Basis and Dimension

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1 Basis and Dimension

Definition 1 *A* basis *of a vector space V, is a set of vectors $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ such that*

1. *$\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ span V,*
2. *$\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ are linearly independent and hence the a_i above are unique.*

Notes

- Point 1 says that any vector in V may be written as a linear combination of vectors from B. So for every vector $\mathbf{u} \in V$ there are scalars $a_1, \ldots, a_n$ such that

$$\mathbf{u} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \ldots + a_n\mathbf{v}_n$$

  The a_i are called the *components* of $\mathbf{u}$ with respect to the basis B.

- Point 2 says that all of the vectors in $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ are needed.
- The plural of basis is bases, pronounced 'base ease'.

Since given a basis $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ of $\mathbb{R}^n$, any vector in $\mathbf{u} \in \mathbb{R}^n$ can be written as

$$\mathbf{u} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \ldots + a_n\mathbf{v}_n$$

for a **unique** set of scalars $a_1, \ldots a_n$, we can then write every $\mathbf{u} \in \mathbb{R}^n$ with respect to the basis B as

$$\mathbf{u}_B = (a_1, a_2, \ldots, a_n)_B.$$

Where the subscript B indicates that the components are with respect to the basis B, not the standard basis.
In general, we can find the coordinates of a vector $\mathbf{u}$ with respect to a given basis B by solving $A_B\mathbf{u}_B = \mathbf{u}$, for $\mathbf{u}_B$, where A_B is the matrix whose columns are the vectors in B. A_B is called the *change of basis matrix* for B.
If B is a basis there must be a unique solution of $A_B\mathbf{u}_B = \mathbf{u}$ for every $\mathbf{u} \in \mathbb{R}^n$. This means that B is a basis if and ony if the REF of A_B does not contain a row of zeros ($\text{rank}(A_B) = n$), or equivalently $\det(A_B) \neq 0$.

Example 2

1. **The Standard Basis for $\mathbb{R}^n$**

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \mathbf{e}_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}.$$

Note that if $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ is any vector in $\mathbb{R}^n$,

$$\mathbf{u} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + \ldots + u_n\mathbf{e}_n$$

and this is consistent with the use of the term component above.

In fact **any** set n linearly independent vectors in $\mathbb{R}^n$ form a basis.

2. Determine whether $\mathbf{v}_1 = (1, 1, 0)$ and $\mathbf{v}_2 = (1, 0, 1)$ and $\mathbf{v}_3 = (3, 1, 2)$ are a basis in $\mathbb{R}^3$.

$$\begin{pmatrix} 1 & 1 & 3 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix} \quad R_2 \to R_2 - R_1 \quad \begin{pmatrix} 1 & 1 & 3 \\ 0 & -1 & -2 \\ 0 & 1 & 2 \end{pmatrix} \quad R_3 \to R_3 + R_2 \quad \begin{pmatrix} 1 & 1 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

Since A_B contains a row of zeros ($\text{rank}(A_B) \neq 3$), this is not a basis.

Alternatively we could have used determinants:

$$\begin{vmatrix} 1 & 1 & 3 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} + 3\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = -1 - 2 + 3 = 0$$

$\det(A_B) = 0$, so B is not a basis.

3. (a) Determine whether $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, where $\mathbf{v}_1 = (1, 1, 0)$ and $\mathbf{v}_2 = (1, 0, 1)$ and $\mathbf{v}_3 = (3, 1, 1)$ is a basis for $\mathbb{R}^3$.

   Any three linearly independent vectors in $\mathbb{R}^3$ are a basis These three vectors will be linearly independent if the determinant of the matrix whose columns are $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ is non-zero.

$$\begin{vmatrix} 1 & 1 & 3 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} = 1$$

   This means that these three vectors can be used in place of the standard basis.

   (b) Find the components of $\mathbf{u} = (1, 2, 2)$ with respect to the basis B above.

   We must solve $\mathbf{u} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3$ for a_1, a_2 and a_3. So

$$\begin{aligned} \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} &= a_1\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + a_2\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + a_3\begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 & 3 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \end{aligned}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 3 & 1 \\ 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 2 \end{array}\right) \quad R_2 \to R_2 - R_1 \quad \left(\begin{array}{ccc|c} 1 & 1 & 3 & 1 \\ 0 & -1 & -2 & 1 \\ 0 & 1 & 1 & 2 \end{array}\right) \quad R_2 \to -R_2$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 3 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 1 & 1 & 2 \end{array}\right) \quad R_3 \to R_3 - R_2 \quad \left(\begin{array}{ccc|c} 1 & 1 & 3 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & -1 & 3 \end{array}\right)$$

Solving gives $a_3 = -3$, $a_2 = -1-2(-3) = 5$, $a_1 = 1-5-3(-3) = 5$. So $\underline{\mathbf{u}_B = (5, 5, -3)_B}$.
Check:

$$\begin{aligned} \mathbf{u} &= 5\mathbf{v}_1 + 5\mathbf{v}_2 - 3\mathbf{v}_3 \\ &= 5(1,1,0) + 5(1,0,1) - 3(3,1,1) \\ &= (5+5-9, 5-3, 5-3) \\ &= (1,2,2) \ \checkmark \end{aligned}$$

(c) Find the coordinates of the arbitrary vector $\mathbf{u} = (a, b, c) \in \mathbb{R}^3$ with respect to this basis.

$$\left(\begin{array}{ccc|c} 1 & 1 & 3 & a \\ 1 & 0 & 1 & b \\ 0 & 1 & 1 & c \end{array}\right) \quad R_2 \to R_2 - R_1 \quad \left(\begin{array}{ccc|c} 1 & 1 & 3 & a \\ 0 & -1 & -2 & b-a \\ 0 & 1 & 1 & c \end{array}\right) \quad R_2 \to -R_2$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 3 & a \\ 0 & 1 & 2 & a-b \\ 0 & 1 & 1 & c \end{array}\right) \quad R_3 \to R_3 - R_2 \quad \left(\begin{array}{ccc|c} 1 & 1 & 3 & a \\ 0 & 1 & 2 & a-b \\ 0 & 0 & -1 & c-(a-b) \end{array}\right)$$

Solving gives

$$\begin{aligned} a_3 &= a - b - c, \\ a_2 &= (a-b) - 2(a-b-c) \\ &= -a + b + 2c, \\ a_1 &= a - (-a+b+2c) - 3(a-b-c) \\ &= -a + 2b + c. \end{aligned}$$

So $\underline{\mathbf{u}_B = (-a+2b+c,\ -a+b+2c,\ a-b-c)_B}$.

4. Find a basis for the plane $x + 2y - z = 0$.

   Any two non colinear vectors in the plane will span it. Take $(1, 1, 3)$ and $(-1, 1, 1)$.

5. Find a basis for the solution set of the system of equations

$$\begin{aligned} x_1 + x_3 + x_4 &= 0 \\ x_2 + x_3 &= 0 \\ x_1 + x_2 + 2x_3 + x_4 &= 0 \end{aligned}$$

$$\begin{pmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 2 & 1 & 0 \end{pmatrix} \quad R_3 \to R_3 - R_1$$

$$\begin{pmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix} \quad R_3 \to R_3 - R_2$$

$$\begin{pmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Let $s, t \in \mathbb{R}$, set

$$x_3 = s,\ x_4 = t,\ \text{thus } x_2 = -s,\ x_1 = -s - t.$$

Solutions are of the form

$$(-s - t, -s, s, t) = s(-1, -1, 1, 0) + t(-1, 0, 0, 1).$$

So all solutions are a linear combination of $\{(-1, -1, 1, 0), (-1, 0, 0, 1)\}$. These vectors forma basis for the solution space.

The solution set to any homogeneous system with non-trivial solutions may be expressed in this way.

2 Dimension

Theorem 3 *If* $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ *and* $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\}$ *are bases for a vector space* V *then* $m = n$.

Corollary 4 *Every basis of* $\mathbb{R}^n$ *contains exactly* n *vectors.*

Definition 5 *The* Dimension *of a vector space* V *is the number of vectors in a basis.*

We write $\dim(V)$.

Example 6

1. $\dim(\mathbb{R}^n) = n$.

   Take the standard basis.

2. $\dim(\mathbb{P}_n) = n + 1$.

Theorem 7 *If* U *is a subspace of a vector space* V, *then* $\dim(U) \leq \dim(V)$.

Example 8

1. Let $U = \{(x, y, z) \mid x - z = 0\} \subseteq \mathbb{R}^3$. ($U$ is the plane $x - z = 0$.) Find the dimension of U.

   Solve $x - z = 0$, Let $s, t \in \mathbb{R}$, set $z = s$, $y = t$, then $x = s$. Solutions are of the form

   $$(s, t, s) = s(1, 0, 1) + t(0, 1, 0)$$

   Thus $\{(0, 1, 0), (1, 0, 1)\}$, is a basis for U, and so $\dim(U) = 2$.

2. Find the dimension of solution set of the system of equations

   $$\begin{aligned} x_1 + x_3 + x_4 &= 0 \\ x_2 + x_3 &= 0 \\ x_1 + x_2 + 2x_3 + x_4 &= 0 \end{aligned}$$

   Above we showed that the solutions are of the form

   $$(-s - t, -s, s, t) = s(-1, -1, 1, 0) + t(-1, 0, 0, 1).$$

   and so $\{(-1, -1, 1, 0), (-1, 0, 0, 1)\}$ forms a basis for the solution space.

   Since there are two vectors in the basis, the dimension is 2.

   (So this is a 2-dimensional plane in $\mathbb{R}^4$.)