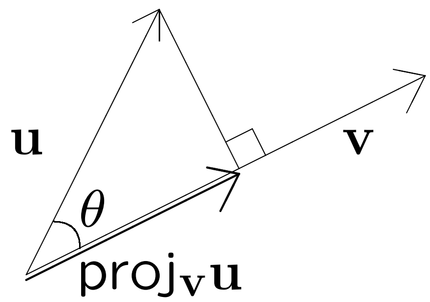
Components and Projections



Given two vectors $\mathbf{u}$ and $\mathbf{v}$, we can ask how far we will go in the direction of $\mathbf{v}$ when we travel along $\mathbf{u}$.

The distance we travel in the direction of $\mathbf{v}$, while traversing $\mathbf{u}$ is called the *component of u with respect to* $\mathbf{v}$ and is denoted $\text{comp}_{\mathbf{v}}\mathbf{u}$.

The vector parallel to $\mathbf{v}$, with magnitude $\text{comp}_{\mathbf{v}}\mathbf{u}$, in the direction of $\mathbf{v}$ is called the *projection of* $\mathbf{u}$ *onto* $\mathbf{v}$ and is denoted $\text{proj}_{\mathbf{v}}\mathbf{u}$.

$$\text{So, } \text{comp}_{\mathbf{v}}\mathbf{u} = ||\text{proj}_{\mathbf{v}}\mathbf{u}||$$

Note $\text{proj}_{\mathbf{v}}\mathbf{u}$ is a vector and $\text{comp}_{\mathbf{v}}\mathbf{u}$ is a scalar.

From the picture $\text{comp}_{\mathbf{v}}\mathbf{u} = ||\mathbf{u}|| \cos\theta$

We wish to find a formula for the projection of $\mathbf{u}$ onto $\mathbf{v}$.

$$\begin{array}{ll} \text{Consider} & \mathbf{u}\cdot\mathbf{v} = ||\mathbf{u}||\,||\mathbf{v}||\cos\theta \\ \text{Thus} & ||\mathbf{u}||\cos\theta = \dfrac{\mathbf{u}\cdot\mathbf{v}}{||\mathbf{v}||} \end{array}$$

$$\text{So } \boxed{\text{comp}_{\mathbf{v}}\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{v}}{||\mathbf{v}||}}$$

The unit vector in the same direction as $\mathbf{v}$ is given by $\frac{\mathbf{v}}{||\mathbf{v}||}$. So

$$\boxed{\text{proj}_{\mathbf{v}}\mathbf{u} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{||\mathbf{v}||^2}\right)\mathbf{v}}$$

Example 1

1. Find the projection of $\mathbf{u} = \mathbf{i} + 2\mathbf{j}$ onto $\mathbf{v} = \mathbf{i} + \mathbf{j}$.

$$\mathbf{u} \cdot \mathbf{v} = 1 + 2 = 3, \;\; ||\mathbf{v}||^2 = \left(\sqrt{2}\right)^2 = 2$$

$$\text{proj}_{\mathbf{v}}\mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{||\mathbf{v}||^2}\right)\mathbf{v} = \frac{3}{2}(\mathbf{i} + \mathbf{j}) = \frac{3}{2}\mathbf{i} + \frac{3}{2}\mathbf{j}$$

2. Find $\text{proj}_{\mathbf{v}}\mathbf{u}$, where $\mathbf{u} = (1, 2, 1)$ and $\mathbf{v} = (1, 1, 2)$

$$\mathbf{u} \cdot \mathbf{v} = 1 + 2 + 2 = 5, \;\; ||\mathbf{v}||^2 = \left(\sqrt{1^2 + 1^2 + 2^2}\right)^2 = 6$$

$$\text{So, } \text{proj}_{\mathbf{v}}\mathbf{u} = \frac{5}{6}(1, 1, 2)$$

3. Find the component of $\mathbf{u} = \mathbf{i} + \mathbf{j}$ in the direction of $\mathbf{v} = 3\mathbf{i} + 4\mathbf{j}$.

$$\mathbf{u} \cdot \mathbf{v} = 3 + 4 = 7, \;\; ||\mathbf{v}|| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$$

$$\text{comp}_{\mathbf{v}}\mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{v}}{||\mathbf{v}||} = \frac{7}{5}$$

4. Find the components of $\mathbf{u} = \mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$ in the directions $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$.

$$\mathbf{u} \cdot \mathbf{i} = 1, \ \mathbf{u} \cdot \mathbf{j} = 3, \ \mathbf{u} \cdot \mathbf{k} = -2,$$

$$||\mathbf{i}|| = ||\mathbf{j}|| = ||\mathbf{k}|| = 1$$

So

$$\text{comp}_{\mathbf{i}}\mathbf{u} = 1, \ \text{comp}_{\mathbf{j}}\mathbf{u} = 3, \ \text{comp}_{\mathbf{k}}\mathbf{u} = -2.$$

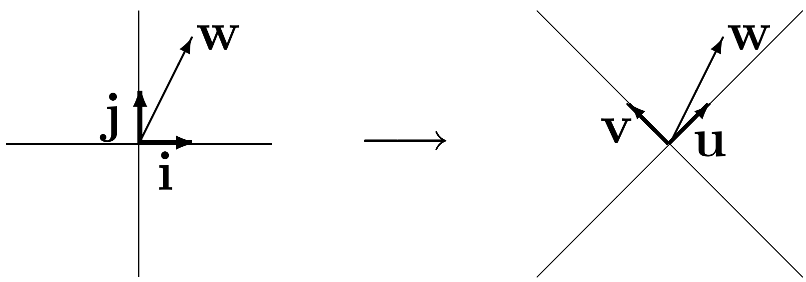
So the use of the term *component* is justified in this context.

Indeed, coordinate axes are arbitrarily chosen and are subject to change.

If $\mathbf{u}$ is a new coordinate vector given in terms of the old set then $\text{comp}_{\mathbf{u}}\mathbf{w}$ gives the component of the vector **w** in the new coordinate system.

Example 2

If coordinates in the plane are rotated by 45^o, the vector **i** is mapped to $\mathbf{u} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$, and the vector **j** is mapped to $\mathbf{v} = -\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$. Find the components of $\mathbf{w} = 2\mathbf{i} - 5\mathbf{j}$ with respect to the new coordinate vectors **u** and **v**. i.e. Express **w** in terms of **u** and **v**.



$$\mathbf{w} \cdot \mathbf{u} = \frac{-3}{\sqrt{2}}, \ \ \mathbf{w} \cdot \mathbf{v} = \frac{-7}{\sqrt{2}}. \ \ ||\mathbf{u}|| = ||\mathbf{v}|| = 1$$

So

$$\text{comp}_{\mathbf{u}}\mathbf{w} = \frac{-3}{\sqrt{2}}, \ \text{comp}_{\mathbf{v}}\mathbf{w} = \frac{-7}{\sqrt{2}}.$$

and

$$\mathbf{w} = \frac{-3}{\sqrt{2}}\mathbf{u} + \frac{-7}{\sqrt{2}}\mathbf{v}$$

Orthogonal Projections

Given a non-zero vector $\mathbf{v}$, we may represent any vector $\mathbf{u}$ as a sum of a vector, $\mathbf{u}_{||}$ parallel to $\mathbf{v}$ and a vector $\mathbf{u}_{\perp}$ perpendicular to $\mathbf{v}$.

So, $\boxed{\mathbf{u} = \mathbf{u}_{||} + \mathbf{u}_{\perp}.}$

Now, $\boxed{\mathbf{u}_{||} = \text{proj}_{\mathbf{v}}\mathbf{u}.}$

and so $\boxed{\mathbf{u}_{\perp} = \mathbf{u} - \text{proj}_{\mathbf{v}}\mathbf{u}.}$

Example 3

Express $\mathbf{u} = 2\mathbf{i}+4\mathbf{j}+2\mathbf{k}$ as a sum of vectors parallel and perpendicular to $\mathbf{v} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$.

$\mathbf{u}\cdot\mathbf{v} = 2+8-2 = 8$, $\;||\mathbf{v}||^2 = \left(\sqrt{1^2+2^2+1^2}\right)^2 = 6$

$$\mathbf{u}_{||} = \text{proj}_{\mathbf{v}}\mathbf{u} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{||\mathbf{v}||^2}\right)\mathbf{v} = \frac{4}{3}(\mathbf{i}+2\mathbf{j}-\mathbf{k})$$

$$\begin{aligned}
\mathbf{u}_\perp &= \mathbf{u} - \text{proj}_{\mathbf{v}}\mathbf{u} \\
&= (2\mathbf{i}+4\mathbf{j}+2\mathbf{k}) - \tfrac{4}{3}(\mathbf{i}+2\mathbf{j}-\mathbf{k}) \\
&= \left(2-\tfrac{4}{3}\right)\mathbf{i} + \left(4-\tfrac{8}{3}\right)\mathbf{j} + \left(2+\tfrac{4}{3}\right)\mathbf{k} \\
&= \tfrac{6-4}{3}\mathbf{i} + \tfrac{12-8}{3}\mathbf{j} + \tfrac{6+4}{3}\mathbf{k} \\
&= \tfrac{2}{3}\mathbf{i} + \tfrac{4}{3}\mathbf{j} + \tfrac{10}{3}\mathbf{k} \\
&= \tfrac{2}{3}(\mathbf{i}+2\mathbf{j}+5\mathbf{k})
\end{aligned}$$

Check

$$\begin{aligned}
\mathbf{u}_{||}\cdot\mathbf{u}_\perp &= \left(\tfrac{2}{3}(\mathbf{i}+2\mathbf{j}+5\mathbf{k})\right)\cdot\left(\tfrac{4}{3}(\mathbf{i}+2\mathbf{j}-\mathbf{k})\right) \\
&= \tfrac{8}{9}((\mathbf{i}+2\mathbf{j}+5\mathbf{k})\cdot(\mathbf{i}+2\mathbf{j}-\mathbf{k})) \\
&= \tfrac{8}{9}(1+4-5) \\
&= 0
\end{aligned}$$

So $\mathbf{u}_{||}$ and $\mathbf{u}_\perp$ are orthogonal.