Adjoints

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1 The Adjoint & Inverses

Definition 1 *Given an $n \times n$ matrix A, the* adjoint *of A is the transpose of the matrix of cofactors.*

$$\begin{pmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \dots & A_{nn} \end{pmatrix}^T = \begin{pmatrix} A_{11} & A_{21} & \dots & A_{n1} \\ A_{12} & A_{22} & \dots & A_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ A_{1n} & A_{2n} & \dots & A_{nn} \end{pmatrix}$$

Example 2

If $A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 3 \\ 1 & 3 & 1 \end{pmatrix}$. find adj$(A)$.

$$A_{11} = \begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} = -8, \qquad A_{12} = -\begin{vmatrix} 1 & 3 \\ 1 & 1 \end{vmatrix} = -(-2) = 2, \qquad A_{13} = \begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} = 2,$$

$$A_{21} = -\begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} = -(-1) = 1, \qquad A_{22} = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0, \qquad A_{23} = -\begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} = -1,$$

$$A_{31} = \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} = 5, \qquad A_{32} = -\begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} = -2, \qquad A_{33} = \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = -1.$$

The matrix of cofactors is given by

$$M = \begin{pmatrix} -8 & 2 & 2 \\ 1 & 0 & -1 \\ 5 & -2 & -1 \end{pmatrix}$$

The adjoint is given by the transpose

$$\text{adj}(A) = M^T = \begin{pmatrix} -8 & 1 & 5 \\ 2 & 0 & -2 \\ 2 & -1 & -1 \end{pmatrix}$$

Theorem 3

$$A^{-1} = \frac{1}{\det(A)}\,\text{adj}(A).$$

Example 4

Continuing the above example:

If $A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 3 \\ 1 & 3 & 1 \end{pmatrix}$. find A^{-1} using the adjoint method,

$$|A| = \begin{vmatrix} 1 & 2 & 1 \\ 1 & 1 & 3 \\ 1 & 3 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} - 2\begin{vmatrix} 1 & 3 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} = -8 + 4 + 2 = -2.$$

$$\text{So } A^{-1} = \frac{1}{\det(A)}\,\text{adj}(A) = -\frac{1}{2}\begin{pmatrix} -8 & 1 & 5 \\ 2 & 0 & -2 \\ 2 & -1 & -1 \end{pmatrix}$$

Check:

$$A^{-1}A = -\frac{1}{2}\begin{pmatrix} -8 & 1 & 5 \\ 2 & 0 & -2 \\ 2 & -1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 3 \\ 1 & 3 & 1 \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix} = I$$