Cross Product

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1 Cross Product

Big Note Cross product works in $\mathbb{R}^3$ **ONLY**

1.1 Definition

Given two non parallel vectors in $\mathbb{R}^3$, $\mathbf{u}$ and $\mathbf{v}$, there is a vector $\mathbf{w}$ which is perpendicular to both of them. The direction of this vector is unique, up to orientation.
The *Cross Product* gives us a way to find a vector in this direction.

Definition 1 *Given two vectors* $\mathbf{u} = (a, b, c)$ *and* $\mathbf{v} = (d, e, f)$*, the* cross product*,*

$$\mathbf{u} \times \mathbf{v} = (bf - ce)\mathbf{i} - (af - cd)\mathbf{j} + (ae - bd)\mathbf{k}$$

Conveniently:

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a & b & c \\ d & e & f \end{vmatrix}$$

Notes

- $\mathbf{u} \times \mathbf{v}$ is a vector.
- Order matters $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$

Example 2
Find $\mathbf{u} \times \mathbf{v}$, where $\mathbf{u} = (1, 3, 1)$ and $\mathbf{v} = (2, 2, 1)$.

$$\begin{aligned} \mathbf{u} \times \mathbf{v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & 1 \\ 2 & 2 & 1 \end{vmatrix} \\ &= (3 - 2)\mathbf{i} - (1 - 2)\mathbf{j} + (2 - 6)\mathbf{k} \\ &= \mathbf{i} + \mathbf{j} - 4\mathbf{k} \end{aligned}$$

Check:

$$\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = (1, 3, 1) \cdot (1, 1, -4) = 1 + 3 - 4 = 0$$
$$\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = (2, 2, 1) \cdot (1, 1, -4) = 2 + 2 - 4 = 0$$

So $\mathbf{u} \times \mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.
Note that

$$\begin{aligned} \mathbf{v} \times \mathbf{u} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & 1 \\ 1 & 3 & 1 \end{vmatrix} \\ &= -\mathbf{i} - \mathbf{j} + 4\mathbf{k} = -(\mathbf{u} \times \mathbf{v}) \end{aligned}$$

1.2 Properties of Cross Product

Theorem 3 (Properties of Cross Product) *Given three vectors* $\mathbf{u}, \mathbf{v}$ *and* $\mathbf{w}$ *in* $\mathbb{R}^3$*:*

1. ($\mathbf{0}$ Absorbant) $\mathbf{u} \times \mathbf{0} = \mathbf{0}$
2. (Antisymmetric) $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$
3. (Distributivity) $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$
4. (Triple Scalar Product) $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \mathbf{w} \cdot (\mathbf{u} \times \mathbf{v})$
5. $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = \mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = 0$. Both $\mathbf{u}$ and $\mathbf{v}$ are orthogonal to $\mathbf{u} \times \mathbf{v}$.

Right Hand Rule On your right hand, if $\mathbf{u}$ points along your forefinger and $\mathbf{v}$ points along your middle finger, then $\mathbf{u} \times \mathbf{v}$ will point along your thumb.

Theorem 4 (Magnitude of $\mathbf{u} \times \mathbf{v}$)

$$||\mathbf{u} \times \mathbf{v}|| = ||\mathbf{u}||\,||\mathbf{v}|| \sin\theta$$

Corollary 5 *Two non-zero vectors* $\mathbf{u}$ *and* $\mathbf{v}$ *are parallel if and only if* $\mathbf{u} \times \mathbf{v} = \mathbf{0}$.

1.3 Triple Scalar product

Definition 6 $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$ *is called the* triple scalar product.

Theorem 7 *Given three vectors* $\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{v} = (v_1, v_2, v_3)$ *and* $\mathbf{w} = (w_1, w_2, w_3)$

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

Example 8
Find the triple scalar product of $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$, where $\mathbf{u} = (1, 3, 1)$, $\mathbf{v} = (2, 2, 1)$ and $\mathbf{w} = (-1, 0, 1)$.

$$\begin{aligned}(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} &= \begin{vmatrix} 1 & 3 & 1 \\ 2 & 2 & 1 \\ -1 & 0 & 1 \end{vmatrix} = -\begin{vmatrix} 3 & 1 \\ 2 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 2 & 2 \end{vmatrix} \\ &= -(3-2) + 2 - 6 \\ &= -5\end{aligned}$$

1.4 Areas and Volumes

Given two vectors, $\mathbf{u}$ and $\mathbf{v}$, they map out a parallelogram with area base $\times$ height $= ||\mathbf{u} \times \mathbf{v}||$
Given three vectors, $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$, they map out a parallelopiped with volume equal to the area of the base $\times$ height $= (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$