# Matrices

## Definitions

### Definition 1

1. A Matrix is an $m \times n$ ($m$ by $n$) array of numbers.

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

2. The entries in a matrix are called the *components* of the matrix and can be written as $a_{ij}$, where $i$ indicates the row number and runs from 1 to $m$, and $j$ indicates the column number and runs from 1 to $n$.

3. A Vector is a $1 \times n$ or $n \times 1$ matrix. That is an ordered set of $n$ numbers.
We say that such a vector is of dimension $n$.

4. A scalar is a number (usually either real or complex).

**Notation 2**

- We generally use uppercase letters from the beginning of the alphabet $(A, B, C \ldots)$ to denote matrices.

- A matrix is identified with its components, given a matrix $A$ with components $a_{ij}$, $1 \le i \le m$, $1 \le j \le n$, we may write

$$A = [a_{ij}]$$

**Example 3**

Find the $4 \times 4$ matrix $A$ with compoents given by $a_{ij} = i + j$.

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{pmatrix}$$

- We generally use lowercase boldface letters from the end of the alphabet $(\mathbf{u}, \mathbf{v}, \mathbf{w} \ldots)$ to denote vectors.

- We use the convention that $\mathbf{u} = (u_1, u_2, \ldots, u_n)$, $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, etc.

- If $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ then the scalars $x_1, x_2, \ldots, x_n$ are called the <u>components</u> of $\mathbf{x}$.

- We denote the set of all vectors of dimension $n$ whose components are real numbers by $\mathbb{R}^n$.

- We denote the set of all vectors of dimension $n$ whose components are complex numbers by $\mathbb{C}^n$.

**Note** This definition of vector differs from the usual 'High School' definition involving magnitude and direction.

# Special Matrices and Vectors

1. **The Identity matrix**
   The identity matrix is a square matrix with 1's down the diagonal, and zeros elsewhere. The $n \times n$ identity matrix is denoted $I_n$.
   $$I_n = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}$$

2. **The Zero Matrix**
   The zero matrix is an $m \times n$ matrix, all of whose entries are 0.
   $$\begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}$$

# Operations on Matrices

1. **Transpose**

   Given an $m \times n$ matrix, $A$, the transpose of $A$ is obtained by interchanging the rows and columns of $A$. We denote the transpose of $A$ by $A^t$, or $A^T$.

   **Notes:**

   - If $A$ is $m \times n$ then $A^T$ will be $n \times m$.
   - If $A = [a_{ij}]$, then $[a_{ij}]^T = [a_{ji}]$.

   **Example 4**

   (a)
$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}^t = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$$

   (b)
$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}^t = \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix}$$

   (c)
$$(1, 2, 3)^t = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

2. **Matrix Addition**

Given two $m \times n$ matrices

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \dots & b_{mn} \end{pmatrix}$$

We may define the sum of $A$ and $B$, $A+B$, to be the sum componentwise, i.e.

$$A+B = \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \dots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \dots & a_{2n}+b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \dots & a_{mn}+b_{mn} \end{pmatrix}$$

Componentwise: $\left[a_{ij}\right] + \left[b_{ij}\right] = \left[a_{ij} + b_{ij}\right]$.

This works for vectors as well.

$$\begin{aligned} \mathbf{u} + \mathbf{v} &= (u_1, u_2, \dots, u_n) + (v_1, v_2, \dots, v_n) \\ &= (u_1 + v_1, u_2 + v_2, \dots, u_n + v_n) \end{aligned}$$

**Note** that matrix addition is only defined if $A$ and $B$ have the same size.

**Example 5**

(a)

$$\begin{aligned} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} + \begin{pmatrix} 10 & 11 & 12 \\ 13 & 14 & 15 \\ 16 & 17 & 18 \end{pmatrix} \\ = \begin{pmatrix} 1+10 & 2+11 & 3+12 \\ 4+13 & 5+14 & 6+15 \\ 7+16 & 8+17 & 9+18 \end{pmatrix} \\ = \begin{pmatrix} 11 & 13 & 15 \\ 17 & 19 & 21 \\ 23 & 25 & 27 \end{pmatrix} \end{aligned}$$

(b)

$$\begin{aligned} (1,2,3) + (4,5,6) &= (1+4, 2+5, 3+6) \\ &= (5,7,9) \end{aligned}$$

## 3. **Matrix Multiplication**

### (a) **Scalar Multiplication**

Given a matrix $A$, and a scalar $k$, we define the <u>scalar product</u> of $k$ with $A$, $kA$ by multiplying each entry of $A$ by $k$.

$$\begin{aligned} kA = \quad & k\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \\ = \quad & \begin{pmatrix} ka_{11} & ka_{12} & \dots & ka_{1n} \\ ka_{21} & ka_{22} & \dots & ka_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ ka_{m1} & ka_{m2} & \dots & ka_{mn} \end{pmatrix} \end{aligned}$$

Componentwise: $k\left[a_{ij}\right] = \left[ka_{ij}\right]$.

**Note** that this works for vectors as well.

$$k\mathbf{u} = k(u_1, u_2, \dots, u_n) = (ku_1, ku_2, \dots, ku_n)$$

**Example 6**

$$10\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = \begin{pmatrix} 10 & 20 & 30 \\ 40 & 50 & 60 \\ 70 & 80 & 90 \end{pmatrix}$$

(b) **Matrix Multiplication**

If $A$ and $B$ are two matrices where $A$ has the same number of columns as $B$ has rows (i.e. $A$ is $m \times n$ and $B$ is $n \times r$) we define the matrix product, $AB$ to be the matrix in which the $i,j^{\text{th}}$ entry is made up of the dot product of the $i^{\text{th}}$ row of $A$ with the $j^{\text{th}}$ column of $B$.

$$A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{pmatrix},$$

$$B = \begin{pmatrix} b_{11} & b_{12} & \ldots & b_{1r} \\ b_{21} & b_{22} & \ldots & b_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \ldots & b_{nr} \end{pmatrix}$$

$$AB = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & \ldots & b_{1r} \\ b_{21} & b_{22} & \ldots & b_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \ldots & b_{nr} \end{pmatrix}$$

$$= \left( \begin{matrix} a_{11}b_{11} + a_{12}b_{21} + \ldots + a_{1n}b_{n1} & \ldots & a_{11}b_{1r} + a_{12}b_{2r} + \ldots + a_{1n} \\ a_{21}b_{11} + a_{22}b_{21} + \ldots + a_{2n}b_{n1} & \ldots & a_{21}b_{1r} + a_{22}b_{2r} + \ldots + a_{2n} \\ \vdots & \ddots & \vdots \\ a_{m1}b_{11} + a_{m2}b_{21} + \ldots + a_{mn}b_{n1} & \ldots & a_{m1}b_{1r} + a_{m2}b_{2r} + \ldots + a_{m} \end{matrix} \right.$$

**Example 7**

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \\ 3 & 2 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} 1\times 9+2\times 6+3\times 3 & 1\times 8+2\times 5+3\times 2 & 1\times 7+2\times 4+3\times 1 \\ 4\times 9+5\times 6+6\times 3 & 4\times 8+5\times 5+6\times 2 & 4\times 7+5\times 4+6\times 1 \\ 7\times 9+8\times 6+9\times 3 & 7\times 8+8\times 5+9\times 2 & 7\times 7+8\times 4+9\times 1 \end{pmatrix}$$

$$= \begin{pmatrix} 9+12+9 & 8+10+6 & 7+8+3 \\ 36+30+18 & 32+25+12 & 28+20+6 \\ 63+48+27 & 56+40+18 & 49+32+9 \end{pmatrix}$$

$$= \begin{pmatrix} 30 & 24 & 18 \\ 84 & 69 & 54 \\ 138 & 114 & 90 \end{pmatrix}$$

**Note** that Matrix multiplication is only defined if $A$ has the same number of columns as $B$ has rows.

# BIG Note

Matrix multiplication is **NOT** commutative.
i.e. It is **NOT** true that $AB = BA$ (where defined).

**Example 8**

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$