Cramer's Rule

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1 Cramer's Rule

Cramer's rule is a method for solving $n \times n$ systems of equations using determinants. Generally it is less preferable than Gaussian elimination or Gauss-Jordan as there are more operations involved. However, in some circumstances it is a preferred method.
Let A be an $n \times n$ matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$$

Consider the system of equations

$$A\mathbf{x} = \mathbf{b}, \text{ where } \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}.$$

Define

$$\begin{aligned} A_1 &= \begin{pmatrix} b_1 & a_{12} & \dots & a_{1n} \\ b_2 & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_n & a_{n2} & \dots & a_{nn} \end{pmatrix} \\ A_2 &= \begin{pmatrix} a_{11} & b_1 & \dots & a_{1n} \\ a_{21} & b_2 & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & b_n & \dots & a_{nn} \end{pmatrix} \\ & \vdots \\ & \vdots \\ & \vdots \\ A_n &= \begin{pmatrix} a_{11} & a_{12} & \dots & b_n \\ a_{21} & a_{22} & \dots & b_n \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & b_n \end{pmatrix} \end{aligned}$$

So, A_i is the matrix A with the i^{th} column replaced by $\mathbf{b}$.
Define $D_i = |A_i|$ - The determinant of A_i.
Define $D = |A|$ - The determinant of A.

Theorem 1 (Cramer's Rule) *Given an $n \times n$ matrix A with det$(A) \neq 0$ and a vector $\mathbf{b}$ then the equation $A\mathbf{x} = \mathbf{b}$ has solutions*

$$x_1 = \frac{D_1}{D}, x_2 = \frac{D_2}{D}, \ldots, x_n = \frac{D_n}{D},$$

Example 2

$$\begin{aligned} x + 2y + z &= 1 \\ 3x + y + 2z &= 0 \\ 2x + y + z &= 0 \end{aligned}$$

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 3 & 1 & 2 \\ 2 & 1 & 1 \end{pmatrix} \text{ and } \mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$A_1 = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 1 \end{pmatrix}, A_2 = \begin{pmatrix} 1 & 1 & 1 \\ 3 & 0 & 2 \\ 2 & 0 & 1 \end{pmatrix}, A_3 = \begin{pmatrix} 1 & 2 & 1 \\ 3 & 1 & 0 \\ 2 & 1 & 0 \end{pmatrix}$$

Expanding down the first, second and third column respectively gives,

$D_1 = \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = 1 - 2 = -1.$	$D_2 = -\begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix} = -(3-4) = 1.$	$D_3 = \begin{vmatrix} 3 & 1 \\ 2 & 1 \end{vmatrix} = 3 - 2 = 1$

$$D = |A| = \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} - 2\begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ 2 & 1 \end{vmatrix} = -1 - (-2) + 1 = 2$$

$$\text{So } x_1 = \frac{D_1}{D} = \frac{-1}{2}, x_2 = \frac{D_2}{D} = \frac{1}{2}, x_3 = \frac{D_3}{D} = \frac{1}{2}.$$