1 Some Useful Sets

1.1 The Empty Set

Definition 1 *The empty set is the set with no elements, denoted by* ϕ*.*

1.2 Number Sets

- $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ - The natural numbers.
- $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$ - The integers.
- $\mathbb{Q} = \{\frac{x}{y} \mid x \in \mathbb{Z} \wedge y \in \mathbb{N}^+\}$ - The rationals.
- $\mathbb{R} = (-\infty, \infty)$ - The Real numbers.
- $\mathbb{I} = \mathbb{R} - \mathbb{Q}$ (all real numbers which are not rational) - The irrational numbers.
- $\mathbb{C} = \{x + yi \mid x, y \in \mathbb{R}\}$ - The Complex numbers.

Note: There are many real numbers which are not rational, e.g. π, $\sqrt{2}$ etc.

2 Complex Numbers

2.1 Introduction

We can't solve the equation $x^2 + 1 = 0$ over the real numbers, so we invent a new number i which is the solution to this equation, i.e. $i^2 = -1$.
Complex numbers are numbers of the form

$$z = x + iy, \quad \text{where } x, y \in \mathbb{R}.$$

The set of complex numbers is represented by $\mathbb{C}$. Generally we represent Complex numbers by z and w, and real numbers by x, y, u, v, so

$$z = x + iy,\ w = u + iv,\ z, w \in \mathbb{C},\ x, y, u, v \in \mathbb{R}.$$

Numbers of the form $z = iy$ (no real part) are called pure *imaginary* numbers.
Complex numbers may be thought of as vectors in $\mathbb{R}^2$ with components (x, y). We can also represent Complex numbers in polar coordinates (r, θ) (θ is the angle to the real (x) axis), in this case we write

$$z = re^{i\theta}. \text{ Thus } x = r\cos\theta, y = r\sin\theta,$$

and we have Demoivre's Theorem.

Theorem 2 (Demoivre's Theorem)

$$re^{i\theta} = r(\cos\theta + i\sin\theta)$$

Example 3

- Put $1-i$ in polar form.
  $\tan\theta = -1$, in fourth quadrant so $\theta = -\frac{\pi}{4}$. $r = \sqrt{1^2+1^2} = \sqrt{2}$. So
  $$1-i = \sqrt{2}e^{-\frac{\pi i}{4}} = \sqrt{2}e^{\frac{7\pi i}{4}}.$$
- Put $2e^{\frac{\pi}{3}}$ in rectangular form.
  $$2e^{\frac{\pi}{3}} = 2\left(\frac{1}{2}+\frac{\sqrt{3}i}{2}\right) = \sqrt{3}+i.$$

2.2 Operations with Complex numbers

Let $z = x+iy = re^{i\theta}$ and $w = u+iv = qe^{i\phi}$ then we have the following operations:

- The imaginary part of z, $\mathrm{Im}(z) = y$.
- The real part of z, $\mathrm{Re}(z) = x$.
- The Complex Conjugate of z, $\overline{z} = x - iy = re^{-i\theta}$.
  Note: Complex conjugation basically means turn every occurence of an i to a $-i$.
- The modulus of z, $|z| = \sqrt{z\overline{z}} = \sqrt{x^2+y^2} = r$.
- The argument of z, $\arg(z) = \tan^{-1} y/x = \theta$.

Note: $z\overline{z} = |z|^2$, so $\overline{z} = |z|^2/z$, so $\overline{z}/|z|^2 = 1/z$ this is used to do division.

Example 4
Let $z = -2+i$ and $w = 1-i$ then:

1. $\mathrm{Re}(z) = -2$, $\mathrm{Im}(z) = 1$, $\mathrm{Re}(w) = 1$ and $\mathrm{Im}(w) = -1$.
2. $|z| = \sqrt{(-2)^2+1^2} = \sqrt{5}$, $\arg(z) = \arctan\left(-\frac{1}{2}\right)$ so $z = \sqrt{5}e^{i\arctan\left(-\frac{1}{2}\right)}$.
3. $|w| = \sqrt{1^2+(-1)^2} = \sqrt{2}$, $\arg(w) = \arctan\frac{-1}{1} = -\frac{\pi}{4}$ so $w = \sqrt{2}e^{\frac{-i\pi}{4}}$.
4. $\overline{z} = -2-i = \sqrt{5}e^{\frac{-5\pi i}{6}}$ and $\overline{w} = 1+i = \sqrt{2}e^{\frac{i\pi}{4}}$.

- Addition $z+w = (x+u)+i(y+v)$ (Includes Subtraction).
- Multiplication $zw = (x+iy)(u+vi) = (xu-yv)+i(xv+yu) = qre^{i(\theta+\phi)}$.
- Division $\dfrac{z}{w} = \dfrac{z\overline{w}}{|w|^2}$.

Example 5
Let $z = -2+i$ and $w = 1-i$ then:

1. $z+w = (-2+1)+(1-1)i = -1$.
2. $zw = (-2+i)(1-i) = -2+2i+i-i^2 = -2+1+3i = -1+3i$.
3. $z/w = z\overline{w}/|w|^2 = \frac{1}{2}(-2+i)(1+i) = \frac{1}{2}(-2-2i+i+i^2) = \frac{1}{2}(-3-i)$.

2.3 Powers

Theorem 6 (Demoivre's Theorem)

$$\left(re^{i\theta}\right)^n = r^n(\cos(n\theta) + i\sin(n\theta))$$

Example 7
Find $(i+i)^{12}$

$$1+i = \sqrt{2}e^{\frac{\pi i}{4}}.$$

So

$$\begin{array}{rcl} (1+i)^{12} & = & \left(\sqrt{2}e^{\frac{\pi i}{4}}\right)^{12} \\ & = & \left(\sqrt{2}\right)^{12} e^{\frac{12\pi i}{4}} \\ & = & 2^6 e^{3\pi i} \\ & = & 64e^{i\pi} \\ & = & -64. \end{array}$$

Note $e^{i\pi} = -1$.

2.4 Roots of Complex Numbers

In order to find the n^{th} root of a complex number $z = x + iy = re^{i\theta}$ we use the polar form, $z = re^{i\theta}$. Since θ is an angle,

$$re^{i\theta} = re^{i(\theta + 2k\pi)}$$

for any intger k. Thus

$$\begin{array}{rcl} z^{\frac{1}{n}} & = & \left(re^{i(\theta+2k\pi)}\right)^{\frac{1}{n}} \\ & = & r^{\frac{1}{n}} e^{\frac{i\theta+2k\pi}{n}} \\ & = & r^{\frac{1}{n}}\left[\cos\left(\frac{\theta+2k\pi}{n}\right) + i\sin\left(\frac{\theta+2k\pi}{n}\right)\right] \end{array}$$

Taking $k = 0, 1, \ldots, n-1$ gives the n roots.
Since $r \geq 0$, $r^{\frac{1}{n}}$ always exists, even for even roots.

Example 8
Find All cube roots of 8.
$8 = 8e^{2k\pi i}$, so, $8^{\frac{1}{3}} = 2e^{\frac{2k\pi i}{3}}$.
Taking $k = 0, 1, 2$ gives 2, $2e^{\frac{2\pi i}{3}}$ and $2e^{\frac{4\pi i}{3}}$ as the three cube roots of 8.

2.5 Fundamental Theorem of Algebra

Note that in $\mathbb{C}$ all numbers have exactly n n^{th} roots.
This leads to the Fundamental Theorem of algebra:

Every polynomial over the Complex numbers of degree n has exactly n roots

$$\text{i.e. if } f(z) = a_0 + a_1 z + \ldots + a_n z^n$$

then there exist $z_1, z_2 \ldots, z_n \in \mathbb{C}$ such that

$$f(x) = (z - z_1)(z - z_2)\ldots(z - z_n).$$

That is f can be decomposed into linear factors.