb{R}^2$**

   A shear has exactly one off diagonal entry non-zero, the diagonal entries are all ones. Thus shears in $\mathbb{R}^2$ have one of two forms, we consider the action on the unit square:

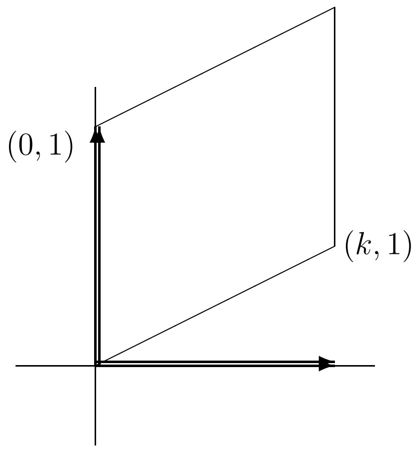



$(k > 0)$

$$\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$$

$T(1,0) = (1,0),\ T(0,1) = (k,1)$

Shear in $x-$direction with factor $k$




$(k > 0)$

$$\begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}$$

$T(1,0) = (1,k),\ T(0,1) = (0,1)$

Shear in $y-$direction with factor $k$

## 2.3 Standard Transformations in $\mathbb{R}^3$

We now consider some basic standard transformations in $\mathbb{R}^3$. A general description of transformations in $\mathbb{R}^3$ is complicated, we consider only some standard ones.

1. **Rotations in $\mathbb{R}^3$**

   For the sense of a rotation in $\mathbb{R}^3$ we use the right hand rule:
   If the rotation is in the direction of the fingers of your right hand, then the axis is along your thumb.

   Similarly, if the axis is along the thumb of your right hand, then the rotation will be in the direction of the fingers on your right hand.

   - **Rotation about the $x-$axis in $\mathbb{R}^3$**

     In this case the $yz-$plane is rotated by *theta*, and the $x-$coordinate is unchanged. We can thus use the rotation in $\mathbb{R}^2$ for the $yz$ part.

$$\boxed{[R_{x,\theta}] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}}$$

- **Rotation about the $y-$axis in $\mathbb{R}^3$**
  In this case the sense of the rotation is reversed, so it corresponds to a rotation in the $xz-$plane of $-\theta$.

$$\boxed{[R_{y,\theta}] = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}}$$

- **Rotation about the $z-$axis**
  This case is equivalent to a rotation in the $xy-$plane by $\theta$.

$$\boxed{[R_{z,\theta}] = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}}$$

For completeness we give the matrix for a generalized rotation about an axis given by a vector $\mathbf{u} = (a, b, c)$ by an angle $\theta$:

$$[R_{z,\theta}] = \begin{pmatrix} a^2(1-\cos\theta)+\cos\theta & ab(1-\cos\theta)-c\sin\theta & ac(1-\cos\theta)+b\sin\theta \\ ab(1-\cos\theta)+c\sin\theta & b^2(1-\cos\theta)+\cos\theta & bc(1-\cos\theta)-a\sin\theta \\ ac(1-\cos\theta)-b\sin\theta & bc(1-\cos\theta)+a\sin\theta & c^2(1-\cos\theta)+\cos\theta \end{pmatrix}$$

**Theorem 17** *If $T_1, T_2, \ldots, T_k$ is a succession of rotations about axes through the origin in $\mathbb{R}^3$, then the resulting mapping can be obtained by a single rotation about some suitable axis through the origin.*

2. **Reflections in $\mathbb{R}^3$**

- **Reflection about $xy-$plane**
  $F_{xy}(x, y, z) = (x, y, -z)$
  $F_{xy}(1, 0, 0) = (1, 0, 0)$, $F_{xy}(0, 1, 0) = (0, 1, 0)$, $F_{xy}(0, 0, 1) = (0, 0, -1)$.

$$\boxed{[F_{xy}] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}}$$

- **Reflection about $xz-$plane**
  $F_{xz}(x, y, z) = (x, -y, z)$
  $F_{xz}(1, 0, 0) = (1, 0, 0)$, $F_{xz}(0, -1, 0) = (0, 1, 0)$, $F_{xz}(0, 0, 1) = (0, 0, 1)$.

$$\boxed{[F_{xz}] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}$$

- **Reflection about $yz-$plane**
  $F_{yz}(x,y,z) = (-x,y,z)$
  $F_{yz}(1,0,0) = (-1,0,0),\ F_{yz}(0,1,0) = (0,1,0),\ F_{yz}(0,0,1) = (0,0,1).$

$$\boxed{[F_{yz}] = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}$$

3. **Projections in $\mathbb{R}^3$**

- **Projection onto the $xy-$plane**
  $P_{xy}(x,y,z) = (x,y,0)$
  $P_{xy}(1,0,0) = (1,0,0),\ P_{xy}(0,1,0) = (0,1,0),\ P_{xy}(0,0,1) = (0,0,0).$

$$\boxed{[P_{xy}] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}}$$

- **Projection onto the $xz-$plane**
  $P_{xz}(x,y,z) = (x,0,z)$
  $P_{xz}(1,0,0) = (1,0,0),\ P_{xz}(0,1,0) = (0,0,0),\ P_{xz}(0,0,1) = (0,0,1).$

$$\boxed{[P_{xz}] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}}$$

- **Projection onto the $yz-$plane**
  $P_{yz}(x,y,z) = (0,y,z)$
  $P_{yz}(0,0,0) = (1,0,0),\ P_{yz}(0,1,0) = (0,1,0),\ P_{yz}(0,0,1) = (0,0,1).$

$$\boxed{[P_{yz}] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}$$

# 3 1-1 and Onto Transformations

**Definition 18** *Given a transformation $T : \mathbb{R}^n \to \mathbb{R}^m$*

1. *$T$ is called <u>one to one</u> (1-1) or <u>injective</u> if For every $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, $T(\mathbf{u}) = T(\mathbf{v}) \Rightarrow \mathbf{u} = \mathbf{v}$. i.e. $T$ maps distinct vectors to distinct vectors.*

2. *$T$ is called <u>onto</u> or <u>surjective</u> if for all $\mathbf{y} \in \mathbb{R}^m$, there exists $\mathbf{x} \in \mathbb{R}^n$ such that $\mathbf{y} = T(\mathbf{x})$.*

3. *If $T$ is both one to one and onto (injective and surjective) it is called a a <u>bijection</u>.*

**Notes** Given a matrix transformation, $T_A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$:

- $T_A$ is one to one if and only if for every $\mathbf{b} \in \mathbb{R}^m$ there is a **unique** $\mathbf{x} \in \mathbb{R}^n$ such that $A\mathbf{x} = \mathbf{b}$.
  $A$ is 1-1 if and only if $A\mathbf{x} = \mathbf{b}$ has unique solution for every $\mathbf{b} \in \mathbb{R}^m$.
- $T_A$ is one to one if and only if $\ker(T_A) = \{\mathbf{0}\}$.
  $A$ is 1-1 if and only if $A\mathbf{x} = \mathbf{0}$ has only the trivial solution $\mathbf{x} = \mathbf{0}$.
- $T_A$ is onto if and only if $\operatorname{ran}(T_A) = \mathbb{R}^m$.
  $A$ is onto if and only if $A\mathbf{x} = \mathbf{b}$ is consistent for every $\mathbf{b}$ in $\mathbb{R}^m$.

Recall that the rank of a matrix $A$, $r(A)$, is the number of leading ones in the REF of $A$.

**Theorem 19** *Given an $m \times n$ matrix $A$, the corresponding matrix transformation $T_A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$*

- *$T_A$ is 1-1 if and only if*
$$r(A) = \text{ number of columns of } A.$$
- *$T_A$ is onto if and only if*
$$r(A) = \text{ number of rows of } A.$$

*Where $r(A)$ is the rank of $A$*

**Example 20**

1. Is $T_A$ 1-1 or onto, where $A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 0 & -1 \\ 3 & 2 & 1 \end{pmatrix}$?

$$\begin{pmatrix} 1 & 2 & 2 \\ 2 & 0 & -1 \\ 3 & 2 & 1 \end{pmatrix} \quad \begin{array}{l} R_2 \longrightarrow R_2 - 2R_1 \\ R_3 \longrightarrow R_3 - 3R_1 \end{array}$$
$$\begin{pmatrix} 1 & 2 & 2 \\ 0 & -4 & -5 \\ 0 & -4 & -5 \end{pmatrix} \quad R_3 \longrightarrow R_3 - R_2$$
$$\begin{pmatrix} 1 & 2 & 2 \\ 0 & -4 & -5 \\ 0 & 0 & 0 \end{pmatrix}$$

Infinite solutions, so $T_A$ is not 1-1. $\operatorname{ran}(T_A) \neq \mathbb{R}^3$, so no onto.

Alternately $r(A) = 2 < 3$. Thus $A$ is neither 1-1 nor onto.

2. Is $T_A$ 1-1 or onto, where $A = \begin{pmatrix} 1 & 2 & 1 \\ 3 & 4 & 3 \end{pmatrix}$?

$$\begin{pmatrix} 1 & 2 & 1 \\ 3 & 4 & 3 \end{pmatrix} \quad R_2 \longrightarrow R_2 - 3R_1$$
$$\begin{pmatrix} 1 & 2 & 2 \\ 0 & -4 & 0 \end{pmatrix}$$

So $r(A) = 2$. Thus $T_A$ is not 1-1. $T_A$, is onto.

3. Is $T_A$ 1-1 or onto, where $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$?

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \qquad R_3 \longrightarrow R_3 - R_1$$
$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} \qquad R_3 \longrightarrow R_3 - R_2$$
$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

So $r(A) = 2$. Thus $T_A$ is 1-1, but not onto.

# 4 Inverse Transformations

Given a transformation $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$, the inverse transformation, denoted $T^{-1}$, is the transformation which undoes the action of $T$. So for any $\mathbf{x} \in \mathrm{dom}(T)$, $T^{-1}(T(\mathbf{x})) = \mathbf{x}$.
Note that not every transformation has an inverase. If the inverse exists, the transformation is called *invertable*.
If $T$ is a matrix transformation, $T_A$, then $T_A$ is invertable if and only if $A$ is a square matrix and $A$ is invertible.
**Notes**

- If $T_A$ is invertible then it is both 1-1 and onto.
- If $A$ is a square matrix and $T_A$ is 1-1 then $A$ is both onto and invertible.
- If $T_A$ is invertible then $(T_A)^{-1} = T_{A^{-1}}$.

**Example 21** *Find the inverse transformation for dilation by a factor $k$ in $\mathbb{R}^3$. $T(x, y, z) = (kx, ky, kz)$, for some fixed $k \in \mathbb{R}$.*

This corresponds to the matrix transformation $T_A$, where

$$[T_A] = A = \begin{pmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{pmatrix}.$$

$$A^{-1} = \begin{pmatrix} \frac{1}{k} & 0 & 0 \\ 0 & \frac{1}{k} & 0 \\ 0 & 0 & \frac{1}{k} \end{pmatrix}.$$

Which gives the inverse transformation

# 5 Orthogonal Operators

Recall that a set of vectors $\{\mathbf{v}_i \mid 1 \leq i \leq n\}$ is *orthogonal* if $\mathbf{v}_i \cdot \mathbf{v}_j = 0$ whenever $i \neq j$ and *orthonormal* if $\mathbf{v}_i \cdot \mathbf{v}_j = \delta_{ij}$, where $\delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$ is the Kronecker delta. Note that $[\delta_{ij}] = I$. So an orthonormal set is an orthogonal set of unit vectors.

**Definition 22** *A linear transformation $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is* orthogonal *if $||T(\mathbf{x})|| = ||\mathbf{x}||$ for every $\mathbf{x} \in \mathbb{R}^n$.*

So orthogonal transformations preserve distances.

**Theorem 23** *A linear transformation $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is* orthogonal *(i.e. $||T(\mathbf{x})|| = ||\mathbf{x}||$ for every $\mathbf{x} \in \mathbb{R}^n$) if and only if*

$$T(\mathbf{x}) \cdot T(\mathbf{y}) = \mathbf{x} \cdot \mathbf{y} \text{ for all } \mathbf{x}, \mathbf{y} \in \mathbb{R}^n.$$

Now, since the angle $\theta$ between two vectors $\mathbf{u}$ and $\mathbf{v}$ is given by $\theta = \arccos\left(\frac{\mathbf{u} \cdot \mathbf{v}}{||\mathbf{u}||\,||\mathbf{v}||}\right)$. If $T$ is an orthogonal transformation, the angle between the images of $\mathbf{u}$ and $\mathbf{v}$, $T(\mathbf{u})$ and $T(\mathbf{v})$ is given by

$$\arccos\left(\frac{T(\mathbf{u}) \cdot T(\mathbf{v})}{||T(\mathbf{u})||\,||T(\mathbf{v})||}\right) = \arccos\left(\frac{\mathbf{u} \cdot \mathbf{v}}{||\mathbf{u}||\,||\mathbf{v}||}\right) = \theta$$

So orthogonal transformations preserve angles.

**Definition 24** *A square matrix $A$ is called orthogonal if and only if $A^{-1} = A^T$*

We want to be sure that these two definitions are compatible.

**Theorem 25** *Given a square matrix $A$ the following are equivalent:*

1. $AA^T = I$. ($A$ is an orthogonal matrix.)
2. $||A\mathbf{x}|| = ||\mathbf{x}||$ for all $\mathbf{x} \in \mathbb{R}^n$. ($T_A$ is an orthogonal transformation.)
3. $A\mathbf{x} \cdot A\mathbf{y} = \mathbf{x} \cdot \mathbf{y}$.
4. The column vectors of are orthonormal. (The column vectors of $A$ form an orthonormal basis of $\mathbb{R}^n$.)

**Proof:**

$1 \Rightarrow 2$ Suppose $A$ is orthogonal, so $AA^T = I$, and consider

$$||A\mathbf{x}||^2 = A\mathbf{x} \cdot A\mathbf{x} = \mathbf{x} \cdot A^T A\mathbf{x} = \mathbf{x} \cdot I\mathbf{x} = \mathbf{x} \cdot \mathbf{x} = ||\mathbf{x}||^2$$

$2 \Rightarrow 3$ This is Theorem 23.

$3 \Rightarrow 4$ Suppose $T_A$ is a map with $A\mathbf{x} \cdot A\mathbf{y} = \mathbf{x} \cdot \mathbf{y}$ for all $\mathbf{x} \in \mathbb{R}^n$. By Theorem 23 this means that $T_A$ preserves distances and angles. In particular an orthonormal set is mapped to an orthonormal set. Now, the columns of $A$ are given by

$$A = \begin{pmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) & \dots & T(\mathbf{e}_n) \end{pmatrix}.$$

$\{\mathbf{e}_i \mid 1 \le i \le n\}$ are an orthonormal set of vectors, ans thus so are the columns of $A$.

$4 \Rightarrow 1$ Suppose that $A$ is a matrix whose column vectors, $\mathbf{c}_1, \mathbf{c}_2, \dots \mathbf{c}_n$, are orthonormal, so $\mathbf{c}_i \cdot \mathbf{c}_j = \delta_{ij}$, where $\delta_{ij} = \begin{cases} 0 & i \ne j \\ 1 & i = j \end{cases}$. Now, the rows of $A^T$ are the columns of $A$, so

$$A^T A = [\mathbf{c}_i \cdot \mathbf{c}_j] = [\delta_{ij}] = I \qquad \square$$

In particular we have the following

**Theorem 26** *A transformation $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is orthogonal if and only if its corresponding matrix $[T]$ is orthogonal, i.e. $[T]^{-1} = [T]^T$*

The transformations which preserve distances and angles are rotations and reflections, so orthogonal transformation are exactly these. In fact in higher dimensions we *define* rotations to be those transformations $T$ for which $\det([T]) = 1$.

**Definition 27** *A square matrix $A$ represents a* rotation *if and only if det$(A) = 1$.*

**Theorem 28** *If $A$ and $B$ are orthogonal matrices, then*

1. $AB$ is orthogonal.
2. $A^{-1}$ is orthogonal.
3. $A^T$ is orthogonal.

**Proof:** Let $A$ and $B$ be orthogonal matrices, so $A^T = A^{-1}$ and $AA^T = BB^T = I$.

**1**
$$(AB)(AB)^T = ABB^TA^T = AIA^T = AA^T = I.$$

**2**
$$\left(A^{-1}\right)^{-1} = \left(A^T\right)^{-1} = \left(A^{-1}\right)^T$$

**3**
$$\left(A^T\right)^{-1} = \left(A^{-1}\right)^{-1} = A = \left(A^T\right)^T$$

**Example 29** *Show that the matrix $A$ given below is orthogonal.*

$$A = \frac{1}{4}\begin{pmatrix} 2 & -3 & \sqrt{3} \\ 2\sqrt{3} & \sqrt{3} & -1 \\ 0 & 2 & 2\sqrt{3} \end{pmatrix}$$

$$\begin{aligned} AA^T &= \frac{1}{16}\begin{pmatrix} 2 & -3 & \sqrt{3} \\ 2\sqrt{3} & \sqrt{3} & -1 \\ 0 & 2 & 2\sqrt{3} \end{pmatrix}\begin{pmatrix} 2 & 2\sqrt{3} & 0 \\ -3 & \sqrt{3} & 2 \\ \sqrt{3} & -1 & 2\sqrt{3} \end{pmatrix} \\ &= \frac{1}{16}\begin{pmatrix} 2^2 + (-3)^2 + \left(\sqrt{3}\right)^2 & 4\sqrt{3} - 3\sqrt{3} - \sqrt{3} & -6 + 2\left(\sqrt{3}\right)^2 \\ 4\sqrt{3} - 3\sqrt{3} - \sqrt{3} & 4\left(\sqrt{3}\right)^2 + \left(\sqrt{3}\right)^2 + 1 & 2\sqrt{3} - 2\sqrt{3} \\ -6 + +2\left(\sqrt{3}\right)^2 & 2\sqrt{3} - 2\sqrt{3} & 4 + 4\left(\sqrt{3}\right)^2 \end{pmatrix} \\ &= \frac{1}{16}\begin{pmatrix} 16 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 16 \end{pmatrix} \\ &= I \end{aligned}$$

The column vectors of $A$ are

$$\mathbf{c}_1 = (2, 2\sqrt{3}, 0) \quad \mathbf{c}_2 = (-3, \sqrt{3}, 2) \quad \mathbf{c}_3 = (\sqrt{3}, -1, 2\sqrt{3})$$

Note that $\mathbf{c}_i \cdot \mathbf{c}_j = \delta_{ij}$.