Inverting Matrices

P. Danziger

1 Properties of Transpose

Transpose has higher precedence than multiplication and addition, so

$$AB^T = A\left(B^T\right) \text{ and } A + B^T = A + \left(B^T\right)$$

As opposed to the bracketed expressions

$$(AB)^T \text{ and } (A+B)^T$$

Example 1
Let $A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 5 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$.
Find AB^T, and $(AB)^T$.

$$AB^T = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 5 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}^T = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 5 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 3 \\ 4 & 7 \end{pmatrix}$$

Whereas $(AB)^T$ is undefined.

Theorem 2 (Properties of Transpose) *Given matrices A and B so that the operations can be preformed*

1. $(A^T)^T = A$
2. $(A+B)^T = A^T + B^T$ *and* $(A-B)^T = A^T - B^T$
3. $(kA)^T = kA^T$
4. $(AB)^T = B^T A^T$

2 Matrix Algebra

Theorem 3 (Algebraic Properties of Matrix Multiplication)

1. $(k+\ell)A = kA + \ell A$ (Distributivity of scalar multiplication I)
2. $k(A+B) = kA + kB$ (Distributivity of scalar multiplication II)

3. $A(B+C) = AB + AC$ (Distributivity of matrix multiplication)
4. $A(BC) = (AB)C$ (Associativity of matrix multiplication)
5. $A + B = B + A$ (Commutativity of matrix addition)
6. $(A + B) + C = A + (B + C)$ (Associativity of matrix addition)
7. $k(AB) = A(kB)$ (Commutativity of Scalar Multiplication)

The matrix 0 is the identity of matrix addition. That is, given a matrix A,

$$A + 0 = 0 + A = A.$$

Further $0A = A0 = 0$, where 0 is the appropriately sized 0 matrix.
Note that it is possible to have two non-zero matrices which multiply to 0.

Example 4

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1-1 & 1-1 \\ -1+1 & -1+1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

The matrix I is the identity of matrix multiplication. That is, given an $m \times n$ matrix A,

$$AI_n = I_m A = A$$

Theorem 5 *If R is in reduced row echelon form then either $R = I$, or R has a row of zeros.*

Theorem 6 (Power Laws) *For any square matrix A,*

$$A^r A^s = A^{r+s} \text{ and } (A^r)^s = A^{rs}$$

Example 7

1. $$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 2 & 2 & 0 \end{pmatrix}^4 = \left(\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 2 & 2 & 0 \end{pmatrix}^2 \right)^2$$

2. Find A^6, where
$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$
$A^6 = A^2 A^4 = A^2 \left(A^2\right)^2$. Now $A^2 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$, so
$$A^2 \left(A^2\right)^2 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix}$$

3 Inverse of a matrix

Given a square matrix A, the *inverse of A*, denoted A^{-1}, is defined to be the matrix such that

$$AA^{-1} = A^{-1}A = I$$

Note that inverses are only defined for square matrices
Note Not all matrices have inverses.
If A has an inverse, it is called *invertible*.
If A is not invertible it is called *singular*.

Example 8

1. $A = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}$ $\qquad A^{-1} = \begin{pmatrix} 5 & -2 \\ -2 & 1 \end{pmatrix}$

   Check: $\begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}\begin{pmatrix} 5 & -2 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

2. $A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$ $\qquad$ Has no inverse

3. $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$ $\qquad A^{-1} = \begin{pmatrix} 3 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$

   Check: $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}\begin{pmatrix} 3 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

4. $A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \\ 3 & 3 & 4 \end{pmatrix}$ $\qquad$ Has no inverse

3.1 Inverses of 2×2 Matrices

Given a 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

A is invertible if and only if $ad - bc \neq 0$ and

$$A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

The quantity $ad - bc$ is called the *determinant* of the matrix and is written $\det(A)$, or $|A|$.

Example 9

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 3 \end{pmatrix} \qquad A^{-1} = \tfrac{1}{-3}\begin{pmatrix} 3 & -2 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} -1 & \frac{2}{3} \\ 1 & -\frac{1}{3} \end{pmatrix}$$

Check: $\tfrac{1}{-3}\begin{pmatrix} 3 & -2 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 3 \end{pmatrix} = -\tfrac{1}{3}\begin{pmatrix} -3 & 0 \\ 0 & -3 \end{pmatrix} = I$

4 Algebra of Invertibility

Theorem 10 *Given an invertible matrix A:*

1. $(A^{-1})^{-1} = A$,
2. $(A^n)^{-1} = (A^{-1})^n \quad (= A^{-n})$,
3. $(kA)^{-1} = \frac{1}{k}A^{-1}$,
4. $(A^T)^{-1} = (A^{-1})^T$,

Theorem 11 *Given two invertible matrices A and B*

$$(AB)^{-1} = B^{-1}A^{-1}.$$

Proof: Let A and B be invertible matricies and let $C = AB$, so $C^{-1} = (AB)^{-1}$.
Consider $C = AB$.
Multiply both sides on the left by A^{-1}:

$$A^{-1}C = A^{-1}AB = B.$$

Multiply both sides on the left by B^{-1}.

$$B^{-1}A^{-1}C = B^{-1}B = I.$$

So, $B^{-1}A^{-1}$ is the matrix you need to multiply C by to get the identity.
Thus, by the definition of inverse

$$B^{-1}A^{-1} = C^{-1} = (AB)^{-1}.$$

5 A Method for Inverses

Given a square matrix A and a vector $\mathbf{b} \in \mathbb{R}^n$, consider the equation

$$A\mathbf{x} = \mathbf{b}$$

This represents a system of equations with coefficient matrix A.
Multiply both sides by A^{-1} on the left, to get

$$A^{-1}A\mathbf{x} = A^{-1}\mathbf{b}.$$

But $A^{-1}A = I_n$ and $I\mathbf{x} = \mathbf{x}$, so we have

$$\mathbf{x} = A^{-1}\mathbf{b}.$$

Note that we have a unique solution. The assumption that A is invertible is equaivalent to the assumption that $A\mathbf{x} = \mathbf{b}$ has unique solution.
During the course of Gauss-Jordan elimination on the augmented matrix $(A|\mathbf{b})$ we reduce $A \to I$ and $\mathbf{b} \to A^{-1}\mathbf{b}$, so $(A|\mathbf{b}) \to (I|A^{-1}\mathbf{b})$.
If we instead augment A with I, row reducing will produce (hopefully) I on the left and A^{-1} on the right, so $(A|I) \to (I|A^{-1})$.
The Method:

1. Augment A with I
2. Use Gauss-Jordan to obtain $(I|A^{-1})$.
3. If I does not appear on the left, A is not invertable.
   Otherwise, A^{-1} is given on the right.

Example 12

1. Find A^{-1}, where

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 5 \\ 3 & 5 & 8 \end{pmatrix}$$

Augment with I and row reduce:

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 5 & 0 & 1 & 0 \\ 3 & 5 & 8 & 0 & 0 & 1 \end{array}\right) \quad \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -1 & -2 & 1 & 0 \\ 0 & -1 & -1 & -3 & 0 & 1 \end{array}\right) \quad R_3 \to R_3 + R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -1 & -2 & 1 & 0 \\ 0 & 0 & -2 & -5 & 1 & 1 \end{array}\right) \quad R_3 \to -\tfrac{1}{2}R_3$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -1 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5/2 & -1/2 & -1/2 \end{array}\right) \quad \begin{array}{l} R_1 \to R_1 - 3R_3 \\ R_2 \to R_2 + R_3 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 0 & -13/2 & 3/2 & 3/2 \\ 0 & 1 & 0 & 1/2 & 1/2 & -1/2 \\ 0 & 0 & 1 & 5/2 & -1/2 & -1/2 \end{array}\right) \quad R_1 \to R_1 - 2R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -15/2 & 1/2 & 5/2 \\ 0 & 1 & 0 & 1/2 & 1/2 & -1/2 \\ 0 & 0 & 1 & 5/2 & -1/2 & -1/2 \end{array}\right)$$

So

$$A^{-1} = \frac{1}{2}\begin{pmatrix} -15 & 1 & 5 \\ 1 & 1 & -1 \\ 5 & -1 & -1 \end{pmatrix}$$

To check inverse multiply together:

$$\begin{aligned} AA^{-1} &= \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 5 \\ 3 & 5 & 8 \end{pmatrix} \tfrac{1}{2} \begin{pmatrix} -15 & 1 & 5 \\ 1 & 1 & -1 \\ 5 & -1 & -1 \end{pmatrix} \\ &= \tfrac{1}{2}\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} = I \end{aligned}$$

2. Solve $A\mathbf{x} = \mathbf{b}$ in the case where $\mathbf{b} = (2, 2, 4)^T$.

$$\begin{aligned} \mathbf{x} &= A^{-1}\mathbf{b} = \tfrac{1}{2}\begin{pmatrix} -15 & 1 & 5 \\ 1 & 1 & -1 \\ 5 & -1 & -1 \end{pmatrix}\begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix} \\ &= \tfrac{1}{2}\begin{pmatrix} -18 \\ 0 \\ 4 \end{pmatrix} = \begin{pmatrix} -9 \\ 0 \\ 2 \end{pmatrix} \end{aligned}$$

3. Solve $A\mathbf{x} = \mathbf{b}$ in the case where $\mathbf{b} = (2, 0, 2)^T$.

$$\begin{aligned} \mathbf{x} &= A^{-1}\mathbf{b} = \tfrac{1}{2}\begin{pmatrix} -15 & 1 & 5 \\ 1 & 1 & -1 \\ 5 & -1 & -1 \end{pmatrix}\begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} \\ &= \tfrac{1}{2}\begin{pmatrix} -20 \\ 0 \\ 8 \end{pmatrix} = \begin{pmatrix} -9 \\ 0 \\ 4 \end{pmatrix} \end{aligned}$$

4. Give a solution to $A\mathbf{x} = \mathbf{b}$ in the general case where $\mathbf{b} = (b_1, b_2, b_3)$

$$\begin{aligned} \mathbf{x} &= \tfrac{1}{2}\begin{pmatrix} -15 & 1 & 5 \\ 1 & 1 & -1 \\ 5 & -1 & -1 \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \\ &= \tfrac{1}{2}\begin{pmatrix} -15b_1 + b_2 + 5b_3 \\ b_1 + b_2 - b_3 \\ 5b_1 - b_2 - b_3 \end{pmatrix} \end{aligned}$$

6 Elementary Matrices

Definition 13 *An* Elementary matrix *is a matrix obtained by preforming a single row operation on the identity matrix.*

Example 14

1. $$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (R_1 \to 2R_1)$$

2. $$\begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (R_2 \to R_2 + 3R_1)$$

3. $$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad (R_1 \leftrightarrow R_2)$$

Theorem 15 *If E is an elementary matrix obtained from I_m by preforming the row operation R and A is any $m \times n$ matrix, then EA is the matrix obtained by preforming the same row operation R on A.*

Example 16

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix}$$

1. $$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix} \sim 2R_2 \text{ on } A$$

2. $$\begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 5 & 4 & 3 \\ 3 & 2 & 1 \end{pmatrix} \sim \begin{array}{c} R_2 \to R_2 + 3R_1 \\ \text{on } A \end{array}$$

3. $$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 3 & 2 & 1 \\ 2 & 1 & 0 \end{pmatrix} \sim \begin{array}{c} R_2 \leftrightarrow R_3 \\ \text{on } A \end{array}$$

6.1 Inverses of Elementary Matrices

If E is an elementary matrix then E is invertible and E^{-1} is an elementary matrix corresponding to the row operation that undoes the one that generated E. Specifically:

- If E was generated by an operation of the form $R_i \to cR_i$ then E^{-1} is generated by $R_i \to \frac{1}{c}R_i$.
- If E was generated by an operation of the form $R_i \to R_i + cR_j$ then E^{-1} is generated by $R_i \to R_i - cR_j$.
- If E was generated by an operation of the form $R_i \leftrightarrow R_j$ then E^{-1} is generated by $R_i \leftrightarrow R_j$.

Example 17

1. $E = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $\qquad E^{-1} = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
2. $E = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $\qquad E^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
3. $E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ $\qquad E^{-1} = E$

6.2 Elementary Matricies and Solving Equations

Consider the steps of Gauss Jordan elimination to find the solution to a system of equations $A\mathbf{x} = \mathbf{b}$. This consists of a series of row operations, each of which is equivalent to multiplying on the left by an elementary matrix E_i.

$$A \overset{\text{Ele. row ops.}}{- - - \longrightarrow} B,$$

Where B is the RREF of A.
So $E_k E_{k-1} \ldots E_2 E_1 A = B$ for some appopriately defined elementary matrices $E_1 \ldots E_k$.
Thus $A = E_1^{-1} E_2^{-1} \ldots E_{k-1}^{-1} E_k^{-1} B$
Now if $B = I$ (so the RREF of A is I), then

$$A = E_1^{-1} E_2^{-1} \ldots E_{k-1}^{-1} E_k^{-1}$$

and $A^{-1} = E_k E_{k-1} \ldots E_2 E_1$

Theorem 18 *A is invertable if and only if it is the product of elementary matrices.*

7 Summing Up Theorem

Theorem 19 (Summing up Theorem Version 1) *For any square $n \times n$ matrix A, the following are equivalent statements:*

1. *A is invertible.*
2. *The RREF of A is the identity, I_n.*
3. *The equation $A\mathbf{x} = \mathbf{b}$ has unique solution (namely $\mathbf{x} = A^{-1}\mathbf{b}$).*
4. *The homogeneous system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution ($\mathbf{x} = \mathbf{0}$)*
5. *The REF of A has exactly n pivots.*
6. *A is the product of elementary matrices.*