Determinants

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1 Determinants

Every $n \times n$ matrix A has an associated scalar value called the *determinant of* A, denoted by $\det(A)$ or $|A|$.
The determinant gives the (hyper)volume of the unit (hyper)cube after it has been transformed by A.
Note that determinant is only defined for square matricies.

1.1 2×2 Determinants

The determinant of a 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is defined to be $\det(A) = |A| = ad - bc$

1.2 Minors

For an $n \times n$ matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$$

For each pair i, j, $1 \leq i, j \leq n$, define the ij^{th} *minor* M_{ij} to be the matrix obtained from A by deleting the i^{th} row and j^{th} column from A.

Example 1

1. If $A = \begin{pmatrix} -3 & 2 & 5 \\ 1 & 0 & -1 \\ 4 & -6 & 7 \end{pmatrix}$. Find M_{ij} for $1 \leq i, j \leq 3$.

$$M_{11} = \begin{pmatrix} 0 & -1 \\ -6 & 7 \end{pmatrix} \quad M_{12} = \begin{pmatrix} 1 & -1 \\ 4 & 7 \end{pmatrix} \quad M_{13} = \begin{pmatrix} 1 & -1 \\ 4 & 7 \end{pmatrix}$$

$$M_{21} = \begin{pmatrix} 2 & 5 \\ -6 & 7 \end{pmatrix} \quad M_{22} = \begin{pmatrix} -3 & 5 \\ 4 & 7 \end{pmatrix} \quad M_{23} = \begin{pmatrix} -3 & 2 \\ 4 & -6 \end{pmatrix}$$

$$M_{31} = \begin{pmatrix} 2 & 5 \\ 0 & -1 \end{pmatrix} \quad M_{32} = \begin{pmatrix} -3 & 5 \\ 1 & -1 \end{pmatrix} \quad M_{33} = \begin{pmatrix} -3 & 2 \\ 1 & 0 \end{pmatrix}$$

2. If $A = \begin{pmatrix} 1 & 5 & 7 & 9 \\ 3 & 4 & 2 & 8 \\ 1 & 1 & 3 & 6 \\ 0 & 2 & 5 & 9 \end{pmatrix}$. Find M_{3i}, for $1 \leq i \leq 4$.

$$M_{31} = \begin{pmatrix} 5 & 7 & 9 \\ 4 & 2 & 8 \\ 2 & 5 & 9 \end{pmatrix} \quad M_{32} = \begin{pmatrix} 1 & 7 & 9 \\ 3 & 2 & 8 \\ 0 & 5 & 9 \end{pmatrix}$$
$$M_{33} = \begin{pmatrix} 1 & 5 & 9 \\ 3 & 4 & 8 \\ 0 & 2 & 9 \end{pmatrix} \quad M_{34} = \begin{pmatrix} 1 & 5 & 7 \\ 3 & 4 & 2 \\ 0 & 2 & 5 \end{pmatrix}$$

1.3 Cofactors

For an $n \times n$ matrix, for each pair i, j, $1 \leq i, j \leq n$, define the ij^{th} *cofactor* by

$$A_{ij} = (-1)^{i+j} |M_{ij}|$$

Notes

1.

$$(-1)^{i+j} = \begin{cases} 1 & \text{when } i+j \text{ is even} \\ -1 & \text{when } i+j \text{ is odd} \end{cases}$$

For example when $n = 6$

$$\begin{pmatrix} 1 & -1 & 1 & -1 & 1 \\ -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 & 1 \\ -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 & 1 \\ -1 & 1 & -1 & 1 & -1 \end{pmatrix}$$

2. $|M_{ij}|$ is the determinant of the ij^{th} minor.

   Note that these minors are of size $(n-1) \times (n-1)$.

Definition 2 (Determinant) *Given any $n \times n$ matrix $A = [a_{ij}]$, the determinant of A, written det(A) or $|A|$ is given by*

$$\begin{aligned} |A| &= \Sigma_{j=1}^{n} a_{1j} A_{1j} \\ &= a_{11}A_{11} + a_{12}A_{12} + \ldots + a_{1n}A_{1n} \end{aligned}$$

Example 3

Find $|A|$, where

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

$$\begin{aligned}
|A| &= a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} \\
&= (-1)^2 \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} + 2 \cdot (-1)^3 \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3 \cdot (-1)^4 \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} \\
&= \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2 \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3 \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} \\
&= (5 \cdot 9 - 6 \cdot 8) - 2(4 \cdot 9 - 6 \cdot 7) + 3(4 \cdot 8 - 5 \cdot 7) \\
&= (45 - 48) - 2(36 - 42) + 3(32 - 35) \\
&= -3 - 2(-6) + 3(-3) \\
&= 0
\end{aligned}$$

So $\det(A) = 0$.

2 Finding Determinants

Finding higher order determinants requires alot of calculations, we want to find ways of limiting the number of calculations involved.

Theorem 4 (Cofactor Expansion) *Given any $n \times n$ matrix $A = [a_{ij}]$, and any fixed row index k*

$$\begin{aligned}
|A| &= \Sigma_{j=1}^{n} a_{kj}A_{kj} \\
&= a_{k1}A_{k1} + a_{k2}A_{k2} + \ldots + a_{kn}A_{kn}
\end{aligned}$$

Thus we may find determinants using any row. This is called expanding long the k^{th} row.
Warning: Remember the signs!

$$\begin{pmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{pmatrix}$$

Example 5
Find $|A|$, where

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 2 \\ 7 & 8 & 9 \end{pmatrix}$$

We expand along the second row (taking advantage of the 0's)

$$|A| = a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23}$$

But since $a_{21} = a_{22} = 0$, this becomes

$$\begin{aligned}
|A| &= a_{23}A_{23} \\
&= (-1)^5 \cdot 2 \begin{vmatrix} 1 & 2 \\ 7 & 8 \end{vmatrix} = -2(8 - 14) = 12
\end{aligned}$$

So $\det(A)$ = -6.

Theorem 6 *Given any $n \times n$ matrix A, the determinant of A is equal to the determinant of the transpose.*

$$|A| = \left|A^T\right|$$

Thus we may find determinants using any column. This is called expanding long the k^{th} column.

Example 7
Find $|A|$, where

$$\begin{pmatrix} 1 & 2 & 0 & 1 \\ 2 & 1 & 2 & 3 \\ 7 & 8 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

We expand down the 3$^{\text{rd}}$ column (taking advantage of the 0's)

$$|A| = a_{13}A_{13} + a_{23}A_{23} + a_{33}A_{33} + a_{43}A_{43}$$

But since $a_{13} = a_{33} = a_{43} = 0$, this becomes

$$\begin{aligned} |A| &= a_{23}A_{23} \\ &= -2\begin{vmatrix} 1 & 2 & 1 \\ 7 & 8 & 0 \\ 1 & 0 & 1 \end{vmatrix} \end{aligned}$$

Expand along third row:

$$\begin{aligned} |A| &= -2\left(\begin{vmatrix} 2 & 1 \\ 8 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 7 & 8 \end{vmatrix}\right) \\ &= -2((0-8) + (8-14)) = -28a \end{aligned}$$

So $\det(A) = -4$.

2.1 Triangular Matrices

Theorem 8 *The determinant of an upper triangular, lower triangular or diagonal matrix is the product of its diagonal entries.*

Example 9

1.

$$\begin{vmatrix} 1 & 3 & 5 & 7 \\ 0 & 9 & 6 & 4 \\ 0 & 0 & 7 & 8 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 1 \cdot 9 \cdot 7 \cdot 1 = 63$$

2.

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 5 & 9 & 2 & 0 \\ 7 & 6 & 4 & 8 \end{vmatrix} = 1 \cdot 3 \cdot 2 \cdot 8 = 48$$

3.

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{vmatrix} = 1 \cdot 2 \cdot 3 \cdot 4 = 63$$

2.2 Row Operations

We know a method (Gaussian Elimination) which will turn any matrix into a triangular matrix (REF is triangular). We need to know the effect of row operations on the determinant.
The effect of the three basic row operations are given in the table below.

Operation	Effect	
$R_i \to cR_i$	$\times c$	$\|A\| \to c\|A\|$
$R_i \to R_i + cR_j$	None	$\|A\| \to \|A\|$
$R_i \leftrightarrow R_j$	$\times(-1)$	$\|A\| \to -\|A\|$

Example 10
Find $|A|$, where

$$A = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 1 & 1 \\ 2 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \end{pmatrix}$$

$$\begin{aligned}
|A| &= \begin{vmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 1 & 1 \\ 2 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \end{vmatrix} && R_1 \leftrightarrow R_2 \\
&= -\begin{vmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 2 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \end{vmatrix} && \begin{array}{l} R_3 \rightarrow R_3 - 2R_1 \\ R_4 \rightarrow R_4 - R_1 \end{array} \\
&= -\begin{vmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & -2 & -1 \\ 0 & 1 & -1 & 0 \end{vmatrix}
\end{aligned}$$

Expand down 1st column

$$\begin{aligned}
&= -\begin{vmatrix} 1 & 2 & 3 \\ 1 & -2 & -1 \\ 1 & -1 & 0 \end{vmatrix} && \begin{array}{l} R_2 \rightarrow R_2 - R_1 \\ R_3 \rightarrow R_3 - R_1 \end{array} \\
&= -\begin{vmatrix} 1 & 2 & 3 \\ 0 & 0 & -4 \\ 0 & -3 & -3 \end{vmatrix}
\end{aligned}$$

Expand along 2nd row

$$\begin{aligned}
&= -4\begin{vmatrix} 1 & 2 \\ 0 & -3 \end{vmatrix} \\
&= 12
\end{aligned}$$

3 Determinants and Solutions to Equations

Theorem 11 (Summing up Theorem Version 2) *For any square $n \times n$ matrix A, the following are equivalent statements:*

1. *A is invertible.*
2. *The homogeneous system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution ($\mathbf{x} = \mathbf{0}$)*
3. *The equation $A\mathbf{x} = \mathbf{b}$ has unique solution (namely $\mathbf{x} = A^{-1}\mathbf{b}$).*
4. *The RREF of A is the identity.*
5. *A is can be expressed as a product of elementary matrices.*
6. *The REF of A has exactly n pivots.*
7. *$det(A) \neq 0$*

4 Algebraic Properties of Determinants

Theorem 12 *Given two $n \times n$ matrices, A and B, $det(AB) = det(A)det(B)$, or*

$$|AB| = |A||B|.$$

Corollary 13 *If A and B are invertible $n \times n$ matrices then AB is invertible.*

Proof: If A and B are invertible then $|A| \neq 0$ and $|B| \neq 0$, so $|AB| = |A||B| \neq 0$. $\square$

Corollary 14 *If A is a non invertible $n \times n$ matrix and B is any $n \times n$ matrix then AB is not invertible.*

Proof: If A is not invertible, so $|A| = 0$, so $|AB| = |A||B| = 0$. $\square$

Corollary 15 *Given an invertible $n \times n$ matrix A, $det(A^{-1}) = (det(A))^{-1} = \frac{1}{det(A)}$, or*

$$\left|A^{-1}\right| = |A|^{-1}.$$

Proof: Consider

$$AA^{-1} = I.$$

Taking determinants of both sides, we have

$$|AA^{-1}| = |I|.$$

But $|AA^{-1}| = |A|\,|A^{-1}|$ and $|I| = 1$, so

$$|A|\,|A^{-1}| = 1.$$

Thus

$$|A^{-1}| = \frac{1}{|A|}. \quad \square$$

Corollary 16 *Given an integer k and an $n \times n$ matrix A, $det\big(A^k\big) = (det(A))^k$*

$$|A^k| = |A|^k$$

Proof: If $k < 0$ then $|A^k| = |A^{-k}|^{-1}$, so we can assume that k is positive.
Now, $|A^k| = |AA^{k-1}| = |A||A^{k-1}|$. Applying this rule iteratively we obtain

$$|A^k| = \underbrace{|A||A|\ldots|A|}_{k} = |A|^k$$

Theorem 17 *Given a scalar c and an $n \times n$ matrix A, $det(cA) = c^n det(A)$, or*

$$|cA| = c^n|A|.$$

We are multiplying each row by c.

Summary Given any $n \times n$ matrices A and B, integer k and scalar c:

- $|A^T| = |A|$
- $|AB| = |A||B|$.
- If A is invertible $|A^{-1}| = |A|^{-1}$.
- $|A^k| = |A|^k$
- $|cA| = c^n|A|$

We can mix and match these rules as desired.

Example 18

Given that $|A| = 2$ and $|B| = 3$, find the following:

1. $\left|AB^T\right|$
$$\left|AB^T\right| = |A|\left|B^T\right| = |A||B| = 2 \cdot 3 = 6$$
2. $|A^{-1}B|$
$$\left|A^{-1}B\right| = \left|A^{-1}\right||B| = |A|^{-1}|B| = \frac{3}{2}$$
3. $|A^2B^{-1}|$
$$\left|A^2B^{-1}\right| = \left|A^2\right|\left|B^{-1}\right| = |A|^2|B|^{-1} = \frac{4}{3}$$
4. $|3A^2B^{-2}|$, where A and B are 3×3.
$$\begin{aligned} |3A^2B^{-2}| &= 3^3\,|A^2|\left|(B^2)^{-1}\right| = 3^3\,|A|^2\,|B^2|^{-1} \\ &= 3^3|A|^2|B|^{-2} = 3^3 \cdot \tfrac{4}{3^2} = 12 \end{aligned}$$