Lines in $\mathbb{R}^3$

We wish to represent lines in $\mathbb{R}^3$. Note that a line may be described in two different ways:

- By specifying two points on the line.
- By specifying one point on the line and a vector parallel to it.

If we are given two points, P and Q on a line, then a vector parallel to it is $\vec{PQ}$.

There are three ways of representing a line algebraically.

- **Vector Representation of a Line**

  Given a point $P = (x_0, y_0, z_0)$ on the line and a vector $\mathbf{v} = (a, b, c)$ parallel to it.

  An arbitrary point $X = (x, y, z)$ on the line will be given by the vector equation:

$$\vec{OX} = \vec{OP} + t\mathbf{v}.$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + t \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

  If we are given two points, P and Q on the line, then take $\mathbf{v} = \vec{PQ}$.

- **Parametric Representation of a Line**

  Given a point $P = (x_0, y_0, z_0)$ on the line and a vector $\mathbf{v} = (a, b, c)$ parallel to it.

  An arbitrary point $X = (x, y, z)$ on the line will be given by the system of equations:

$$\begin{array}{rcrcl} x & = & x_0 & + & ta \\ y & = & y_0 & + & tb \\ z & = & z_0 & + & tc \end{array}$$

  If we are given two points, P and $Q = (x_1, y_1, z_1)$ on the line, then take $\mathbf{v} = \vec{PQ}$ and this becomes.

$$\begin{array}{rcrcl} x & = & x_0 & + & t(x_1 - x_0) \\ y & = & y_0 & + & t(y_1 - y_0) \\ z & = & z_0 & + & t(z_1 - z_0) \end{array}$$

- **Symmetric Representation of a Line**

  Solving for t gives the Symmetric Representation of a Line.

$$t = \frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

  Note that this is only definied if a, b and c are non-zero.

Example 1

1. Find the equation of the line ℓ joining $P = (1, 1, 1)$ to $Q = (1, 0, 1)$ in vector form.

   $\vec{PQ} = (0, -1, 0)$, so the equation os given by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + t \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$$

2. Find the equation in parametric form of the line ℓ above.

$$\begin{aligned} x &= 1 \\ y &= 1 - t \\ z &= 1 \end{aligned}$$

3. Find the equation in symmetric form of the line ℓ above.

   This does not exist.

4. Does $R = (1, 2, 2)$ lie on ℓ?

   Substituting $R = (1, 2, 2)$ for $X = (x, y, z)$ we get
   $$\begin{aligned} 1 &= 1 \\ 2 &= 1 - t \\ 2 &= 1 \end{aligned}$$
   Which has no solution, so R does not lie on the line.

5. Does $S = (1, 2, 1)$ lie on ℓ?

   Substituting $S = (1, 2, 1)$ for $X = (x, y, z)$ we get
   $$\begin{aligned} 1 &= 1 \\ 2 &= 1 - t \\ 1 &= 1 \end{aligned}$$
   Which is true when $t = -1$, so this lies on the line.

6. With the parameterization above at what point will we be when $t = -2$

   When $t = -2$ we will be at $(1, 3, 1)$.

Planes in $\mathbb{R}^3$

We wish to represent planes in $\mathbb{R}^3$. Note that a plane may be described in three different ways:

- By specifying three points on the plane.
- By specifying one point in the plane and two vectors parallel to it.
- By specifying one point in the plane and a vector perpendicular to it.

The third form is preferable since it needs the least information.

Let π be a plane described by a vector $\mathbf{n} = (a, b, c)$ orthogonal to it and a point $P = (x_0, y_0, z_0)$ which lies in it.

Consider a point $Q = (x, y, z)$ on the plane π.

Since $\mathbf{n}$ is orthogonal to the plane $\mathbf{n} \cdot \mathbf{v} = 0$ for any vector $\mathbf{v}$ parallel to the plane.

Now $\vec{PQ} = (x - x_0, y - y_0, z - z_0)$ is in the plane.

So $\mathbf{n} \cdot \vec{PQ} = 0$, or

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

This is called the *point normal form* of the equation of a plane.

Setting $d = ax_0 + by_0 + cz_0 = \mathbf{n} \cdot \vec{OP}$, we get

$$ax + by + cz = d$$

This is called the *standard form* of the equation of a plane.

Example 2

1. Find the equation of the plane π which is orthogonal to the vector $\mathbf{n} = (1, 1, 2)$ and through the point $P = (1, 0, 1)$

$$d = \mathbf{n} \cdot \vec{OP} = (1, 1, 2) \cdot (1, 0, 1) = 3$$

Equation is $\mathbf{n} \cdot \vec{OX} = d$

$$x + y + 2z = 3$$

2. Is $Q = (1, 2, 3) \in \pi$?

   Put $X = Q$ in the equation to get

   $$1 + 2 + 2 \times 3 = 9 \neq 3$$

   So equation is inconsistent and $Q \notin \pi$.

3. Is $Q = (3, 2, -1) \in \pi$?

   Put $X = Q$ in the equation to get

   $$3 + 2 + 2 \times (-1) = 3$$

   So equation is consistent and $Q \in \pi$.

If to vectors $\mathbf{u}$ and $\mathbf{v}$ parallel to the plane are given we can take solve the equations

$$\mathbf{n} \cdot \mathbf{u} = 0$$
$$\mathbf{n} \cdot \mathbf{v} = 0.$$

Generally this will result in an infinite solution set (a line through the origin). Any vector parallel to this line will work for $\mathbf{n}$ as all are perpendicular to the plane. The magnitude of $\mathbf{n}$ will affect the value of d.

Alternately we can use the cross product $\mathbf{n} = \mathbf{u} \times \mathbf{v}$. (see Section 4.3).

If a plane is defined three points P, Q and R in the plane then $\vec{PQ}$, $\vec{PR}$ and $\vec{QR}$ are all vectors parallel to the plane and the method outlined above may be used.

Example 3

Find the equation of the plane π parallel to $\mathbf{u} = (1, 0, 1)$ and $\mathbf{v} = (2, -1, 2)$, through $P = (1, 1, 1)$.

Let $\mathbf{n} = (n_1, n_2, n_3)$ be the normal vector to π.

$$\begin{aligned}\mathbf{u} \cdot \mathbf{n} = 0 &\Rightarrow (1, 0, 1) \cdot (n_1, n_2, n_3) = \\ &\quad n_1 + n_3 = 0. \\ \mathbf{v} \cdot \mathbf{n} = 0 &\Rightarrow (2, -1, 2) \cdot (n_1, n_2, n_3) = \\ &\quad 2n_1 - n_2 + 2n_3 = 0.\end{aligned}$$

The solution to the set of simultaneous equations

$$\begin{aligned}n_1 + n_3 = 0. \\ 2n_1 - n_2 + 2n_3 = 0.\end{aligned}$$

are vectors of the form $(t, 0, -t)$, for any $t \in \mathbb{R}$. We arbitrarily pick $t = 1$, so $\mathbf{n} = (1, 0, -1)$.

Alternately, using cross product

$$\mathbf{n} = \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 1 \\ 2 & -1 & 2 \end{vmatrix} = \mathbf{i} - \mathbf{k}$$

Now, $d = \mathbf{n} \cdot \vec{OP} = (1, 0, -1) \cdot (1, 1, 1) = 1 - 1 = 0$.

So the equation of the plane is

$$x - z = 0$$

Example 4
Find the equation of the plane through the points $P = (1, 1, 1)$, $Q = (0, 1, 2)$ and $R = (1, 1, 2)$

$\vec{PQ} = (-1, 0, 1)$ and $\vec{QR} = (1, 0, 0)$ are two vectors in the plane. Let $\mathbf{n} = (n_1, n_2, n_3)$ be the normal vector to the plane.

$$\vec{PQ} \cdot \mathbf{n} = 0 \Rightarrow (-1, 0, 1) \cdot (n_1, n_2, n_3) = -n_1 + n_3 = 0.$$
$$\vec{QR} \cdot \mathbf{n} = 0 \Rightarrow (1, 0, 0) \cdot (n_1, n_2, n_3) = n_1 = 0.$$

The solution to the set of simultaneous equations

$$-n_1 + n_3 = 0, \text{ and } n_1 = 0$$

are vectors of the form $(0, t, 0)$, for any $t \in \mathbb{R}$. We arbitrarily pick $t = 1$, so $\mathbf{n} = (0, 1, 0)$.

Alternately, using cross product:

$$\mathbf{n} = \vec{PQ} \times \vec{QR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} = (0, 1, 0)$$

Now, $d = \mathbf{n} \cdot \vec{OP} = (0, 1, 0) \cdot (1, 1, 1) = 1$.

So the equation of the plane is $y = 1$

Example 5

1. Find the point of intersection of the plane $x - z = 0$ with the line ℓ:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + t \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

The line can be expressed as

$$x = 2 + 2t, \;\; y = 1 - t, \;\; z = 1 + t.$$

Intersection (x, y, z) for x, y and z which satisfy both the equation of the line and that of the plane. Substitute equation for the line into the equation for the plane and solve for t.

$$\underbrace{(2 + 2t)}_{x} - \underbrace{(1 + t)}_{z} = 0$$

$$t + 1 = 0$$

So the solution is $t = -1$.

Substituting back in the equation for the line ℓ we get the point of intersection:
$(2 - 2(-1), 1 - (-1), 1 - 1) = (0, 2, 0)$

2. Find the point of intersection of π with the line

   ℓ: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + t \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$

   Intersect when these are the same x, y and z. $x = 1 + t$, $y = 1 + 2t$, $z = 1 + t$, substitute in equation for plane and solve for t

   $$\underbrace{(1+t)}_{x} - \underbrace{(1+t)}_{z} = 0$$

   $$0 = 0$$

   Many solutions, so the line is in the plane.

3. Find the point of intersection of π with the line

   ℓ: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + t \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$

   Intersect when these are the same x, y and z. $x = 2 + t$, $y = 1 + 2t$, $z = 1 + t$, substitute in equation for plane and solve for t

   $$(2+t) - (1+t) = 0$$

   $$1 = 0$$

   No solution, so the line is parallel to the plane, but not in it.