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Introduction

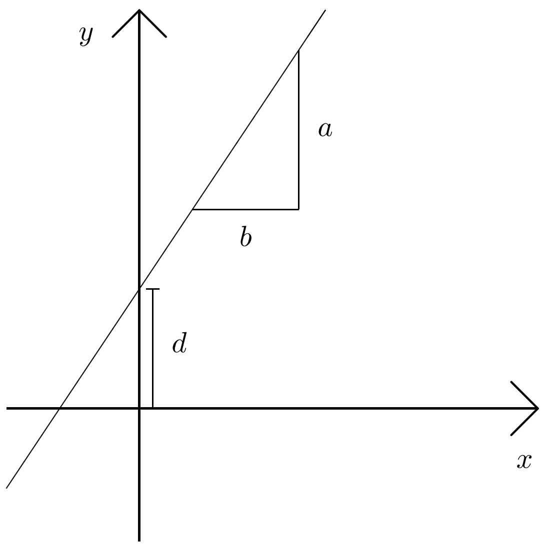
P. Danziger

1 Linear Algebra

Linear	–	of line or line like
Algebra	–	Manipulation, Solution or Transformation

Thus Linear Algebra is about the Manipulation, Solution and Transformation of 'line like' objects.
We will also investigate the spaces in which they live.
What we consider a 'line' will be driven by algebraic considerations and may deviate significantly from your current notion of what a line is.
Throughout this course there are generally two ways of looking at the objects we are considering, Geometrically (as a picture) or Algebraically (as equations).
Indeed, much of what we will be doing is to take our geometric understanding, translate that to an algebraic form and to generalise that form.
However, it is often useful to refer back to the geometric underpinnings from time to time.

1.1 Lines in the Plane (2D)



Any line in the plane can be written as $y = mx + d$ (*);

Here $m = \frac{a}{b}$ is the slope;
d is the y intercept.

The line is defined as the set of *points* (x, y) which satisfy the equation (*).

There are some problems with this approach:
If the line is horizontal $m = 0$ (which is OK)
but
If the line is vertical m is undefined (which is not)
Another problem is that it is not clear how to generalise this to 3, or more, dimensions.

1.2 Linear Equations

We rearrange $y = mx + d$ (recalling $m = \frac{a}{b}$ and redefining a to $-a$ for convenience) to get

$$ax + by = c$$

Now $m = -\frac{a}{b}$ and $d = \frac{c}{b}$.
This is called a *linear equation in two variables*
x and y are called variables or unknowns and
a, b and c are called constants.
An equation is called *linear* if there are no higher order terms in the unknowns, like x^2 or y^3 or xy.
Now horizontal lines have $a = 0$ and vertical lines have $b = 0$.
We call the set of all possible pairs (x, y) (the plane) $\mathbb{R}^2$. i.e.

$$\mathbb{R}^2 = \{(x, y) \mid x, y \in \mathbb{R}\}$$

1.3 Generalising to Higher Dimension

We can easily generalise this form to higher dimensions:

$$ax + by + cz = d \quad \text{(3D)}$$

The set of points (x, y, z) which satisfy this equation in fact defines a *plane* in 3 dimensions.
We call the set of all possible triples (x, y, z) $\mathbb{R}^3$.
Or given n variables x_1, $x_2, \ldots, x_n$

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = b$$

The set of points $(x_1, x_2, \ldots, x_n)$ which satisfy this equation in fact defines an $n-1$ dimensional object called a *hyperplane* in $\mathbb{R}^n$, where

$$\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) \mid x_i \in \mathbb{R} \text{ for each } 1 \leq i \leq n\}.$$

We usually think of a line as a 1 dimensional object, but with this generalisation we view it as a $2 - 1 = 1$ dimensional object in the plane.

1.4 A Small Problem

There is still a small problem with this notation, but it is not as serious.
Given any number $k \in \mathbb{R}$, the equations

$$ax + by = c \tag{1}$$

and

$$kax + kby = kc \tag{2}$$

represent the same line since any point (x, y) that satisfies 1 also satisfies 2.
1 and 2 are called linear multiples of each other.
Thus in this form, we may have more than one equation representing the same line.

Similarly in 3 dimensions:

$$ax + by + cz = d \text{ and } kax + kby + kcz = kd$$

represent the same plane since any point (x, y, z) that satisfies one equation also satisfies the other.
In n dimensions

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = b$$

and

$$ka_1x_1 + ka_2x_2 + \ldots + ka_nx_n = kb$$

represent the same hyperplane since any point $(x_1, x_2, \ldots, x_n)$ that satisfies one also satisfies the other.
As with the 2 dimensional case the two equations are called *linear multiples* of each other.

2 Parametric and Vector Forms

2.1 Equations in 2 Variables

Given an equation of the form

$$ax + by = c$$

We may solve in terms of a new variable t.
let $t \in \mathbb{R}$, set $x = t$, $y = \frac{c-at}{b}$.
Thus x and y are both expressed in terms of a single variable t. t is called a *parameter* and tells us how far along the line we are.
A line has 'one degree of freedom' and hence is described by one parameter.
We could instead of taken $x = 2t$, $y = \frac{c-2at}{b}$. This is called a reparameterization.
In the second parameterisation we 'move along' the line twice as fast.
We can also write this solution as $(x, y) = (2t, \frac{c-2at}{b})$, or more properly $(x, y) \in \{(2t, \frac{c-2at}{b}) \mid t \in \mathbb{R}\}$.
We can also express the solution $(x, y) = (2t, \frac{c-2at}{b})$ in terms of vectors $(x, y) = (0, \frac{c}{b}) + t(2, \frac{-2a}{b})$.
Here $(0, \frac{c}{b})$ is the constant part and is a point on the line, $(2, \frac{-2a}{b})$ is the coefficient of t and is a a vector parallel to the line, t is a free parameter (a scalar).

Example 1
Express $x + 2y = 1$ in parametric and vector form
Let $t \in \mathbb{R}$ and set $y = t$, then $x = 1 - 2t$.
In vector form $(x, y) = (1 - 2t, t) = (1, 0) + t(-2, 1)$.

2.2 Equations in 3 Unknowns

A three dimensional linear equation

$$ax + by + cz = d$$

represents a plane in three dimensions.
In 2D a line is a $2 - 1 = 1$ dimensional object in 2-space; in 3D a linear equation represents a $3 - 1 = 2$ dimensional object in 3-space.

We can 'solve' this equation with 2 parameters: Let $s, t \in \mathbb{R}$, $y = s$, $z = t$, $x = d - bs - ct$.
This is called a *2-parameter solution*.
We can also write this as

$$\begin{aligned}(x, y, z) &= (d - bs - ct, s, t) \\ &= (d, 0, 0) + s(-b, 1, 0) + t(-c, 0, 1)\end{aligned}$$

or more properly

$$(x, y, z) \in \{(d - bs - ct, s, t) \mid s, t \in \mathbb{R}\}$$

Example 2

Find $x + 2y - z = 3$ in parametric and vector form.
Let $s, t \in \mathbb{R}$ and set $y = s$ and $z = t$, then $x = 3 - 2s + t$.
In vector form:

$$\begin{aligned}(x, y, z) &= (1 - 2s + t, s, t) \\ &= (1, 0, 0) + s(-2, 1, 0) + t(1, 0, 1).\end{aligned}$$

In this example, $(1, 0, 0)$ is a point on the plane, and $(-2, 1, 0)$ and $(1, 0, 1)$ are two vectors parallel to the plane (but not parallel to each other).