1.2

Gaussian Elimination

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1 m Equations in n Unknowns

Given n variables $x_1, x_2, \ldots, x_n$ and $n+1$ constants $a_1, a_2, \ldots, a_n, b$ the equation

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = b$$

represents an $n-1$ dimensional object in n-space, called a hyperplane.
We want to consider the situation where we have m such equations

$$\begin{array}{rcl} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n & = & b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n & = & b_2 \\ \vdots & & \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n & = & b_m \end{array}$$

This is called a system of m (linear) equations in n unknowns (or variables).
We want to find solutions of this system of equations.

Theorem 1 *Given a system of m equations in n unknowns:*

- *If $m < n$ then the number of parameters in the solution will be* at least *$n - m$.*

  (Thus if there is a unique solution we must have $m \geq n$.)

- *If $m > n$ the system is called* overprescribed*.*

  Overprescribed systems either have no solution or they contain reduncancy. redundancy means that we can find $(m - n)$ equations which can be dropped without affecting the solution.

If a system of equations has no solution it is called *inconsistent*
If a system of equations has at least one solution it is called *consistent*

1.1 Coefficient Matrices and Augmented Matrices

The x_i actually carry no information, the system is completely described by the a_{ij} and b_i, $i = 1, \ldots m$, $j = 1, \ldots, n$.
We thus use the *matrix of coefficients*, wich is an $m \times n$ array containing the coefficients of the equations.

$$\begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{pmatrix}$$

We also have the *Augmented Matrix*, which includes the b_i on the right:

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array}\right)$$

The augmented matrix contains all the information necessary to solve the system.

1. Find the matrix of coefficients and the augmented matrix for the following system.

$$\begin{array}{ccccccc} x & + & 2y & - & 3z & = & 1 \\ & + & y & + & z & = & 1 \\ x & + & y & + & z & = & 0 \end{array}$$

   This system of equations has coefficient matrix:

$$\left(\begin{array}{ccc} 1 & 2 & -3 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{array}\right)$$

   and Augmented matrix:

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{array}\right)$$

2. Find the augmented matrix for the following system.

$$\begin{array}{ccccccc} x & + & & - & 2z & = & 1 \\ & + & y & - & z & = & 0 \end{array}$$

   This system of equations has Augmented matrix:

$$\left(\begin{array}{ccc|c} 1 & 0 & -2 & 1 \\ 0 & 1 & -1 & 0 \end{array}\right)$$

3. Given the following augmented matrix find the original system of equations.

$$\left(\begin{array}{cc|c} 1 & 2 & -3 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{array}\right)$$

   The system is

$$\begin{array}{ccccc} x & + & 2y & = & -3 \\ & & y & = & 1 \\ x & + & y & = & 0 \end{array}$$

This is a system of 3 equations in 2 unknowns.
It is inconsistent (no solution), since by the second equation $y = 1$, the third equation then tells us that $x = -1$, but then the first equation states (substituting in $x = -1$ and $y = 1$): $-1 + 2 = 3$, which is not true.
Note that each ow of the augmented matrix corresponds to one of the original equations.
Each column contains the all the coefficients of a given variable in the system. We say that this column *corresponds* to this variable.

Example 2

$$\begin{array}{rcrcr} x & + & 2y & = & -3 \\ & & y & = & 1 \\ x & + & y & = & 0 \end{array} \qquad \left(\begin{array}{cc|c} 1 & 2 & -3 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{array}\right)$$

The first row corresponds to x, the second corresponds to y and the third corresponds to the constants.

2 Elementary Row Operations

There are three basic operations we can preform on equations, these correspond to *Row Operations* on the corresponding matrices.

1. We can multiply an equation by a constant ≡ Multiply a row by a constant.
2. Add a multiple of one equation to another ≡ replace a row by itself plus a multiple of another row.
3. Interchange the order of equations ≡ Interchange two rows.

Notation We generally denote the i^{th} row of the matrix by R_i. Let c be a constant, and $1 \leq i, j \leq m$ then
$R_i \to R_i + cR_j$ means replace Row i by row i plus c times row j.
$R_i \to cR_i$ means replace row i with c times row i.
$R_i \leftrightarrow R_j$ means interchange row i with row j.
Note that preforming any of these operations does not change the solution to the original system of equations.
When using row operations always indicate the operation you have used!

Example 3

1.

$$\left(\begin{array}{ccc|c} 1 & 1 & 3 & 3 \\ 2 & 2 & 3 & 3 \\ 1 & 1 & 1 & 1 \end{array}\right) \quad \begin{array}{rcl} R_2 & \to & R_2 - 2R_1 \\ R_3 & \to & R_3 - R_1 \end{array} \quad \longrightarrow \quad \left(\begin{array}{ccc|c} 1 & 1 & 3 & 3 \\ 0 & 0 & -3 & -3 \\ 0 & 0 & -2 & -2 \end{array}\right)$$

2.

$$\left(\begin{array}{ccc|c} 1 & 1 & 3 & 3 \\ 2 & 2 & 3 & 3 \\ 1 & 1 & 1 & 1 \end{array}\right) \quad R_1 \leftrightarrow R_2 \longrightarrow \left(\begin{array}{ccc|c} 2 & 2 & 3 & 3 \\ 1 & 1 & 3 & 3 \\ 1 & 1 & 1 & 1 \end{array}\right)$$

3.

$$\left(\begin{array}{ccc|c} 1 & 1 & 3 & 3 \\ 2 & 2 & 3 & 3 \\ 1 & 1 & 1 & 1 \end{array}\right) \quad R_2 \rightarrow 2R_2 \longrightarrow \left(\begin{array}{ccc|c} 2 & 2 & 3 & 3 \\ 4 & 4 & 6 & 6 \\ 1 & 1 & 1 & 1 \end{array}\right)$$

Never operate on the same row twice in one step.

3 Row Echelon Form

Definition 4 *1. A matrix is in* <u>*Row Echelon Form*</u> *(REF) if all of the following hold:*

(a) Any rows consisting entirely of 0's appear at the bottom.

(b) In any non-zero row the first number, from the left, is a one. Called the leading one *or* pivot*.*

(c) In any two successive non-zero rows the leading one on top is to the left of the one on the bottom.

2. A matrix is in <u>*Reduced Row Echelon Form*</u> *(RREF) if it is in REF (all of the above hold) and any column containing a leading one is zero in all other entries.*

Example 5

1. The following are in REF

$$\begin{pmatrix} \boxed{1} & 1 & 3 \\ 0 & \boxed{1} & 1 \\ 0 & 0 & \boxed{1} \end{pmatrix} \quad \begin{pmatrix} 0 & \boxed{1} & 3 & 3 \\ 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \boxed{1} & 1 & 0 \\ 0 & 0 & \boxed{1} \end{pmatrix} \quad \begin{pmatrix} \boxed{1} & 2 \\ 0 & \boxed{1} \\ 0 & 0 \end{pmatrix}$$

$\boxed{1}$ indicates a pivot.

2. The following are **NOT** in REF

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 & 3 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 3 \\ 3 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$$

3. The following are in RREF

$$\begin{pmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{1} \end{pmatrix} \quad \begin{pmatrix} 0 & \boxed{1} & 3 & 0 \\ 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \boxed{1} & 2 & 0 \\ 0 & 0 & \boxed{1} \end{pmatrix} \quad \begin{pmatrix} \boxed{1} & 0 \\ 0 & \boxed{1} \\ 0 & 0 \end{pmatrix}$$

$\boxed{1}$ indicates a pivot. All of the 0's in these examples are forced.

4. The following are **NOT** in RREF

$$\begin{pmatrix} \boxed{1} & 0 & 2 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{1} \end{pmatrix} \quad \begin{pmatrix} 0 & \boxed{1} & 3 & 0 \\ 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} \boxed{1} & 2 & 3 \\ 0 & 0 & \boxed{1} \end{pmatrix} \quad \begin{pmatrix} \boxed{1} & 2 \\ 0 & \boxed{1} \\ 0 & 0 \end{pmatrix}$$

4 The Gaussian Algorithm

The following Algorithm reduces an $m \times n$ matrix to REF by means of elementary row operations alone.

1. For Each row i (R_i) from 1 to m
   (a) If any row j below row i has non zero entries to the left of the first non zero entry in row i exchange row i and j ($R_i \leftrightarrow R_j$) [Ensure We are working on the leftmost nonzero entry.]
   (b) Preform $R_i \to \frac{1}{c} R_i$ where $c =$ the first non-zero entry of row i. [This ensures that row i starts with a one.]
   (c) For each row j (R_j) below row i (Each $j > i$)
      i. Preform $R_j \to R_j - dR_i$ where
         $d =$ the entry in row j which is directly below the pivot in row i. [This ensures that row j has a 0 below the pivot of row i.]
   (d) If any 0 rows have appeared exchange them to the bottom of the matrix.

5 The Gaussian-Jordan Algorithm

The following Algorithm reduces an $n \times m$ matrix to RREF by means of elementary row operations alone.

1. Preform Gaussian elimination to get the matrix in REF
2. For each non zero row i (R_i) from n to 1 (bottom to top)
   (a) For each row j (R_j) above row i (Each $j < i$)
      i. Preform $R_j \to R_j - bR_i$ where
         $b =$ the value in row j directly above the pivot in row i. [This ensures that row j has a zero above the pivot in row i]

5.1 Gaussian Elimination

To Solve a system of equations we preform the following steps:

1. Translate the system to its augmented matrix A.
2. Use Gaussian elimination to reduce A to REF. Note that the REF form of A has the same solution set.
3. For each column which does **not** contain a pivot introduce a parameter and set the corresponding variable equal to that parameter.
4. Substitute the parameters back into the remaining non zero equations, this will produce a solution for the remaining variables.

The number of pivots in the REF of a matrix A is called the *rank of* A and is denoted by r or $r(A)$. Note that the number of parameters in the solution is equal to $n - r$.

Example 6 *Solve the following system of equations.*

$$\begin{array}{ccccccc} x_1 & + & 2x_2 & + & x_3 & = & 3 \\ x_1 & + & 3x_2 & + & 2x_3 & = & 5 \\ & & 2x_2 & + & x_3 & = & 6 \end{array}$$

Row reduce augmented matrix to REF

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 1 & 3 & 2 & 5 \\ 0 & 2 & 1 & 6 \end{array}\right) \quad R_2 \to R_2 - R_1$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 2 & 1 & 6 \end{array}\right) \quad R_3 \to R_3 - 2R_2$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & -2 \end{array}\right)$$

For Gaussian elimination use back substitution:

$$\begin{array}{rcll} x_1 + 2x_2 + x_3 & = & 3 & (1) \\ x_2 + x_3 & = & 2 & (2) \\ x_3 & = & -2 & (3) \end{array}$$

From (3) $x_3 = -2$,
From (2) $x_2 = 2 - x_3 = 2 - (-2) = 4$ and
From (1) $x_1 = 3 - 2x_2 - x_3 = 3 - 2(4) - (-2) = -3$.

5.2 Gaussian-Jordan

Instead of using back substitution as in Gaussian elimination, we can continue reducing until A is in RREF.
As before, for each column which does **not** contain a pivot introduce a parameter and set the corresponding variable equal to that parameter.
But now we may read off the other variables with no further work.

Example 7 *Solve the following system of equations.*

$$\begin{array}{rcrcrcl} x_1 & + & x_2 & + & 3x_3 & = & 3 \\ 2x_1 & + & 2x_2 & + & 3x_3 & = & 3 \\ x_1 & + & x_2 & + & x_3 & = & 1 \end{array}$$

We write out the Augmented matrix and use Gaussian-Jordan to reduce it to RREF.

$$\left(\begin{array}{ccc|c} 1 & 1 & 3 & 3 \\ 2 & 2 & 3 & 3 \\ 1 & 1 & 1 & 1 \end{array}\right) \quad \begin{array}{lcl} R_2 & \to & R_2 - 2R_1 \\ R_3 & \to & R_3 - R_1 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 3 & 3 \\ 0 & 0 & -3 & -3 \\ 0 & 0 & -2 & -2 \end{array}\right) \quad R_2 \to -\tfrac{1}{3}R_2$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 3 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -2 & -2 \end{array}\right) \quad R_3 \to R_3 + 2R_2$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 3 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad R_1 \to R_1 - 3R_2$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

We let the variable corresponding to the column not containing a pivot (the second column which corresponds to x_2) be the free variable.
Let $t \in \mathbb{R}$, set $x_2 = t$, then $x_3 = 1$ (from row 2) and
$x_1 = -x_2 = -t$ (from row 1).
Or $(x_1, x_2, x_3) = (-t, t, 1)$

Example 8 *Solve the following system of equations.*

$$\begin{array}{rcl} x_1 + 3x_2 - 2x_3 + 2x_5 & = & 0 \\ 2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 & = & -1 \\ 5x_3 + 10x_4 + 15x_6 & = & 5 \\ 2x_1 + 6x_2 + 8x_4 + 4x_5 + 18x_6 & = & 6 \end{array}$$

The Augmented Matrix is:

$$\left(\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 2 & 6 & 0 & 8 & 4 & 18 & 6 \end{array}\right)$$

First leading 1 is in the 1,1 position, already 1.
Get all 0's below this leading 1 position.

$$\begin{array}{c} R_2 \longrightarrow R_2 - 2R_1 \\ R_4 \longrightarrow R_4 - 2R_1 \end{array} \left(\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{array}\right)$$

Get leading 1 in second row.

$$R_2 \longrightarrow -R_2 \left(\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{array}\right)$$

Get all 0's below second leading 1.

$$\begin{array}{c} R_3 \longrightarrow R_3 - 5R_2 \\ R_4 \longrightarrow R_4 - 4R_2 \end{array} \left(\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 2 \end{array}\right)$$

Move row of 0's to bottom:

$$R_3 \leftrightarrow R_4 \left(\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 6 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Get next leading 1.

$$R_3 \longrightarrow \frac{1}{6}R_3 \left(\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Matrix is now in Row Echelon Form.

Gauss Elimination

We now use back substitution. The Matrix translates to the following system of equations:

$$\begin{array}{rcl} x_1 + 3x_2 - 2x_3 + 2x_5 & = & 0 \\ x_3 + 2x_4 + 3x_6 & = & 1 \\ x_6 & = & \frac{1}{3} \end{array}$$

For each variable corresponding to a column not containing a leading 1, we assign a free variable.
Let $s, t, r \in \mathbb{R}$.
Let $x_2 = s, x_4 = t, x_5 = r$.
Then the equations imply: $x_6 = \frac{1}{3}$
$x_3 = 1 - 2x_4 - 3x_6 = 1 - 2t - 1 = -2t$ So $x_3 = -2t$.
$x_1 = -3x_2 + 2x_3 - 2x_5 = -3s + 2(-2t) - 2r$. So $x_1 = -3s - 4t - 2r$.
Thus the final solution is:
$(x_1, x_2, x_3, x_4, x_5, x_6) = (-3s - 4t - 2r, s, -2t, t, r, \frac{1}{3})$

Gauss-Jordan

We continue the algorithm to get the matrix in Reduced Row Echelon Form.
Get 0's above rightmost leading 1 (in column 6).

$$R_2 \longrightarrow R_2 - 3R_3 \left(\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Get 0's above next leading 1 (in column 3).

$$R_1 \longrightarrow R_1 + 2R_2 \left(\begin{array}{cccccc|c} 1 & 3 & 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

The Matrix is now in Reduced Row Echelon Form.
The Matrix translates to the following system of equations:

$$\begin{array}{rcl} x_1 + 3x_2 + 4x_4 + 2x_5 & = & 0 \\ x_3 + 2x_4 & = & 0 \\ x_6 & = & \frac{1}{3} \end{array}$$

For each variable corresponding to a column not containing a leading 1, we assign a free variable.
Let $s, t, r \in \mathbb{R}$.
Let $x_2 = s, x_4 = t, x_5 = r$.
Then the matrix implies: $x_6 = \frac{1}{3}$
$x_3 = -2t$
$x_1 = -3x_2 - 4x_4 - 2x_5 = -3s - 4t - 2r$.
Thus the final solution is
$(x_1, x_2, x_3, x_4, x_5, x_6) = (-3s - 4t - 2r, s, -2t, t, r, \frac{1}{3})$.