Triangular Matrices

Definition 1 *Given an $n \times n$ matrix A*

- *A is called <u>upper triangular</u> if all entries* **below** *the main diagonal are 0.*

- *A is called <u>lower triangular</u> if all entries* **above** *the main diagonal are 0.*

- *A is called <u>diagonal</u> if only the diagonal entries are non-zero.*

If D is a diagonal matrix with diagonal entries $d_1, d_2, \ldots d_n$, we may write it as $\text{diag}(d_1, d_2, \ldots, d_n)$

Example 2

1. Upper Triangular

$$a) \quad \begin{pmatrix} 1 & 3 & 5 & 7 \\ 0 & 9 & 6 & 4 \\ 0 & 0 & 7 & 8 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad b) \quad \begin{pmatrix} 1 & 3 & 7 & 9 \\ 0 & 0 & 2 & 5 \\ 0 & 0 & 9 & 6 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

2. Lower Triangular

$$a) \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 5 & 9 & 2 & 0 \\ 7 & 6 & 4 & 8 \end{pmatrix} \qquad b) \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 7 & 0 & 0 & 0 \\ 5 & 0 & 0 & 0 \\ 7 & 6 & 4 & 8 \end{pmatrix}$$

3. Diagonal

$$a) \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 8 \end{pmatrix} = \text{diag}(1, 3, 2, 8)$$

$$b) \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 \end{pmatrix} = \text{diag}(1, 0, 0, 5)$$

4. $I_n = diag(1, 1, \ldots, 1)$, where there are n 1's.

Notes:

1. A matrix in REF is upper triangular.

2. The transpose of an upper triangular matrix is lower triangular and visa versa.

3. The product of two Upper triangular matricies is upper triangular.

4. The product of two Lower triangular matricies is Lower triangular.

5. The product of two Diagonal matricies is Diagonal.

6. The transpose of a Diagonal matrix is Diagonal.

Theorem 3 *A diagonal, upper or lower triangular matrix is invertable if and only if its diagonal entries are all non-zero.*

Diagonal Matrices

Theorem 4 *Given two diagonal matricies $D =$ diag$(d_1, \ldots, d_n)$ and $E =$ diag$(e_1, \ldots, e_n)$:*

1. *$DE =$ diag$(d_1 e_1, d_2 e_2 \ldots, d_n e_n)$*

2. *For any positive integer k,*
$$D^k = \textit{diag}\left(d_1^k, d_2^k \ldots, d_n^k\right).$$

3. *D is invertable if and only if all the diagonal entries are non-zero and*
$$D^{-1} = \textit{diag}\left(\frac{1}{d_1}, \ldots, \frac{1}{d_n}\right).$$

4. *$D + E =$ diag$(d_1 + e_1, d_2 + e_2 \ldots, d_n + e_n)$*

5. *Diagonal matrices are both upper and lower triangular. Further, any matrix which is both upper and lower triangular is diagonal.*

Symmetric Matrices

Definition 5 *An $n \times n$ matrix A is called symmetric if it is equal to its transpose, i. e. $A = A^T$. It is called antisymmetric if it is equal to the negative of its transpose, i. e. $A = -A^T$.*

Note that any diagonal matrix is symmetric.

Example 6

1.

$$\begin{pmatrix} 1 & 2 & 4 \\ 2 & 2 & 5 \\ 4 & 5 & 3 \end{pmatrix}$$

2. A mileage chart shows the distance between cities. Such a chaart is symmetric since the distance between city A and city B is the same as the distance from city B to City A.

Southern Ontario Distances Chart (In KM) 1 KM = 0.6 Miles	Brantford	Collingwood	Elora	Fort Erie	Goderich	Grand Bend	Hamilton	Kitchener-Waterloo	London	Niagara Falls	Niagara-on-the-Lake	Oakville	Orangeville	Owen Sound	Point Pelee	Port Dover	Port Stanley	Sarnia	Port Elgin	St. Catharines	St. Thomas	Tobermory	Toronto	Windsor
Brantford	0	234	70	149	155	144	39	39	91	112	130	70	116	186	262	53	119	161	200	108	112	302	95	277
Collingwood	234	0	110	282	175	220	200	140	310	256	262	169	75	58	477	273	336	409	107	240	325	170	145	486
Elora	70	110	0	208	128	149	76	30	136	177	153	94	42	121	303	114	164	239	125	131	151	224	114	317
Fort Erie	149	282	208	0	282	316	95	166	237	32	55	118	193	305	402	117	264	336	300	43	252	408	153	413
Goderich	155	175	128	282	0	49	172	115	91	235	258	201	140	123	279	207	130	116	89	239	126	239	214	301
Grand Bend	144	220	149	316	49	0	186	105	60	371	296	236	182	196	248	169	99	67	136	274	96	267	203	218
Hamilton	39	200	76	95	172	186	0	64	126	63	75	36	98	179	298	60	165	196	200	53	146	294	63	319
Kitchener-Waterloo	39	140	30	166	115	105	64	0	91	123	145	88	77	144	263	84	130	161	153	122	125	263	98	284
London	91	310	136	237	91	60	126	91	0	193	216	157	168	214	182	109	39	91	186	145	29	330	175	193
Niagara Falls	112	256	177	32	235	371	63	123	193	0	28	92	161	242	377	109	224	259	274	17	224	357	133	382
Niagara-on-the-Lake	130	262	153	55	258	296	75	145	216	28	0	97	172	284	382	128	243	316	280	23	231	387	132	393
Oakville	70	169	94	118	201	236	36	88	157	92	97	0	79	191	323	109	183	255	208	76	171	294	40	333
Orangeville	116	75	42	193	140	182	98	77	168	161	172	79	0	101	341	161	207	256	151	150	189	295	74	361
Owen Sound	186	58	121	305	123	196	179	144	214	242	284	191	101	0	361	240	274	239	53	263	272	116	203	434
Point Pelee	262	477	303	402	279	248	298	263	182	377	382	323	341	361	0	301	172	187	342	360	177	535	361	74
Port Dover	53	273	114	117	207	169	60	84	109	109	128	109	161	240	301	0	98	234	242	106	93	341	153	311
Port Stanley	119	336	164	264	130	99	165	130	39	224	243	183	207	274	172	98	0	127	244	222	15	387	220	182
Sarnia	161	409	239	336	116	67	196	161	91	259	316	255	256	239	187	234	127	0	214	294	115	336	293	108
Port Elgin	200	107	125	300	89	136	200	153	186	274	280	208	151	53	342	242	244	214	0	258	232	122	239	318
St. Catharines	108	240	131	43	239	274	53	122	145	17	23	76	150	263	360	106	222	294	258	0	210	366	111	371
St. Thomas	112	325	151	252	126	96	146	125	29	224	231	171	189	272	177	93	15	115	232	210	0	376	208	187
Tobermory	302	170	224	408	239	267	294	263	330	357	387	294	295	116	535	341	387	336	122	366	376	0	295	439
Toronto	95	145	114	153	214	203	63	98	175	133	132	40	74	203	361	153	220	293	239	111	208	295	0	371
Windsor	277	486	317	413	301	218	319	284	193	382	393	333	361	434	74	311	182	108	318	371	187	439	371	0

Theorem 7 *Given symmetric $n \times n$ matrices A and B then:*

1. *A^T is symmetric.*

2. *$A + B$ and $A - B$ are symmetric.*

3. *For any scalar k kA is symmetric.*

A^T is symmetric since $\left(A^T\right)^T = A$, for any matrix A.

Example 8

1.
$$\begin{pmatrix} 1 & 2 & 4 \\ 2 & 2 & 5 \\ 4 & 5 & 3 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 1+0 & 2+1 & 4+1 \\ 2+1 & 2+1 & 5+1 \\ 4+1 & 5+2 & 3+3 \end{pmatrix} =$$

2.
$$2 \begin{pmatrix} 0 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 2 \cdot 0 & 2 \cdot 1 & 2 \cdot 1 \\ 2 \cdot 1 & 2 \cdot 2 & 2 \cdot 2 \\ 2 \cdot 1 & 2 \cdot 2 & 2 \cdot 3 \end{pmatrix} = \begin{pmatrix} 0 & 2 & 2 \\ 2 & 4 & 4 \\ 2 & 4 & 6 \end{pmatrix}$$

Theorem 9 *If A is an invertable symmetric matrix then:*

1. *A^{-1} is symmetric;*

2. *AA^T and A^TA are also invertable.*

Note that if A is symmetric then $AA^T = A^2$, so $(AA^T)^{-1} = A^{-2} = (A^{-1})^2$

Example 10

$$\text{Let } A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 5 & 3 \\ 1 & 3 & 3 \end{pmatrix}$$

1. $$A^{-1} = \begin{pmatrix} 6 & -3 & 1 \\ -3 & 2 & -1 \\ 1 & -1 & 1 \end{pmatrix} \text{ (Exercise)}$$

   Which is also symmetric.

2.

$$AA^T = A^2 = \begin{pmatrix} 6 & 15 & 10 \\ 15 & 38 & 26 \\ 10 & 26 & 19 \end{pmatrix}$$

$$(AA^T)^{-1} = (A^{-1})^2 = \begin{pmatrix} 6 & -3 & 1 \\ -3 & 2 & -1 \\ 1 & -1 & 1 \end{pmatrix}^2 = \begin{pmatrix} 46 & -25 & 10 \\ -25 & 14 & -6 \\ 10 & -6 & 3 \end{pmatrix}$$

$$\begin{pmatrix} 6 & 15 & 10 \\ 15 & 38 & 26 \\ 10 & 26 & 19 \end{pmatrix} \begin{pmatrix} 46 & -25 & 10 \\ -25 & 14 & -6 \\ 10 & -6 & 3 \end{pmatrix} = I$$

Note: Not all symmetric matrices are invertible.
For example $\begin{pmatrix} 1 & 2 & 1 \\ 2 & 5 & 3 \\ 1 & 3 & 2 \end{pmatrix}$ is not invertible.

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & 5 & 3 \\ 1 & 3 & 2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \quad \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \end{array}$$